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ZETA FUNCTIONS ON THE REAL GENERAL LINEAR GROUP

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In this paper, the zeta functions on the real general linear group, formed relative to arbitrary positive definite zonal spherical functions, are defined and studied. The main result asserts that such functions satisfy a functional equation and can be meromorphically continued to the whole complex plane.

1. Introduction. Let $\boldsymbol{R}$ denote the field of real numbers. Let $M = M(n, \boldsymbol{R})$ denote the n^2-dimensional vector space over $\boldsymbol{R}$ comprised of all $n \times n$ matrices with real coefficients, $G = GL(n, \boldsymbol{R})$ the subset of M consisting of those elements with nonzero determinant. As is known, G is a locally compact unimodular group whose Haar measure $d_G(x)$ $(x \in G)$ is given, up to a constant factor, by

$$d_G(x) = |x|^{-n} d_M(x) \qquad (|x| = |\det(x)|) ,$$

$d_M(x)$ being normalized Lebesgue measure on M. Let

$$K = \{x \in G: x \cdot {}^t x = 1\} ,$$

the 't' denoting transpose—then the orthogonal group K is a maximal compact subgroup of G.

Given x, y in M, let us agree to write (x, y) for $\mathrm{tr}(x \cdot {}^t y)$. Viewing M as an (additive) locally compact abelian group, the Fourier transform $\hat{f}$ of any element f belonging to the Schwartz space $\mathscr{C}(M)$ of M (i.e., the C^∞, rapidly decreasing functions on M) is defined by the rule:

$$\hat{f}(x) = \int_M f(y) e^{2\pi\sqrt{-1}(x,y)} d_M(y) \qquad (x \in M) .$$

With these conventions, one then has the relation

$$\hat{\hat{f}}(x) = f(-x) \qquad (x \in M)$$

valid for all f in $\mathscr{C}(M)$.

By $\mathscr{I}(M)$ we shall understand the subspace of $\mathscr{C}(M)$ made up of those f which are biinvariant under K, i.e., which have the property that

$$f(k_1 x k_2) = f(x) \qquad (x \in G)$$

for all $k_1, k_2 \in K$. This being so, fix an f in $\mathscr{I}(M)$. Then, relative to a given positive definite zonal spherical function Φ on G, the *zeta* *function* $\zeta(f, \Phi, s)$ of f with respect to Φ is that function of the complex variable s defined by:

$$\zeta(f, \Phi, s) = \int_G f(x)\Phi(x)\,|x|^s d_G(x)\ .$$

It is elementary that the integral on the right converges absolutely and defines a holomorphic function of s whenever $\mathrm{Re}\,(s) > n - 1$. The problem which we shall consider in this note is that of effecting a meromorphic continuation of $\zeta(f, \Phi, s)$ to the whole complex plane. In so doing we shall derive a functional equation relating $\zeta(f, \Phi, s)$ and $\zeta(\hat{f}, \bar{\Phi}, n - s)$ ($\bar{\Phi}$ the complex conjugate of Φ). [In passing, note that the integral defining $\zeta(\hat{f}, \bar{\Phi}, n - s)$ converges for all s such that $\mathrm{Re}\,(s) < 1$.]

Precisely stated, the theorem that will be established can be formulated as follows. Let $s_1, \cdots, s_n$ be the parameters of our given positive definite zonal spherical function Φ (see §2 below) and put

$$\gamma(s_1, \cdots, s_n) = \pi^{-\mathrm{Re}(\sum_i s_i) + 2^{-1}(n^2 - n)}\ .$$

For complex z, we shall agree to write

$$\Gamma_R(z) = \pi^{1/2 - z} \frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{1 - z}{2}\right)}\ ,$$

Γ the classical Gamma function.

THEOREM. *Let* $f \in \mathscr{S}(M)$*—then the zeta function* $\zeta(f, \Phi, s)$ *of* f *with respect to* Φ*, initially defined by the rule*

$$\zeta(f, \Phi, s) = \int_G f(x)\Phi(x)\,|x|^s d_G(x)$$

for $\mathrm{Re}\,(s) > n - 1$*, can be meromorphically continued to the whole s-plane and, for* $\mathrm{Re}\,(s) \leqq 0$*, the functional equation*

$$\zeta(f, \Phi, s) = \left\{\gamma(s_1, \cdots, s_n) \prod_{i=1}^{n} \Gamma_R(s - s_i)\right\} \zeta(\hat{f}, \bar{\Phi}, n - s)$$

obtains.

