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ON THE REST POINTS OF A NONLINEAR NONEXPANSIVE SEMIGROUP

CHI-LIN YEN

Let X be a reflexive Banach space and T a nonlinear nonexpansive semigroup on X. The results which we shall prove are the following:

THEOREM 1. **Suppose that for any closed convex set M with the property that $T(t)M \subseteqq M$ for all $t \geqslant 0$, M contains a precompact orbit. Then T has a rest point. Moreover, the set of all rest points of T is connected.**

THEOREM 2. **Suppose that X is strictly convex and T has a bounded orbit. If there is an unbounded increasing sequence $\{u_i\}$ of positive numbers and point x such that $\lim_{i\to\infty} T(u_i)x$ exists then T has a rest point. Moreover, if $\{t_i\}$ is an unbounded increasing sequence of positive numbers such that**

$$y = w - \lim_{i\to\infty} \frac{1}{t_i} \int_0^{t_i} T(t)x\, dt$$

exists, then $y \in F$.

Let X be a Banach space. By a nonlinear nonexpansive strongly continuous semigroup T on X (or briefly, a semigroup T on X) we mean that T is a mapping from $[0, \infty) \times X$ into X such that

(i) for any $x \in X$, $t_1 \geqslant 0$, and $t_2 \geqslant 0$, $T(t_1)T(t_2)x = T(t_1 + t_2)x$;

(ii) for any $x \in X$, $\lim_{t\to 0+} T(t)x = T(0)x = x$;

(iii) for any $x \in X$, $y \in X$, and $t \geqslant 0$, $|T(t)x - T(t)y| \leqslant |x - y|$.

Throughout this paper T will denote a semigroup on X. We shall give some definitions as follows:

(1) For $x \in X$ the orbit of x is the set $O_x = \{T(t)x;\ t \geqq 0\}$

(2) $F = \{x;\ T(t)x = x \text{ for all } t \geqslant 0\}$, and if $x \in F$ then x is called a rest point of T.

(3) $P = \{x;\ \text{there is } t_0 > 0 \text{ such that } T(t_0)x = x\}$.

(4) $A = \{x;\ O_x \text{ is precompact}\}$.

(5) $L = \{x;\ \text{there is a sequence } \{t_i\} \text{ of positive numbers such that } t_i \uparrow \infty \text{ and } \lim_{i\to\infty} T(t_i)x \text{ exists}\}$.

Clearly, $L \supseteqq A \supseteqq P \supseteqq F$. Moreover, if $F \neq \phi$ then O_x is bounded for all $x \in X$. The question arises "Is the converse true?" M. Crandall and A. Pazy [2] give an affirmative answer, when X is a Hilbert space. However, the converse is not true in general (see R. Martin [4]). In this paper some sufficient conditions will be given such that $F \neq \phi$.

Our main results are the following:

THEOREM 1. *Let X be a reflexive Banach space. Suppose that for any closed convex set M with the property that $T(t)M \subseteqq M$ for all $t \geqslant 0$, $M \cap A \neq \phi$. Then $F \neq \phi$. Moreover, F is connected.*

THEOREM 2. *Let X be a srictly convex reflexive Banach space. If T has a bounded orbit and $L \neq \phi$, then $F \neq \phi$. Moreover, if $t_i \uparrow \infty$ and $y = w - \lim_{i\to\infty} 1/t_i \int_0^{t_i} T(t)x\,dt$ for some $x \in X$, then $y \in F$.*

As an application of Theorem 1 one can verify that if X is a reflexive Banach space and T has a bounded orbit, then $F \neq \phi$ provided that either of the following holds: (i) there is a $t_0 > 0$ such that $T(t_0)$ is weakly continuous function on X or (ii) X has the property that every m-dissipative Lipschitz continuous function on X is demiclosed (f is demiclosed if $x_n \to x_0$ strongly then $y_0 = fx_0$). It is known that if X is a uniformly convex space, the condition (ii) is fulfilled, (see F. Browder [1]).

