PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 2, 1973

SIMPLE PERIODIC RINGS

WILLIAM MITCHELL

Let A be a power-associative ring, and suppose that for each $a \in A$ there exists an integer $n = n(a) > 1$ such that $a^n = a$. Such a ring A is called a *periodic ring*. In this paper the structure of all simple periodic rings of characteristic not 2 or 3 is determined. This solves a problem posed by Osborn [Varieties of Algebras, Advances in Mathematics, to appear]. It follows from these results and from Osborn's that every flexible periodic ring with no elements of additive order 2 or 3 is a Jordan ring.

Let A denote a simple periodic ring. It is shown in [1] that every element of a periodic ring has finite additive order, and it is known that any simple ring has a well-defined characteristic. Thus A must be an algebra over Z_p, the integers modulo p, for some prime p. We suppose that $p \neq 2$ or 3 in order to use the results of [1]. Definitions not given here may be found in [1].

Let the multiplication give in A be denoted by juxtaposition and let the operation "o" in A be defined by $aob = 1/2(ab + ba)$ for $a, b \in A$. The algebra formed by taking the elements of A under the same operation of addition but under the new multiplication "o" is denoted by A^+. It is shown in [1] that A^+ is a simple Jordan algebra and that if A is not a field than A^+ is a periodic Jordan algebra of capacity 2. By a Jordan algebra J of capacity 2 we mean a simple Jordan algebra in which there exist two orthogonal idempotents e_1, e_2 adding to the unity quantity and having the property that the Peirce subspaces $J_1(e_1)$ add $J_1(e_2)$ are Jordan division algebras. Periodic Jordan algebras of capacity 2 are characterized in [1] by

PROPOSITION. *Let Φ be a periodic field of characteristic not 2, let μ be a nonsquare in Φ and let Φ_2 denote the ring of 2×2 matrices over Φ. Then the Jordan subalgebra of Φ^+ consisting of the set*

$$J = \left\{ \begin{pmatrix} \alpha & \beta\mu \\ \beta & \gamma \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in \Phi \right\}$$

is a simple periodic Jordan algebra of capacity 2 over Φ. Conversely, every simple periodic Jordan ring of capacity 2 and characteristic not 2 is isomorphic to J for some choice of Φ.

In view of this proposition we may identify the elements of A with 2×2 matrices of the form

$$\begin{pmatrix} \alpha & \beta\mu \\ \beta & \gamma \end{pmatrix}$$

over some periodic field Φ where μ is a fixed nonsquare of Φ. Let juxtaposition of two such matrices denote the product regarding them as elements of A, let "o" denote the product regarding them as in A^+, and let "$\cdot$" denote the usual matrix product. Then by the above proposition these products are connected by

$$ab + ba = 2(aob) = a\cdot b + b\cdot a \tag{1}$$

for all $a, b \in A$. If $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $g = \begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix}$, then the set $\{e, f, g\}$ will be a vector-space basis for A^+ over Φ. Since elements of Φ do not act as scalars on the algebra A, we cannot make A an algebra over Φ, but have to work with it as an algebra over Z_p. If $\{\omega_1, \omega_2, \cdots\}$ is a basis of Φ over Z_p, then $\{\omega_1 e, \omega_2 e, \cdots, \omega_1 f, \omega_2 f, \cdots, \omega_1 g, \omega_2 g, \cdots\}$ form a basis of A over Z_p, and we must determine how these different basis elements multiply together.

Since we know the multiplication in A^+, we also know the product of any two elements of A that commute. Most of our calculations will be to show that appropriate pairs of basis elements of A commute. As any two nonzero elements of Φ may be expressed as powers of a single element of Φ, it will suffice to determine the products $\lambda^i e\lambda^j e$, $\lambda^i e\lambda^j f$, $\lambda^i e\lambda^j g$, $\lambda^i f\lambda^j f$, $\lambda^i f\lambda^j g$ and $\lambda^i g\lambda^j g$ for arbitrary $\lambda \in \Phi$.

