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EQUATIONS WITH OPERATORS FORMING A RIGHT ANGLE

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The operator B in a complex Hilbert space H is said to form an angle θ with the (stronger) operator A if $D(A) \subset D(B)$ and, for every x in $D(A)$, $(Ax, Bx)_H$ belongs to the cone $K(\theta)$ of all complex z with $|\arg(z)| \leqq \theta$. If A and B are closed maximal accretive operators and B forms a right angle with A, then $A + B$ is closed maximal accretive and the Cauchy problem for each of the equations $u'(t) + (A + B)u(t) = f(t)$ and $(I + B)u'(t) + Au(t) = f(t)$ is well-posed. Applications to partial differential equations are indicated in the second part.

1. Global perturbations. A linear operator $B\colon D(B) \to H$, $D(B) \subset H$, is accretive if $\mathrm{Re}(Bx, x)_H \geqq 0$ for all $x \in D(B)$ and maximal accretive if is accretive and has no proper accretive extension in H. An accretive operator B is closed if and only if the range $R(I + B)$ is closed. A maximal accretive operator is closed if and only if it has dense domain. For an accretive operator B, $R(I + B) = H$ if and only if B is closed and maximal accretive. These results are given in [6].

Let B be a closed and accretive operator in H. Then $R(I + B)$ is closed in H and $I + B$ is a bijection of $D(B)$ onto $R(I + B)$. Hence the set $J \equiv D(B)$ with the inner-product $(x, y)_J \equiv ((I + B)x, (I + B)y)_H$ is a Hilbert space. J is a subset of H and $(x, x)_J \geqq (x, x)_H$ for $x \in J$, so J is topologically imbedded in H.

Let A be an operator in H and assume $D(A) \subset D(B)$ and $R(A) \subset R(I + B)$. Define by $T \equiv (I + B)^{-1}A\colon D(A) \to J$ an operator in J.

LEMMA 1. *If A is closed then T is closed.*

Proof. Let $x_n \in D(A)$ and $\lim Tx_n = y$, $\lim x_n = x$ in J. These imply $\lim Ax_n = (I + B)y$ and $\lim x_n = x$ in H, respectively, so the result follows.

LEMMA 2. *If T is accretive in J and $R(I + A) \supset R(I + B)$, then $R(I + T)$ is dense in J.*

Proof. Let $x \in J$ be orthogonal to $R(I + T)$ and choose $z \in D(A)$ such that $(I + A)z = (I + B)x$. Then $0 = \mathrm{Re}(x, (I + T)z)_J = \mathrm{Re}((I + A)z, (I + B + A)z)_H = |(I + A)z|_H^2 + \mathrm{Re}(Tz, z)_J + \mathrm{Re}(z, Bz)_H - \mathrm{Re}(Az, z)_H$. This implies $\mathrm{Re}(Az, z)_H \geqq |(I + A)z|_H^2$ and thus $0 \geqq |z|^2 + \mathrm{Re}(Az, z)_H +$

$|Az|_H^2$, so $(I + A)z = (I + B)x = 0$. Hence $x = 0$.

THEOREM 1. *Let B be a closed accretive operator in H, A a closed operator with $R(A) \subset R(1 + B) \subset R(1 + A)$. Assume $I + B$ forms a right angle with A:* $\mathrm{Re}\,(Ax, (I + B)x)_H \geqq 0$ *for all $x \in D(A) \subset D(B)$. Then T is closed maximal accretive on J and $R(I + B + A) = R(I + B)$.*

Proof. The right angle condition is precisely the statement that T is accretive on J. Lemma 1 implies $R(I + T)$ is closed and hence (by Lemma 2) equal to J.

COROLLARY 1. *Let B be a closed maximal accretive operator in H, A a closed operator with $R(I + A) = H$, and assume $I + B$ forms a right angle with A. Then $R(I + B + A) = H$. If A is accretive (hence, maximal accretive) then $B + A$ is closed maximal accretive.*

We note here that if any two of the following three conditions hold, then so does the third: B forms a right angle with A, A is accretive (I forms a right angle with A), $I + \alpha B$ forms a right angle with A for every $\alpha > 0$. In particular the Corollary 1 is close to a result of [4].