As an application of Theorem 2 one can verify that if X is a strictly convex, reflexive Banach spach and $A \neq \phi$ then $F \neq \phi$. Furthermore, if $x \in A$ then for some unbounded increasing sequence $\{t_i\}$ of positive numbers $\lim_{t_i\to\infty} 1/t_i \int_0^{t_i} T(u)x\,du$ exists and is an element of F. This result generalizes that of D. Rutedge [5] in which X is a Hilbert space and $P \neq \phi$.

We need two known lemmas to prove our theorems and we state them below without proof. Lemma 1 was put in the present form by M. Crandall and A. Pazy [2] and Lemma 2 due to R. de Marr [3].

LEMMA 1. *Let $x \in X$ such that $|T(t)x| \leqslant M$ for all $t \geqslant 0$. Then $K = \bigcup_{\tau\geqslant 0} \bigcap_{t\geqslant\tau} \{y; |y - T(t)x| \leqslant |x| + M\}$ is a nonempty convex subset of X such that $T(t)K \subseteqq K$ for all $t \geqq 0$.*

LEMMA 2. (R. de Marr). *Let C be a compact subset of X such that $r = \operatorname{diam} C > 0$. Then there is an $x_0 \in \operatorname{clco} C$ and a positive number $r_1 < r$ such that $|y - x_0| \leqslant r_1$ whenever $y \in C$.*

We will use the following two lemmas and the above twe lemmas to prove Theorem 1.

LEMMA 3. *Let M be a closed subset of X such that $T(t)M \subseteqq M$ for all $t \geqslant 0$. If $M \cap A \neq \phi$, then there is a compact subset C of M such that $T(t)C = C$.*

Proof. Let $x \in M \cap A$. Then $\bar{O}_x$ is a compact subset of M and $T(t_1)\bar{O}_x \subseteqq T(t_2)\bar{O}_x$ whenever $t_1 \geqslant t_2 \geqslant 0$. Hence $C = \bigcap_{t>0} T(t)\bar{O}_x$ is a nonempty compact subset of M. Furthermore, $T(t)C = C$ for all $t \geqslant 0$.

LEMMA 4. *Let* $x_0, x_1 \in X$ *and* $\lambda \in [0, 1]$. *Then*

$$M_\lambda = \{y \in X;\ |x_0 - y| = \lambda |x_1 - x_0|,\ |x_1 - y| = (1 - \lambda)|x_1 - x_0|\}$$

is a nonempty closed convex bounded subset of X. *Moreover, if* $x_0, x_1 \in F$ *then* $T(t)M_\lambda \subseteqq M_\lambda$.

Proof.

$$M_\lambda = \{y \in X;\ |x_0 - y| \leqslant \lambda |x_0 - x_1|\} \cap \{y \in X;\ |x_1 - y| \leqslant (1 - \lambda)|x_0 - x_1|\}$$

contains $\lambda x_1 + (1 - \lambda)x_0$. Thus M_λ is a nonempty closed convex bounded subset of X.

Since $T(t)x_i = x_i$ for all $t \geqslant 0$, $i = 0, 1$ thus for any $y \in M_\lambda$,

$$|x_0 - T(t)y| = |T(t)x_0 - T(t)y| \leqq \lambda |x_0 - x_1|$$

and

$$|x_0 - T(t)y| = |T(t)x_1 - T(t)y| \leqslant (1 - \lambda)|x_0 - x_1|,$$

that is, $T(t)y \in M_\lambda$.

Now we prove Theorem 1.

Proof of Theorem 1. By Lemma 1 there is a nonempty closed bounded convex set M such that $T(t)M \subseteqq M$. Let $\{M_\alpha\}$ be a chain of subset of M such that

(i) M_α is a nonempty closed bounded convex set satisfying $T(t)M_\alpha \subseteqq M_\alpha$ for all α.

(ii) $M_\alpha \subseteqq M_\beta$ if $\alpha \geqslant \beta$.