The proof of this result is similar in spirit to the analogous statement proved by Tate in his thesis for the case $n = 1$ (cf. Lang [4, Chapter 7]). The main technical tool is Lemma 2 in §2 below. To actually carry out the meromorphic continuation for *all* f belonging to $\mathscr{S}(M)$ (and not just those having compact support), we shall use an argument due to Stein [7]; see §3 infra.

As for related results and possible extensions of the theory, we point out that there is no difficulty treating $M(n, \boldsymbol{C})$ and $GL(n, \boldsymbol{C})$ ($\boldsymbol{C}$ the field of complex numbers); in fact Stein [7] has already considered this case when $\Phi \equiv 1$. On the other hand, Gelbart [2] in his thesis treated $M(n, \boldsymbol{R})$ and $GL(n, \boldsymbol{R})$ but considered there only the special situation when $\Phi \equiv 1$. For a historical survey of the subject in general, the reader may find the paper of Andrianov [1] to be of value.

Finally, for details on zonal spherical functions and related questions, we refer the reader to Helgason [3, Chapter 10].

2. THE MAIN LEMMA. We shall retain the notations introduced in §1; in particular, the symbols G, K, and M have the same meaning as there. Let

$$B = \{b = (b_{ij}) \in G\colon b_{ij} = 0 (i > j) \quad \text{and} \quad b_{ii} > 0\} .$$

Then one has the 'Iwasawa decomposition' $G = KB = BK$ with $K \cap B = \{1\}$. Relative to this decomposition of G, it will be convenient to normalize the Haar measure $d_G(x)$ on G as follows. First normalize the Haar measure $d_K(k)$ on the compact group K by the condition

$$\mathrm{vol}(K) = \int_K d_K(k) = 1 .$$

Second normalize the right and left invariant Haar measures $d_B^r(b)$ and $d_B^l(b)$, respectively, on B by the conditions

$$d_B^r(b) = 2^n \prod_{i=1}^n b_{ii}^{-i} \prod_{i=1}^n db_{ii} \prod_{i<j} db_{ij} ,$$

$$d_B^l(b) = \prod_{i=1}^n b_{ii}^{(2i-n-1)} d_B^r(b) ,$$

db_{ii}, db_{ij} being normalized Lebesque measure on the ambient 1-dimensional space over $\boldsymbol{R}$. Finally normalize the Haar measure $d_G(x)$ on G via the stipulation

$$d_G(x) = d_K(k) d_B^r(b) = d_B^l(b) d_K(k) .$$

Let Φ be a positive definite zonal spherical function on G—then Φ possesses the following properties:

(1) $\Phi(1) = 1$, $\Phi(k_1 x k_2) = \Phi(x)$ $(k_1, k_2 \in K; x \in G)$;

(2) $\Phi(x^{-1}) = \overline{\Phi(x)}$ $(x \in G)$;

(3) $|\Phi(x)| \leqq 1$ $(x \in G)$;

(4) $\Phi(x)\Phi(y) = \int_K \Phi(xky) d_K(k)$ $(x, y \in G)$.

In addition there exists a continuous homomorphism

$$\chi: B \to C^*$$

of B into the multiplicative group of complex numbers such that

$$\Phi(x) = \int_K \chi(xk) d_K(k) \qquad (x \in G)$$

where χ has been extended from B to G by the prescription

$$\chi(x) = \chi(kb) = \chi(b) \quad (x \in G,\, x = kb(k \in K,\, b \in B)) .$$

Because χ has the form

$$b \mapsto \prod_{i=1}^{n} |\, b_{ii} |^{-(s_i - (i-1))} \qquad (b = (b_{ij}) \in B)$$

for certain complex parameters $s_1, \cdots, s_n$, we shall sometimes write $\chi_{(s_i)}$ for χ and then denote the associated positive definite zonal spherical function Φ by $\Phi_{(s_i)}$ so that

$$\Phi_{(s_i)}(x) = \int_K \chi_{(s_i)}(xk) d_K(k) \qquad (x \in G) .$$

Now fix once and for all a positive definite zonal spherical function Φ on G. Given $f \in \mathscr{S}(M)$, define a function $\zeta(f, \Phi, s)$ of the complex variable s by:

$$\zeta(f, \Phi, s) = \int_G f(x)\Phi(x) |\, x\,|^s d_G(x) .$$

LEMMA 1. *The integral defining* $\zeta(f, \Phi, s)$ $(f \in \mathscr{S}(M))$ *is absolutely convergent for* $\mathrm{Re}\,(s) > n - 1$.