The closed maximal accretive operators are characterized as the negatives of infinitesimal generators of strongly-continuous semigroups of contractions, so Corollary 1 gives a sufficient condition for the well-posedness of a Cauchy problem [5].

COROLLARY 2. *Let A and B be closed maximal accretive operators on H and assume $I + B$ forms a right angle with A. For each $u_0 \in D(A)$ and continuously differentiable $f\colon [0, \infty) \to H$, there is a unique continuously differentiable $u\colon [0, \infty) \to H$ with $u(0) = u_0$, $u(t) \in D(A)$ for $t > 0$ and*

$$u'(t) + (A + B)u(t) = f(t)\,. \tag{1}$$

This is a perturbation of the Cauchy problem for the equation

$$u'(t) + Au(t) = f(t) \tag{2}$$

by an (unbounded) operator B which is weaker than A [2]. This result is known to hold when B is replaced by a strongly continuously differentiable map $t \to B(t)$ of $[0, \infty)$ into the space of continuous linear operators on H [5]. Thus the term $B(t)x(t)$ can be added to (1) and a well-posed problem is obtained. Perturbations of a "local" type are known without our right angle condition [1, 2, 4]. See [1, 3, 6] for applications of (1) to parabolic and hyperbolic differential equations.

In the proof of Theorem 1 we showed that T is closed maximal accretive on J, so $-T$ generates a strongly continuous semigroup of contractions on J. This yields the following result.

COROLLARY 3. *Let B be a closed accretive operator in H, A a closed operator with $R(A) \subset R(I+B) \subset R(I+A)$. Assume $I+B$ forms a right angle with A. Then for $u_0 \in D(A)$, continuously differentiable f: $[0, \infty) \to H$ and strongly continuously differentiable $B(\cdot)$ from $[0, \infty)$ to the space of continuous linear operators from J to H, there is a unique continuously differentiable u: $[0, \infty) \to J$ with $u(0) = u_0$, $u(t) \in D(A)$ and $u'(t) \in D(B)$ for $t > 0$ and*

$$u'(t) + Bu'(t) + Au(t) + B(t)u(t) = f(t) . \tag{3}$$

Proof. It suffices to note that (3) is equivalent to the equation $u'(t) + Tu(t) + (I+B)^{-1}B(t)u(t) = (I+B)^{-1}f(t)$.

The equation (3) arises in applications wherein $B = cA$, A is a realization of partial differential operator in spatial variables, and c is a complex number [7, 9]. Our hypotheses hold if A is a closed accretive operator and $\operatorname{Re}(c) \geqq 0$.

Our second major result is a refinement of Theorem 1 under the (stronger) hypothesis that $I+B$ forms an acute angle with A.

THEOREM 2. *Let B be a closed accretive operator in H, A a closed operator with $R(A) \subset R(I+B) \subset R(I+A)$. Assume $I+B$ forms an angle $\theta < \pi/2$ with A. Then $-T$ generates an analytic semigroup on J.*

Proof. Since T is closed maximal accretive, $(\lambda + T)^{-1}$ is in the space $\mathscr{L}(J)$ of bounded linear maps on J and $\|(\lambda + T)^{-1}\| \leqq (\operatorname{Re}(\lambda))^{-1}$ whenever $\operatorname{Re}(\lambda) > 0$. It suffices to show that the operators $\{\lambda(\lambda + T)^{-1}$: $\operatorname{Re}(\lambda) > 0\}$ are uniformly bounded in $\mathscr{L}(J)$ [10].