The element e is the identity for A^+ and the power-associativity of A implies that e is the identity for A. Hence $ee = e$, $ef = f = fe$, and $eg = g = ge$. Using (1) we also have

$$ff = fof = f\cdot f = f \text{ and } gg = gog = g\cdot g = \mu e\ .$$

Since A is power-associative, it satisfies the identity $x^2x = xx^2$. Defining the notation $(x, y, z) \equiv (xy)z - x(yz)$ and $[x, y] = xy - yx$, this identity can be written either as $(x, x, x) = 0$ or $[x^2, x] = 0$. The latter form has the linearization

$$Q(x, y, z) \equiv [xoy, z] + [yoz, x] + [zox, y] = 0 \tag{2}$$

which will be of fundamental importance in subsequent calculations. We note at this point that (2) is symmetric in all variables.

We first compute the value of $\lambda^i e\lambda^j e$. Substituting $x = \lambda^{i-1}e$, $y = \lambda e$, $z = \lambda^j e$ in (2) and using the fact that $\lambda^r eo\lambda^s e = \lambda^{r+s}e$ for r, s positive integers, we obtain

$$[\lambda^i e, \lambda^j e] + [\lambda^{j+1}e, \lambda^{i-1}e] + [\lambda^{i+j-1}e, \lambda e] = 0\ . \tag{3}$$

This equation is generalized in

LEMMA 1. *$[\lambda^i e, \lambda^j e] + [\lambda^{j+l} e, \lambda^{i-l} e] + l[\lambda^{i+j-1} e, \lambda e] = 0$ where i, j, l are positive integers.*

Proof. The proof is by induction on l. The case $l = 1$ is (3). Now assume

$$[\lambda^i e, \lambda^j e] + [\lambda^{j+l} e, \lambda^{i-l} e] + l[\lambda^{i+j-1} e, \lambda e] = 0 \tag{4}$$

holds. Substituting $x = \lambda^{i-l-1} e$, $y = \lambda e$, $z = \lambda^{i+l} e$ in (2) gives the relation

$$[\lambda^{i-l} e, \lambda^{j+l} e] + [\lambda^{j+l+1} e, \lambda^{i-l-1} e] + [\lambda^{i+j-1} e, \lambda e] = 0 . \tag{5}$$

Adding (4) and (5) gives

$$[\lambda^i e, \lambda^j e] + [\lambda^{j+(l+1)} e, \lambda^{i-(l+1)} e] + (l+1)[\lambda^{i+j-1} e, \lambda e] = 0 ,$$

and hence the lemma holds by induction on l.

The substitution $i = p$, $j = (k-1)p$, $l = p$ in (4) gives $[\lambda^p e, \lambda^{(k-1)p} e] + 0 + p[\lambda^{kp-1} e, \lambda e] = 0$. Since A has characteristic p this last equation becomes

$$[\lambda^p e, \lambda^{(k-1)p} e] = 0 . \tag{6}$$

Let n be a positive integer such that $\lambda^{p^n} = \lambda$. Replacing λ by $\lambda^{p^{n-1}}$ in (6) shows that

$$[\lambda e, \lambda^{k-1} e] = 0, k \text{ any positive integer} . \tag{7}$$

Since i, j, l are arbitrary positive integers in (4), (4) and (7) combine to give $[\lambda^i e, \lambda^j e] + [\lambda^{j+l} e, \lambda^{i-l} e] + l[\lambda^{i+j-1} e, \lambda e] = [\lambda^i e, \lambda^j e] + [\lambda^{j+l} e, \lambda^{i-l}] = 0$, and setting $i = l$ in this last equation gives

$$[\lambda^i e, \lambda^j e] = 0, i, j \text{ positive integers} . \tag{8}$$

The value of $\lambda^i f \lambda^j f$ is computed next. The substitution $x = \lambda e$, $y = z = f$ in (2) gives $[\lambda e o f, f] + [f \circ f, \lambda e] + [f o \lambda e, f] = 0$ and the middle commutator is $[e, \lambda e] = 0$. Hence this last equation reduces to $2[\lambda e o f, f] = [\lambda f, f] = 0$, and replacing λ by λ^i generalizes this to