Since M_α is weak-compact, thus $\bigcap_\alpha M_\alpha \neq \phi$. Further,

$$T(t)(\bigcap_\alpha M_\alpha) \subseteqq \bigcap_\alpha M_\alpha .$$

By Zorn's lemma there is a maximal element, say M_0, in the collection $\mathscr{F} = \{M_1;\ M_1$ is a nonempty closed bounded convex subset of M such that $T(t)M_1 \subseteqq M_1\}$. We want to show that M_0 contains exactly one point. Suppose not. By hypothesis, $M_0 \cap A$ contains at least one point, say x. By Lemma 3 there is a compact subset C of M_0 such that $T(t)C = C$. By Lemma 2 there is a point $x_0 \in \operatorname{clco} C \subseteqq M_0$ such that $|y - x_0| \leqslant r_1 < r = \operatorname{diam} C$ for all $y \in C$. Consider the set $M' = \bigcap_{y \in C} \{z \in M_0;\ |z - y| \leqslant r_1\}$.

We see that M' is a nonempty closed bounded convex subset of M_0 such that $T(t)M \subseteqq M$. Since $r = \operatorname{diam} C$ and C is compact, thus there are $x_1, x_2 \in C$ such that $|x_1 - x_2| = r$. By the definition of M' and the fact that $r_1 < r$, we have $x_i \notin M'$ for $i = 1, 2$. Thus $M' \neq M_0$ and the maximality of M_0 is contradicted. Thus M_0 must contain exactly one point which lies in F. This shows that if M is a closed convex set satisfying $T(t)M \subseteqq M$ for all $t \geqslant 0$ then $M \cap F \neq \phi$.

Next we want to show that F is connected. Suppose not. Then there are two disjoint closed subsets A and B of X such that $A \cup B \supseteqq F$, $A \cap F \neq \phi$ and $B \cap F \neq \phi$. Let $A' = A \cap F$ and $B' = B \cap F$. Since F is closed thus A' and B' are closed. For $x_1 \in A'$, $D(x_1, B') = \inf\{|x_1 - y|; y \in B'\} = k > 0$. Thus, there is a $y_1 \in B'$ such that $|x_1 - y_1| < 5/4\,K$. It follows from Lemma 4 and the above paragraph there is $z_1 \in M^1 = \{z \in X; |z - x_1| = |z - y_1| = 1/2\,|x_1 - y_1|\}$ such that $z_1 \in F = A' \cup B'$. Since $|z_1 - x_1| = 1/2\,|x_1 - y_1| < 5/8\,K$, $z_1 \in A'$. Let $x_2 = z_1$. Then there is a $y_1 \in B'$ such that

$$|x_2 - y_2| \leqslant \operatorname{Min}\left\{\frac{5}{4} D(x_2, B'), |x_2 - y_1|\right\}.$$

Similarly, there is $x_3 \in M^2 = \{z \in X; |z - x_2| = |z - y_2| = 1/2\,|x_2 - y_2|\}$ such that $x_3 \in F$. By the same argument we have $x_3 \in A'$. We assume we have chosen $x_{n+1} \in M^n = \{z \in X; |z - x_n| = |z - y_n| = 1/2\,|x_n - y_n|\}$ and $x_{n+1} \in A'$ and $y_n \in B'$ such that

$$|y_n - x_n| \leqslant \operatorname{Min}\left\{\frac{5}{4} D(x_n, B'), |x_n - y_{n-1}|\right\}$$

for all $n \leqslant k - 1$ where $k \geqslant 3$. We can choose y_k, x_{k+1} as follows:

Since $D(x_k, B') \leqslant |x_k - y_{k-1}|$, there is a $y_k \in B'$ such that

$$|x_k - y_k| \leqslant \operatorname{Min}\left\{\frac{5}{4} D(x_k, B'), |x_k - y_{k-1}|\right\}$$

and let $x_{k+1} \in A'$ such that

$$x_{k+1} \in M^k = \left\{z \in X; |z - x_k| = |z - y_k| = \frac{1}{2}|x_k - y_k|\right\}.$$

Note that

$$|x_{n+1} - y_{n+1}| \leqslant |x_{n+1} - y_n| = \frac{1}{2}|x_n - y_n| \leqslant \cdots \leqslant \left(\frac{1}{2}\right)^n |x_1 - y_1| < \left(\frac{1}{2}\right)^n \left(\frac{5}{4}K\right)$$

and

$$|x_{n+1} - x_n| = |x_{n+1} - y_n| < \left(\frac{1}{2}\right)^n \left(\frac{5}{4} K\right).$$

Thus, $\{x_n\}$ is a Cauchy sequence and so $\{x_n\}$ converges to some point, say x_0 in A'. Also $D(x_{n+1}, B') \leqq |x_{n+1} - y_{n+1}| < (1/2)^n ((5/4)K) \to 0$, so $D(x_0, B') = 0$. Since B' is closed $x_0 \in B'$. This is a contradiction to $\phi = A \cap B \ni x_0$. Therefore, F is connected.

