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SIMULTANEOUS APPROXIMATION AND INTERPOLATION IN L_1 AND $C(T)$

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Given a dense subspace of M of a Banach space X, an element x in X and a finite collection of linear functions in X^*, the problem of simultaneous approximation and interpolation is to interpolate x at the given functionals in X^* by an element m of M, with the restriction that the norms of x and m be equal and their difference in norm be arbitrarily small. A solution is given for the space L_1 with dense subspace, the simple functions in L_1, and any collection of functions in L_∞. In addition the problem is studied in the space $C(T)$, with any dense subalgebra and any finite collection of linear functionals in $C(T)^*$.

In [1] the concept of simultaneous approximation and interpolation which preserves the norm, (SAIN), was introduced.

DEFINITION [1]. Let X be a normed linear space, M a dense subset of X, L a finite dimensional subspace of X^*. The triple (X, M, L) has property (SAIN) if for every x in X and $\varepsilon > 0$ there exists y in M such that $\|x - y\| < \varepsilon$, $\|x\| = \|y\|$ and $\lambda(x) = \lambda(y)$ for all λ in L.

Other papers concerned with this topic are [4], [5], and [6].

In [5] it was shown that if L is any finite dimensional subspace of l_∞ and if M is the subspace of l_1 consisting of the elements having only finitely many nonzero components, then (l_1, M, L) had property (SAIN). In this paper, we let M be the subspace of simple functions in L_1. We show (L_1, M, T) has property (SAIN) for any finite dimensional subspace T in L_∞.

In [1], the space $C(T)$ is studied, where T is a compact Hausdorff space. One finds there

THEOREM 4.1. *Let A be a dense subalgebra of $C(T)$ and $t_1, \cdots, t_n$ in T. Then $(C(T), A, \{\delta_{t_1}, \cdots, \delta_{t_n}\})$ has property (SAIN). (δ_t is the linear functional on $C(T)$ given by point evaluation at t.)*

When arbitrary linear functionals in $C(T)$ are used, examples in [1] show that $(C(T), A, \{\nu\})$ may or may not have property (SAIN) depending on ν.

In this paper we wish to find sufficient conditions on f in $C(T)$ and M dense in $C(T)$ such that given $\{\nu_1, \cdots, \nu_n\}$ in $C(T)^*$ and $\varepsilon > 0$ there exists m in M such that $\|f - m\| < \varepsilon$, $\|f\| = \|m\|$ and

$$\int f d\nu_i = \int m d\nu_i, \; i = 1, \cdots, n \,.$$

In particular one finds that if f attains its norm at most a finite number of times, then any dense subalgebra of $C(T)$ will satisfy these conditions.

In this paper the following notation and terminology is used. X is to denote a real normed linear space. X^* is to denote the continuous dual of X, $U(X)$ and $S(X)$, the closed unit ball and its boundary in X. A set E contained in a set F is F-extremal if whenever $tx + (1-t)y$ is in E, with $0 < t < 1$ and x, y in F then x, y are in E. A hyperplane H supports a set K, if it bounds K and intersects K. The real valued function $\operatorname{sgn}(\cdot)$: Reals $\rightarrow \{-1, 0, 1\}$ is defined via $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = x/|x|$, $x \neq 0$. Then convex hull of a set A is to be denoted by $\operatorname{co}(A)$. All other notation will correspond to that of [3].

1. Minimal closed $U(X)$ extremal subsets.

DEFINITION 1.1. $F(x)$ is to denote the minimal closed $U(X)$-extremal set containing x. $Q(x)$ is the intersection of all $U(X)$ supporting hyperplanes at x.

THEOREM 1.1. *Let X be a normed linear space, M a dense subspace of X and $L = \operatorname{span}\{\varphi_1, \cdots, \varphi\}$ a finite dimensional subspace of X^*, and x in $S(X)$. If $F(x) \cap M$ is dense in $F(x)$ then given $\varepsilon > 0$ there exists m in $S(X)$ such that $\varphi_i(x) = \varphi_i(m)$, $i = 1, \cdots, n$ and $\|x - m\| < \varepsilon$.*

Proof. Define the continuous function $\varphi: F(x) \rightarrow R^n$ via $\varphi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$. Assume that $F(x) \subset \varphi_i^{-1}(\varphi_i(x))$ for $i = 0, 1, \cdots, k$ and that this is the largest set of linearly independent elements of L for which this is true. If no such set exists, $k = 0$. In R^{n-k} we assert the existence of $m_\alpha \in F(x) \cap M$ with $\|x - m_\alpha\| < \varepsilon$ such that

$$(\varphi_{k+1}(x), \cdots, \varphi_n(x)) \in \operatorname{co}(\varphi_{k+1}(m_\alpha), \cdots, \varphi_n(m_\alpha) \,|\, \alpha \in A) \,,$$