$$[\lambda^i f, f] = 0, i \text{ a positive integer} . \tag{9}$$

The substitution $x = \lambda^{i-1} f$, $y = \lambda e$, $z = \lambda^j_f$ in (2) gives

$$[\lambda^i f, \lambda^j f] + [\lambda^{j+1} f, \lambda^{i-1} f] + [\lambda^{i+j-1} e, \lambda e] = 0$$

and (8) shows that the last commutator of this equation is 0. Hence this equation reduces to $[\lambda^i f, \lambda^j f] + [\lambda^{j+1} f, \lambda^{i-1} f] = 0$, and setting $i = 1$ and using (9) reduces this last expression to

$$[\lambda f, \lambda^j f] = 0, j \text{ a positive integer} . \tag{10}$$

The substitution $x = \lambda^{i-1}e,\ y = \lambda f,\ z = \lambda^j f$ in (2) gives the equation

$$[\lambda^{i-1}eo\lambda f, \lambda^j f] + [\lambda f o \lambda^j f, \lambda^{i-1}e] + [\lambda^j f 0 \lambda^{i-1}e, \lambda f] = 0\ ,$$

and evaluating the Jordan product in these commutators in terms of Jordan product of 2×2 matrices converts this last equation to

$$[\lambda^i f, \lambda^j f] + [\lambda^{j+1}e, \lambda^{j-1}e] + [\lambda^{i+j-1}f, \lambda f] = 0$$

which by (8) and (10) reduces to the equation

(11) $$[\lambda^i f, \lambda^j f] = 0,\ i, j \text{ positive integers}\ .$$

We note that just as in (8), equation (11) implies

$$\begin{aligned}\lambda^i f o \lambda^j f &= \frac{1}{2}(\lambda^i f \lambda^j f + \lambda^j f \lambda^i f) = \frac{1}{2}(2\lambda^i f \lambda^j f) = \lambda^i f \lambda^j f \\ &= \lambda^{i+j} f\ .\end{aligned}$$

The next case to calculate is the product $\lambda^i g \lambda^j g$. The substitution $x = \lambda^{i-1}g,\ y = \lambda e,\ z = \lambda^j g$ in (2) gives

$$[\lambda^i g, \lambda^j g] + [\lambda^{j+1}g, \lambda^{j-1}g] + [\lambda^{i+j-1}(\mu e), \lambda e] = 0\ .$$

But $\lambda^{i+j-1}\mu$ and λ may be expressed as powers of a single element of Φ. But then (7) shows that the last commutator in the above equation vanishes and that equation thus reduces to $[\lambda^i g, \lambda^j g] + [\lambda^{j+1}g, \lambda^{i-1}g] = 0$ and setting $i = 1$ in this last equation yields

(12) $$[\lambda g, \lambda^j g] + [\lambda^{j+1}g, g] = 0\ .$$

Since the substitution $x = \lambda e,\ y = z = g$ in (2) gives $2[\lambda g, g] + [\mu e, \lambda e] = 0$, (3) shows that this reduces to $[\lambda g, g] = 0$, or more generally that $[\lambda^i g, g] = 0$ for i a positive integer. Hence (12) reduces to

(13) $$[\lambda g, \lambda^j g] = 0,\ j \text{ a positive integer}\ .$$

Finally, the substitution $x = \lambda^{i-1}e,\ y = \lambda g,\ z = \lambda^j g$ in (2) gives

$$[\lambda^i g, \lambda^j g] + [\lambda^{j+1}(\mu e), \lambda^{i-1}e] + [\lambda^{i+j-1}g, \lambda g] = 0$$

and the second commutator of this equation vanishes by (8) and the third commutator vanishes by (13) so that the equation reduces to