In order to prove Theorem 2 we need the following lemmas.

LEMMA 5. *If $x_0 \in X$ such that $x_0 = \lim_{i\to\infty} T(t_i)x$ for some $x \in X$ and some unbounded increasing sequence $\{t_i\}$ of positive numbers, then there is an unbounded increasing sequence $\{s_i\}$ of positive numbers, such that*

$$\lim_{1\to\infty} T(s_i)x_0 = x_0 .$$

Indication of proof. By an inductive process, for each i, choose n_{i+1} such that $t_{n_{i+1}} - t_{i+1} \geqslant 1 + t_{n_i} - t_i$, $i = 1, 2, 3, \cdots$ and $n_1 = 1$. Let $s_i = t_{n_i} - t_i$. Then,

$$\begin{aligned} | T(s_i)x_0 - x_0 | &\leqq | T(s_i)T(t_i)x - x_0 | + 2 | T(t_i)x - x_0 | \\ &= | T(t_{n_i})x - x_0 | + 2 | T(t_i)x - x_0 | \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty . \end{aligned}$$

That is, $\lim_{i\to\infty} T(s_i)x_0 = x_0$.

LEMMA 6. *Let X be a strictly convex Banach space. If*

$$\lim_{i\to\infty} T(s_i)x_0 = x_0$$

for some increasing unbounded sequence $\{s_i\}$ of positive numbers, then for any n, any $\lambda_1, \cdots, \lambda_n$ such that $\lambda_i \geqslant 0$, $\sum_{i=1}^n \lambda_i = 1$ and any $x_1, \cdots, x_n$ in 0_{x_0},

$$T(t)\left(\sum_{i=1}^n \lambda_i x_i\right) = \sum_{i=1}^n \lambda_i T(t)x_i \quad \text{for all } t \geqslant 0 . \tag{1}$$

Indication of proof. Clearly, (1) is true for the case $n = 1$. Using inductive argument we may assume that (1) holds for all $n \leqslant k$ where $k \geqslant 1$. We shall show that (1) holds for the case $n = k + 1$, that is, for any λ_i, $\lambda_i \neq 1$, $\sum_{i=1}^{k+1} \lambda_i = 1$, and any $x_1, \cdots, x_{k+1}$ in 0_{x_0},

$$T(t)\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = \sum_{i=1}^{k+1} \lambda_i T(t)x_i .$$

Let $y = \sum_{i=1}^{k+1} \lambda_i x_i$, $z = (1 - \lambda_1)^{-1} \sum_{i=1}^{k+1} \lambda_i x_i$. Then $y = \lambda_1 x_1 + (1 - \lambda_1)z$, and

(2) $$|T(t)y - T(t)x_1| \leqslant |y - x_1|,\ |T(t)y - T(t)z| \leqslant |y - z| \quad \text{for all } t \geqslant 0 .$$

(3) $$\begin{aligned}|T(t)x_1 - T(t)z| &\leqslant |T(t)y - T(t)x_1| + |T(t)y - T(t)z| \\ &\leqslant |y - x_1| + |y - z| = |x_1 - z| .\end{aligned}$$

Since $|T(t_i)x_1 - T(t_i)z| \downarrow |x_1 - z|$ as $i \to \infty$, thus we have

$$|T(t)y - T(t)x_1| + |T(t)y - T(t)z| = |T(t)x_1 - T(t)z| .$$

By the strict convexity of X, (2) and (3) we have that

$$T(t)y = \lambda_1 T(t)x_1 + (1 - \lambda_1)T(t)z .$$

By the inductive hypothesis,

$$T(t)y = \sum_{i=1}^{k+1} \lambda_i T(t)x_i .$$

LEMMA 7. *Let x_0, X be as in Lemma 6. If there is an unbouded increasing sequence $\{u_i\}$ of positive numbers such that*

$$y = w - \lim_{i\to\infty} \frac{1}{u_i}\int_0^{u_i} T(t)x_0\,dt,\ \textit{then}\ y \in F .$$