Proof. Since $|\Phi(x)| \leqq 1$ for all $x \in G$, it is clear that one need only verify that the integral

$$\int_G |f(x)| \cdot |\, x\,|^s d_G(x) \qquad (f \in \mathscr{S}(M))$$

is convergent for $\mathrm{Re}\,(s) > n - 1$. This, however, is well-known and easy to prove (see, e.g. Gelbart [2, p. 36]).

Let $I_c^\infty(G)$ denote the subspace of $C_c^\infty(G)$ consisting of those functions which are biinvariant under K—then it is clear that $I_c^\infty(G) \subset \mathscr{S}(M)$. We come now to the main technical lemma for the present investigation.

LEMMA 2. *If* $f, g \in I_c^\infty(G)$ *and if* $\mathrm{Re}\,(s) < 1$, *then*

$$\zeta(f, \Phi, s)\zeta(\hat{g}, \bar{\Phi}, n - s) = \zeta(\hat{f}, \bar{\Phi}, n - s)\zeta(g, \Phi, s) .$$

Proof. To begin with, observe that since $f, g \in I_c^\infty(G)$, the zeta functions $\zeta(f, \Phi, s)$, $\zeta(g, \Phi, s)$ are actually well-defined for all values of s. On the other hand, the Fourier transforms $\hat{f}$, $\hat{g}$ evidently lie in $\mathscr{S}(M)$ and so the zeta functions $\zeta(\hat{f}, \bar{\Phi}, n - s)$, $\zeta(\hat{g}, \bar{\Phi}, n - s)$ are well-defined whenever $\operatorname{Re}(s) < 1$ (cf. Lemma 1). Therefore, all the zeta functions appearing in the statement of our lemma make sense for $\operatorname{Re}(s) < 1$. To establish the asserted equality, note first that

$$\begin{aligned}\zeta(f, \Phi, s) &= \int_G f(x)\Phi(x)\,|x|^s d_G(x)\\ &= \int_G f(x)\Big(\int_K \chi(xk)d_K(k)\Big)|x|^s d_G(x)\\ &= \int_G f(x)\chi(x)\,|x|^s d_G(x)\\ &= \int_B\int_K f(kb)\chi(kb)\,|kb|^s d_K(k)d_B^r(b)\\ &= \int_B f(b)\chi(b)\,|b|^s d_B^r(b)\ .\end{aligned}$$

In a similar way we find that

$$\begin{aligned}\zeta(\hat{g}, \bar{\Phi}, n - s) &= \int_G \hat{g}(x)\bar{\Phi}(x)\,|x|^{n-s} d_G(x)\\ &= \int_G \hat{g}(x)\Big(\int_K \chi(x^{-1}k)d_K(k)\Big)|x|^{n-s} d_G(x)\\ &= \int_G \hat{g}(x)\chi(x^{-1})\,|x|^{n-s} d_G(x)\\ &= \int_K\int_B \hat{g}(bk)\chi(k^{-1}b^{-1})\,|bk|^{n-s} d_B^l(b)d_K(k)\\ &= \int_B \hat{g}(b)\chi(b^{-1})\,|b|^{n-s} d_B^l(b)\ .\end{aligned}$$

Hence:

$$\begin{aligned}&\zeta(f, \Phi, s)\zeta(\hat{g}, \bar{\Phi}, n - s)\\ &\quad= \int_B\int_B f(b')\hat{g}(b'')\chi(b')\chi(b'')^{-1}\,|b'|^s\,|b''|^{n-s} d_B^r(b')d_B^l(b'')\\ &\quad= \int_B f(b')\chi(b')\,|b'|^s\Big(\int_B \hat{g}(b'b'')\chi(b'b'')^{-1}\,|b'b''|^{n-s} d_B^l(b'')\Big)d_B^r(b')\\ &\quad= \int_B\Big(\int_B f(b')\hat{g}(b'b'')\,|b'|^n d_B^r(b')\Big)\chi(b'')^{-1}\,|b''|^{n-s} d_B^l(b'')\ .\end{aligned}$$