(14) $$[\lambda^i g, \lambda^j g] = 0,\ i, j \text{ positive integers}\ .$$

To determine products of the form $\lambda^i e \lambda^j g$ make the substitution $x = \lambda^j g,\ y = \lambda^{i-1}f,\ z = \lambda f$ in (1) and thus obtain

$$[\lambda^j g o \lambda^{i-1}f, \lambda f] + [\lambda^{i-1}f o \lambda f, \lambda^j g] + [\lambda f o \lambda^j g, \lambda^{i-1}f] = 0\ .$$

Since $\lambda^r f o \lambda^s g = \lambda^t g o \lambda^v f = 0$ for positive integers r, s, t, v the above equation reduces to

$$(15) \qquad [\lambda^i e, \lambda^j g] = 0,\ i, j \text{ positive integers}.$$

To compute products of the form $\lambda^i e \lambda^j f$ substitute $x = \lambda^j f$, $y = \mu^{-1}(\lambda g)$, $z = \lambda^{i-1} g$ in (1) to obtain

$$(16) \qquad [\lambda^i e, \lambda^j f] = 0,\ i, j \text{ positive integers}.$$

In remains to determine the products of the form $\lambda^i f \lambda^j g$. In order to do this the information obtained from (2) and the power-associative identity $(x^2 x)x = x^2 x^2$ must be studied. We first prove a preliminary lemma concerning power-associative algebras in general.

LEMMA 2. *Let A be an algebra over a field of characteristic $\neq 2$ for which A^+ is power-associative and let $x^2 x = x x^2$ hold in A. Then A is power-associative.*

Proof. The partial linearization $2[xoy, x] + (x^2, y] = 0$ of $x^2 x = x x^2$ holds in A. Setting $y = x^2$ gives $2[xox^2, x] = 0$. Hence $(x^2 o x)x = x(x^2 o x)$. Since $x^2 o x = x o x^2 = x^3$ we have $x^3 x = x x^3$ so that $x^3 x = (x x^2)x = x(x x^2) = x(x^2 x)$. But $(x^2 x)x = 1/4[(x^2 x)x + (x x^2)x + x(x^2 x) + x(x x^2)] = (x^2 o x) o x$. However $(x^2 o x) o x = (x o x) o (x o x)$ because A^+ is power-associative. Thus $(x^2 x)x = (x o x) o (x o x) = x^2 x^2 = x(x x^2)$. Hence $x^2 x^2 = (x^2 x)x = x(x^2 x)$ holds in A and A is power-associative.

The products $\lambda^i f \lambda^j g$ in A are now investigated in

LEMMA 3. *If $\lambda \in \Phi$ then*

$$(17) \qquad (\lambda^i f)(\lambda^j g) = f(\lambda^{i+j} g).$$

Thus all products of the form $\lambda^i f \lambda^i g$ are determined in A once the products $f \lambda^k g$ are determined in A. Conversely if A is an algebra such that A^+ is a periodic Jordan algebra of capacity 2 and in which (8), (11), (14), (15), (16) *and* (17) *hold, then A is power-associative.*

Proof. The substitution $x = \lambda^i f$, $y = \lambda^j g$ in $2[xoy, x] + [x^2, y] = 0$ yields $\lambda^{2i} e \lambda^j g = \lambda^j g \lambda^{2i} e$, which is just a special case of (15). The substitution $x = \lambda^i g$, $y = \lambda^j f$ in $2[xoy, x] + [x^2, y] = 0$ yields a special case of (16). Hence no information about $\lambda^i f \lambda^j g$ is obtained from the partial linearization of $x x^2 = x^2 x$. To obtain information from (2) for $\lambda^i f \lambda^j g$ at least one of x, y, z must be of the form $\lambda^i f$ and at least one of the remaining two variables must be of the form $\lambda^j g$. Since (2) is symmetric in x, y, z we may assume $x = \lambda^i f$, $y = \lambda g$ in any of the substitutions into (2) which will be of interest. For these substitutions we note at the beginning that $\lambda^i f o \lambda^j g = 0$ so that $\lambda^i f \lambda^j g = -\lambda^j g \lambda^i f$. We now consider the cases.