Proof. Let

$$y_i = \frac{1}{u_i}\int_0^{u_i} T(t)x_0\,dt .$$

For $\varepsilon > 0$, $r > 0$ fixed, there is an $N > 0$ such that if $M \geqslant |T(t)x_0|$ for all $t \geqslant 0$,

$$\frac{rM}{u_i} < \frac{\varepsilon}{3} \text{ whenever } i \geqslant N .$$

It follows from Lemma 6 that

$$T(r)y_i = \frac{1}{u_i}\int_r^{u_i+r} T(t)x_0\,dt = y_i + \frac{1}{u_i}\left(\int_{u_i}^{u_i+r} - \int_0^r\right)T(t)x_0\,dt .$$

Thus $|T(r)y_i - y_i| < 2\varepsilon/3$ for all $i \geqslant N$. Since $y = w - \lim_{i\to\infty} y_i$, there exists a $k > 0$, $\lambda_1, \lambda_2, \cdots, \lambda_k \geqslant 0$ such that $\sum_{i=1}^k \lambda_i = 1$ and $|y - \sum_{i=1}^k \lambda_i y_{i+N-1}| < \varepsilon/6$. Hence,

$$\begin{aligned}|T(r)y - y| &\leqslant \left|T(r)y - \sum_{i=1}^k \lambda_i T(r)y_{i+N-1}\right| \\ &\quad + \left|\sum_{i=1}^k \lambda_i (T(r)y_{i+N-1})\right| + \left|y - \sum_{i=1}^k \lambda_i y_{i+N-1}\right| \\ &< 2\varepsilon/6 + 2\varepsilon/3 = \varepsilon .\end{aligned}$$

Since ε and r are arbitrary positive numbers, thus $y \in F$.

LEMMA 8. *Let $\{t_i\}$ be an unbounded increasing sequence of positive numbers and x in X. If T has a bounded orbit and*

$$x_0 = \lim_{i \to \infty} T(t_i)x ,$$

then

$$\lim_{u \to \infty} \frac{1}{u} \int_0^n (T(t)x - T(t)x_0)dt = 0 .$$

Proof. For $\varepsilon > 0$ be given there is an positive integer n such that

$$| T(t_i)x - x_0 | < \varepsilon \qquad \text{for all } i \geqslant n .$$

Let u be any positive number great than t_n. Then

$$\begin{aligned} \left| \frac{1}{u} \int_0^u (T(t)x - T(t)x_0)dt \right| &\leqslant \frac{1}{u} \int_0^{u-t_n} | T(t)T(t_n)x - T(t)x_0 | dt \\ &\quad + \frac{1}{u} \int_0^{t_n} | T(t)x | dt + \frac{1}{u} \int_{u-t_n}^{u} | T(t)x_0 | dt \\ &< \frac{u - t_n}{u} \in \\ &\quad + \frac{1}{u} \left(\int_0^{t_n} | T(t)x | dt + \int_{u-t_n}^{u} | T(t)x_0 | dt \right) . \end{aligned}$$

Since orbits are bounded the last term in above inequality will tend to 0 as $u \to \infty$. Hence, we prove the assertion.

Proof of Theorem 2. By Lemma 5, Lemma 7 and reflexivity of X, there is an increasing unbounded sequence $\{u_i\}$ of positive numbers such that

$$w - \lim_{i \to \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 \, dt$$

exists and is in F, where $x_0 = \lim_{i \to \infty} T(t_i)x$. Also, it follows from Lemma 8

$$\lim_{i \to \infty} \frac{1}{u_i} \int_0^{u_i} (T(t)x - T(t)x_0) \, dt = 0 .$$

Thus,

$$w - \lim_{i \to \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x \, dt = w - \lim_{i \to \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 \, dt \qquad \text{is in } F .$$

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ACADEMIA SINICA