The acute angle assumption implies the existence of a $k > 0$ such that

$$\operatorname{Re}(Tx, x)_J \geqq k|\operatorname{Im}(Tx, x)_J|,\ x \in D(A) , \tag{4}$$

and we may assume $k \leqq 1$. Letting $\lambda = \sigma + i\tau$, $\sigma > 0$, and $x \in D(A)$ we have

$$\operatorname{Re}((\lambda + T)x, x)_J = \sigma(x, x)_J + \operatorname{Re}(Tx, x)_J \tag{5}$$

and

$$|\operatorname{Im}((\lambda + T)x, x)_J| \geqq |\tau|(x, x)_J - |\operatorname{Im}(Tx, x)_J| . \tag{6}$$

If it were not true that

$$(7) \qquad |\mathrm{Im}\,((\lambda + T)x, x)_J| \geqq (|\tau|/2)(x, x)_J ,$$

then from (4), (6) and the negation of (7) we have

$$(8) \qquad \mathrm{Re}\,(Tx, x)_J \geqq (k|\tau|/2)(x, x)_J .$$

Thus, at least one of (7), (8) holds and this gives

$$|((\lambda + T)x, x)_J| \geqq (k|\tau|/2)(x, x)_J .$$

From this last estimate follows the inequality

$$||(\lambda + T)^{-1}|| \leqq (2/k|\tau|) .$$

But we already have this quantity bounded by $(1/\sigma)$ (cf. (5)), so we obtain finally,

$$||\lambda(\lambda + T)^{-1}|| \leqq 4/k, \mathrm{Re}\,(\lambda) > 0 .$$

COROLLARY. *For each* $u_0 \in D(B)$ *and Hölder continuous* f: $[0, \infty) \to H$, *there is a unique continuously differentiable* u: $[0, \infty) \to H$ *for which* $u(0) = u_0$, $u(t) \in D(A)$ *for* $t > 0$ *and* (3) *is satisfied* [1, 2].

2. Applications. The applications of the abstract Cauchy problem for (2) are well known [1, 3, 6] so we shall restrict our discussion to applications of (3). No attempt will be made to be comprehensive in any sense, but we shall give three elementary examples for which generalizations are obvious.

Let $H = L^2(0, 1)$, the Lebesgue square-summable (equivalence classes of) functions on the unit interval, and let $H^k(0, 1)$ be the Sobolev space of elements of $L^2(0, 1)$ whose derivatives through order k are in $L^2(0, 1)$ [1]. Let c be a complex number with $|c| \leqq 1$ and define $B = d/dx$ on $D(B) = \{\phi \in H^1(0, 1): \phi(0) = c\phi(1)\}$. Then B is closed maximal accretive in H. Let $A \equiv B$; then we have

$$\mathrm{Re}\,(A\phi, (I + B)\phi)_H = (1 - |c|^2)|\phi(1)|^2/2 + \int_0^1 |\phi'|^2$$

for $\phi \in D(A) = D(B)$, so $I + B$ forms a right angle with A. $J \equiv D(B)$ is a closed subspace of $H^1(0, 1)$, so we may define $B(t): J \to H$ by $B(t)\phi = b_1(t)\phi' + b^2(t)\phi$. $B(\cdot)$ is strongly continuously differentiable if b_1 and b_2 are continuously differentiable. Finally, let F be continuously differentiable on $[0, 1] \times [0, \infty)$ and define $f(t) = F(\cdot, t)$. Then f is continuously differentiable from $[0, \infty)$ to H. Thus, Corollary 3 implies that for each $u_0 \in D(B)$ there is a unique (generalized) solution $u(x, t)$ of

$$u_t + u_{xt} + u_x + b_1(t)u_x + b_2(t)u = F \tag{9}$$

in $(0, 1)\times(0, \infty)$ for which $u(x, 0) = u_0(x)$ and $u(0, t) = cu(1, t)$ for $t \geqq 0$. Thus our results apply to the hyperbolic equation (9) with boundary conditions specified on the characteristics. Furthermore, we can use the Poincaré inequality

$$\int_0^1 |\phi|^2 \leqq 2|\phi(1)|^2 + 4\int_0^1 |\phi'|^2, \ \phi \in H^1(0, 1)$$

to show that $I + B$ forms an acute angle with A when $|c| < 1$. This permits us to relax the smoothness requirements on $B(t)$ and $f(t)$ in (9).