Case 1. $x = \lambda^i f$, $y = \lambda^j g$, $z = \lambda^k e$ in (2) gives

$$[\lambda^{j+k} g, \lambda^i f] + [\lambda^{i+k}, \lambda^j g] = 0 \tag{18}$$

Case 2. $x = \lambda^i f$, $y = \lambda^j g$, $z = \lambda^k f$ in (2) gives a special case of (15).

Case 3. $x = \lambda^i f$, $y = \lambda^j g$, $z = \lambda^k g$ in (2) gives a special case of (16). Hence (18) is the only new relation derived from (2). But (18) expanded out is $\lambda^{j+k} g \lambda^i f + \lambda^{i+k} f \lambda^j g = \lambda^i f \lambda^{j+k} g + \lambda^j g \lambda^{i+k} f$, which by anticommutativity of $\lambda^r f$ and $\lambda^s g$ becomes $2(\lambda^{j+k} g \lambda^i f + \lambda^{i+k} f \lambda^j g) = 0$. Thus $\lambda^{i+k} f \lambda^j g = -\lambda^{j+k} g \lambda^i f = \lambda^i f \lambda^{j+k} g$ and setting $i = 0$ yields the relation $(\lambda^k f)(\lambda^j g) = f(\lambda)^{k+j} g)$, which is just (17).

Formula (17) shows that once products of the form $f \lambda^i g$ are defined all products of the form $\lambda^r f \lambda^s g$, r, s positive integers, are determined. However the substitutions that we have just made above into (2) do not give any information about the value of the products $f \lambda^i g$ themselves. We have extracted all the information that can be extracted from (2) here. Put another way, equations (8), (11), (14), (15), (16), and (17) imply that (2) holds, using the linearity of (2) and the fact that any two nonzero elements of Φ are powers of a common element. By lemma 2, if (2) holds and if A^+ is power-associative, then A is power-asssociative. This establishes the last statement of Lemma 3.

Stating the relations that we have derived in equations (8), (11), (14), (15), (16) and (17) for our original basis $\{\omega_2 e, \omega_2 e, \cdots, \omega_1 f, \omega_2 f, \cdots, \omega_1 g, \omega_2 g, \cdots\}$, we obtain

$$\begin{aligned}[\omega_i e, \omega_j e] = [\omega_i e, \omega_j f] = [\omega_i e, \omega_j g] &= [\omega_j f, \omega_j f] \\ = [\omega_i g, \omega_j g] &= 0\,,\end{aligned} \tag{19}$$

$$(\omega_i f)(\omega_j g) = f(\omega_i \omega_j g) = -(\omega_j g)(\omega_i f)\,. \tag{20}$$

Using (1) we obtain from (19) that

$$\begin{aligned}\omega_i e \omega_j f &= \frac{1}{2}(\omega_i e \omega_j f + \omega_j f \omega_i e) = \frac{1}{2}(\omega_i e \cdot \omega_j f + \omega_j f \cdot \omega_i e) \\ &= \frac{1}{2}\omega_i \omega_j (e \cdot f + f \cdot e) = \omega_i \omega_j f\,.\end{aligned}$$

Doing the same thing for each other pair of basis elements that commute by (19), we obtain

$$\begin{aligned}\omega_i e \omega_j e = \omega_i \omega_j e,\ \omega_i e \omega_j f &= \omega_i \omega_j f,\ \omega_i e \omega_j g = \omega_i \omega_j g, \\ = \omega_i f \omega_j f &= \omega_i \omega_j e,\ \omega_i g \omega_j g = \omega_i \omega_j \mu e\,,\end{aligned} \tag{21}$$

for all positive integers i and j. The products of the form $f(\omega_i g)$ can be chosen completely arbitrarily and the other products involving f and g determined by (20), since we have seen that (2) is satisfied as long as (19) and (20) are satisfied. Clearly A^+ is power-associative if (21) and $\omega_i f o \omega_j g = 0$ hold, and the latter is implied by (20). We sum up our result as