A an arbitrary index set. If not, then in R^{n-k} there exists a linear functional τ, a linear combination of the φ_i, $i > k$ such that without loss of generality $\tau(m) \leqq \tau(x)$ for all $m \in F(x) \cap M$ such that $\|x - m| < \varepsilon$. But this implies $\tau(m) \leqq \tau(x)$ for all $m \in F(x) \cap M$, since if there exists $m_0 \in F(x) \cap M$ with $\|x - m_0\| > \varepsilon$ then the set

$$\{y \in F(x) \,|\, \tau(y) > \tau(x), \; \|y - x\| < \varepsilon\}$$

is $F(x)$ relatively open and nonempty (choose a suitable combination of x and m_0) and hence contains m in $F(x) \cap M$ contradicting $\tau(m) \leqq \tau(x)$ with $\|m - x\| < \varepsilon$. Since $F(x) \cap M$ is dense in $F(x)$ this implies $\tau(y) \leqq \tau(x)\ \forall y \in F(x)$. Let $K = \{y \in F(x) \mid \tau(y) = \tau(x)\}$. K is convex closed and $F(x)$-extremal since $tz + (1-t)y$ in K implies $t\tau(z) + (1-t)\tau(y) = \tau(x)$ with $\tau(z) \leqq \tau(x)$, $\tau(y) \leqq \tau(x)$. Hence $\tau(z) = \tau(y) = \tau(x)$ and $z, y \in K$. Hence K is closed $U(X)$-extremal and $K = F(x)$. Thus $F(x) \subset \tau^{-1}(\tau(x))$. Since τ is linearly independent of φ_i, $i = 1, \cdots, k$, this contradicts the maximal choice of φ_i at the start of the proof. Therefore

$$(\varphi_{k+1}(x), \cdots, \varphi_n(x)) \in \operatorname{co}(\varphi_{k+1}(m_\alpha), \cdots, \varphi_n(m_\alpha) \mid \alpha \in A)$$

with $\|x - m_\alpha\| < \varepsilon$. This yields the result by the convexity of M and $\varphi(M)$.

In a recent paper of Deutsch and Lindahl [2], they showed that in certain spaces that the set $Q(x)$, the intersection of all $U(X)$ supporting hyperplanes at x, is equal to the closure of the minimal extremal subset containing x. Thus $Q(x)$ is equal to the minimal closed extremal subset containing x. This occurs, in particular [2, Theorem 4.2], if (T, Σ, ν) is a σ-finite measure space, in $L_1(T, \Sigma, \nu)$. Also, this occurs [2, Theorem 3.3] in the space $C_0(T)$, the space of continuous functions vanishing at infinity, T locally compact.

THEOREM 2.1. *Let (T, Σ, ν) be a σ-finite measure space with $L_1^*(T, \Sigma, \nu) = L_\infty(T, \Sigma, \nu)$. Let M be the dense subspace of L_1 consisting of the simple functions. Then (L_1, M, H) has property (SAIN) for any finite dimensional subspace H contained in L_∞.*

Proof. Given x in $S(L_1)$. By [2, Theorem 4.2], $Q(x) = \{z \in S(L_1) \mid \int z \operatorname{sgn}(x) = 1\}$ and $Q(x) = F(x)$. M is dense in $Q(x)$ and by Theorem 1.1 the result follows.

THEOREM 2.2. *Let T be a compact Hausdorff space, $C(T)$ the space of real valued continuous functions on T. Let f in $S(C(T))$ be such that $Q(f) = \bigcap_{i=1}^{r} \varphi_i^{-1}(\|f\|)$ with φ_i in $\mathrm{rca}(T)$. If*

$$(C(T), M, \{\varphi_i \mid i = 1, \cdots, n\})$$

has property SAIN then given any finite collection μ_i in $\mathrm{rca}(T)$, $\varepsilon > 0$ there exists m in M such that $\|f - m\| < \varepsilon$, $\|f\| = \|m\|$ and $\int f\, d\mu_i = \int m\, d\mu_i$.

Proof. By [2, Theorem 3.3] $Q(f) = \{x \in C(T) \mid x(t) = f(t)$ for $t \in T$ such that $|f(t)| = 1\}$ and $Q(f) = F(f)$. $(C(T), M, \{\varphi_i \mid i = 1, \cdots, n\})$ having property (SAIN) implies $F(f) \cap M$ is dense in $F(f)$. By Theorem 1.1 the result follows.

COROLLARY 2.1. *Let f be in $S(C(T))$ $\varepsilon > 0$. If $|f|$ attains is norm finitely often then given μ_i, $i = 1, \cdots, n$ in rca (T) there exists p in A (any dense subalgebra of $C(T)$) such that $\|p - f\| < \varepsilon$ $\|p\| = \|f\|$ and $\int p d\mu_i = \int f d\mu_i$.*

Proof. By [1, Theorem 4.1] quoted in the introduction of this article $((C(T), A, \{\delta_t \mid f(t) \mid = 1\})$ has property (SAIN). But $Q(f) = \cap \{\delta_t^{-1}(f(t)) \mid |f(t)| = 1\}$. Hence apply Theorem 2.2.

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