For our second example we take H, B, $B(t)$ and $f(t)$ as above and define $A \equiv -(d/dx)^2$ on $D(A) = \{\phi \in H^2(0, 1): \phi(0) = c\phi(1), \bar{c}\phi'(0) = \phi'(1)\}$. Then A is closed maximal accretive in H and

$$\mathrm{Re}\,(A\phi, (I + B)\phi) = \int_0^1 |\phi'|^2 + |\phi'(0)|^2(1 - |c|^2)/2 \ ,$$

for $\phi \in D(A) \subset D(B)$, so $I + B$ forms a right angle with A. As before, we have for each $u_0 \in D(A)$ a unique solution of

$$u_t + u_{xt} + b_1(t)u_x + b_2(t)u = F \tag{10}$$

in $(0, 1)\times(0, \infty)$ for which $u(x, 0) = u_0(x)$ and $u(0, t) = cu(1, t)$, $\bar{c}u_x(0, t) = u_x(1, t)$ for $t \geqq 0$. We cannot improve the result to obtain an acute angle above, but this is expected since we would then have a regularity result (see below) too strong for the hyperbolic equation (10).

For our final example let G be a bounded open subset of $\boldsymbol{R}^n$ with G on one side of its infinitely differentiable boundary ∂G. $H^k(G)$ is the Sobolev space of (equivalence classes of) functions all of whose derivatives through order k are in $L^2(G)$. Let $\Delta \equiv \sum_{i=1}^n(\partial/\partial x_i)^2$ be the Laplacian operator on the domain $D(\Delta) = \{\phi \in H^2(G): \phi = 0 \text{ on } \partial G\}$. Then for each complex b with $\mathrm{Re}\,(b) \geqq 0$, the operator $B \equiv -b\Delta$ with $D(B) = D(\Delta)$ if $b \neq 0$ and $D(B) = H$ if $b = 0$ is closed maximal accretive in $H \equiv L^2(G)$. Let $\mathrm{Re}\,(a) \geqq 0$ for the nonzero complex number a and define $A \equiv a\Delta^2$ on $D(A) \equiv \{\phi \in H^4(G): \phi = \Delta\phi = 0 \text{ on } \partial G\}$. Then A is closed maximal accretive in H. From the divergence theorem we obtain

$$(A\phi, (I + B)\phi)_H = a(\Delta\phi, \Delta\phi)_H = a\bar{b}\int_G \Sigma|\partial\Delta\phi/\partial x_i|^2$$

for $\phi \in D(A)$. Thus $I+B$ forms a right (acute) angle with A if $\mathrm{Re}\,(a\bar{b}) \geqq 0$ (resp., $\mathrm{Re}\,(a\bar{b}) > 0$ and $\mathrm{Re}\,(a) > 0$), and for each $u_0 \in D(A)$ (resp., $u_0 \in D(B)$) there is a unique (generalized) solution of

$$u_t - b\Delta u_t + a\Delta^2 u = 0 \tag{11}$$

in $G \times (0, \infty)$ for which $u(x, 0) = u_0(x)$ and $u(x, t) = \Delta u(x, t) = 0$ for $x \in \partial G$ and $t > 0$. Nonhomogeneous terms and perturbations by first order spatial derivatives can be added to (11) when $b \neq 0$. When Theorem 2 applies, the solution of (3) with $B(t) = f(t) = 0$ belongs to the domain of every power of the generator $-T$. Hence, when $1 - b\Delta$ forms an acute angle with $a\Delta^2$, the solution $u(t) = u(x, t)$ of (11) belongs to $((1 - b\Delta)^{-1}a\Delta^2)^{-n}[D(A)] \subset H^{2n+2}(G)$ for every $t > 0$ and $n > 0$. Thus $u(x, t)$ is by Sobolev's lemma a C^∞ function of x. Further, one can show by standard techniques [1, 3, 8] that $u(x, t)$ is infinitely differentiable in x and t and is a genuine (pointwise) solution of (11).

The last example illustrates the technique when A is a polynomial with coefficients in the right half-plane in a self-adjoint operator B.

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