THEOREM 1. *Let Φ be a subfield of the algebraic closure of Z_p the integers modulo p for $p \neq 2$ or 3 and let $\{\omega_1, \omega_2, \cdots\}$ be a basis of Φ over Z_p. Let e, f, g be three symbols and let A be the vector space over Z_p with basis consisting of the set of symbols $\{\omega_1 e, \omega_2 e, \cdots, \omega_1 f, \omega_2 f, \cdots, \omega_1 g, \omega_2 g, \cdots\}$. If $\beta_1, \beta_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$ and $\delta_1, \delta_2, \cdots$ are any three sequence of elements of Φ, then make A into an algebra by defining $f(\omega_i g) = \beta_i e + \gamma_i f + \delta_i g$ and by letting* (20) *and* (21) *hold. Then A is a simple periodic ring. Conversely, every simple periodic ring of characteristic not* 2 *or* 3 *either arises in this manner or is a field.*

Suppose now that our simple periodic ring A is flexible, that is, $(x, y, x) = 0$ for all $x, y \in A$. Then we can establish

THEOREM 2. *Let A be a simple periodic ring of characteristic $\neq$ 2 or 3 and let A satisfy the flexible law. Then A is a Jordan ring.*

Proof. We may clearly assume that A is not a field. Let A be as constructed in Theorem 1. For $\omega_1, \omega_2 \in \Phi$ the substitution $\omega_1 f$ for x and $\omega_2 g$ for y in the flexible law yields

$$(\omega_1 f \omega_2 g)(w_1 f) = -(\omega_1 f)(\omega_1 f \omega_2 g) . \tag{22}$$

The substitution $x = y = \omega_2 g$, $z = \omega_1 f$ in the full linearization $(x, y, z) + (z, y, x) = 0$ of the flexible law gives

$$\begin{aligned} 0 = (\omega_2^2 \mu e)(\omega_1 b) - (\omega_2 g)(\omega_2 \omega_1 g f) \\ + (\omega_1 \omega_2 f g)(\omega_2 g) - (\omega_1 f)(\omega_2^2 \mu e) , \end{aligned}$$

or

$$(\omega_1 \omega_2 f g)(\omega_2 g) = -(\omega_1 \omega_2 f g)(\omega_1 \omega_2 f g) . \tag{23}$$

Let $\omega_1 f \omega_2 g = \beta e + \gamma f + \delta g$ with $\beta, \gamma, \delta \in \Phi$. Equation (22) implies that $(\beta e + \gamma f + \delta g)(\omega_1 f) = -(\omega_1 f)(\beta e + \gamma f + \delta g)$ and using the fact that $\phi_1 f$ and $\phi_2 g$, $\phi_1 \in \Phi$ anticommute in A, this last equation reduces to $2\omega_1 \beta f + 2\omega_1 \lambda e = 0$. Since e and f are linearly independent over Ω we conclude that $\omega_1 \beta = \omega_1 \gamma = 0$, so that $\beta = \gamma = 0$. Thus $\omega_1 f \omega_2 g = \delta g$ for some $\delta \in \Omega$. Equation (23) shows that $\delta g \omega_2 g = -\omega_2 g \delta g$ so that $2\omega_2\delta\mu e = 0$. Thus $\delta = 0$. Hence $\omega_1 f \omega_2 g = 0$ for every $0 \neq \omega_1, \omega_2 \in \Omega$. Since (19) holds in A we conclude that products of pairs of basis elements of A commute. Hence the multiplication in A is the same as in A^+, so A is Jordan. This completes the proof of the theorem.

Theorem 2 may be used to establish the result that a periodic flexible ring with no elements of additive order 2 or 3 must be a Jordan algebra (see [1]).

Reference

1. J. Marshall Osborn, *Varieties of algebras*, Advances in Mathematics, to appear.

Received August 31, 1971.

University of Wisconsin