Let us consider the inner integral in the last equality. Thus, we can write

$$\begin{aligned}&\int_B f(b')\hat{g}(b'b'')\,|b'|^n d_B^r(b')\\ &\qquad= \int_B\int_K f(kb')\hat{g}(kb'b'')\,|kb'|^n d_K(k)d_B^r(b)\end{aligned}$$

$$= \int_G f(x)\hat{g}(xb'')\,|x|^n d_G(x)$$

$$= \int_G \hat{f}(xb'')g(x)\,|x|^n d_G(x) \qquad \text{(Plancherel's Theorem)}$$

$$= \int_B \hat{f}(b'b'')g(b')\,|b'|^n d_B^r(b') .$$

It therefore follows that

$$\begin{aligned}\zeta(f, \Phi, s)&\zeta(\hat{g}, \bar{\Phi}, n - s)\\ &= \int_B \Big(\int_B \hat{f}(b'b'')g(b')\,|b'|^n d_B^r(b')\Big)\chi(b'')^{-1}\,|b''|^{n-s} d_B^l(b'')\\ &= \int_B g(b')\,|b'|^n \Big(\int_B \hat{f}(b'b'')\chi(b'')^{-1}\,|b''|^{n-s} d_B^l(b'')\Big) d_B^r(b') .\end{aligned}$$

Now make the change of variable $b'' \mapsto (b')^{-1}b''$ in the last relation. This gives:

$$\begin{aligned}\zeta(f, \Phi, s)&\zeta(\hat{g}, \bar{\Phi}, n - s)\\ &= \Big(\int_B g(b')\chi(b')\,|b'|^s d_B^r(b')\Big)\cdot\Big(\int_B \hat{f}(b'')\chi(b'')^{-1}\,|b''|^{n-s} d_B^l(b'')\Big)\\ &= \zeta(g, \Phi, s)\zeta(\hat{f}, \bar{\Phi}, n - s) ,\end{aligned}$$

which is the desired relation.

COROLLARY. *There exists a scalar $c(s)$, depending only on s, such that*

$$\zeta(f, \Phi, s) = c(s)\zeta(\hat{f}, \bar{\Phi}, n - s)$$

for all $f \in I_c^\infty(G)$ whenever $\mathrm{Re}(s) < 1$.

In order to determine $c(s)$ explicitly, it first will be necessary to extend Lemma 2 to a different class of functions and then to summarize the results of some well-known computations.

Let Δ denote the differential operator on M associated with the polynomial $[\det(x)]^2$ $(x \in M)$: If $x = (x_{ij}) \in M$, then

$$\Delta = [\det(\partial/\partial x_{ij})]^2 .$$

Let Δ^N denote the Nth iterate of $\Delta (N = 1, 2, \cdots)$—then it follows from the "operational calculus" for Fourier transforms that

$$(\Delta^N f)^\wedge(x) = (2\pi\sqrt{-1})^{2nN}|x|^{2N}\hat{f}(x) \qquad (x \in M)$$

for all $f \in \mathscr{C}(M)$.

By $[n/2]$ we shall understand, as usual, the greatest integer in $n/2$.

LEMMA 3. *There exists a scalar $C(s)$, depending only on s, such that*

$$\zeta(\Delta^{[n/2]}f, \Phi, s) = C(s)\zeta((\Delta^{[n/2]}f)^{\Lambda}, \bar{\Phi}, n - s)$$

for all $f \in \mathscr{S}(M)$ *whenever* $n - 1 < \mathrm{Re}\,(s) < 2[n/2] + 1$.

Proof. Let $f \in \mathscr{S}(M)$—then, since $\Delta^{[n/2]}f \in \mathscr{S}(M)$, the integral defining $\zeta(\Delta^{[n/2]}f, \Phi, s)$ converges for $\mathrm{Re}\,(s) > n - 1$ (cf. Lemma 1). On the other hand,

$$(\Delta^{[n/2]}f)^{\Lambda}(x) = (2\pi\sqrt{-1})^{2n[n/2]}\,|x|^{2[n/2]}\hat{f}(x) \qquad (x \in M)$$

and so the integral defining $\zeta((\Delta^{[n/2]}f)^{\Lambda}, \bar{\Phi}, n - s)$ converges for $\mathrm{Re}\,(s) < 2[n/2] + 1$. Consequently both sides of the asserted equality in our lemma make sense in the strip $n - 1 < \mathrm{Re}\,(s) < 2[n/2] + 1$. This being so, the argument used in the proof of Lemma 2 implies that

$$\zeta(\Delta^{[n/2]}f, \Phi, s)\zeta((\Delta^{[n/2]}g)^{\Lambda}, \bar{\Phi}, n - s) = \zeta((\Delta^{[n/2]}f)^{\Lambda}, \bar{\Phi}, n - s)\zeta(\Delta^{[n/2]}g, \Phi, s)$$

for all $f, g \in \mathscr{S}(M)$ whenever $n - 1 < \mathrm{Re}\,(s) < 2[n/2] + 1$. The lemma now follows at once.

Let us now recall some computations which go back to Maass [5, p. 3] and Selberg [6, p. 59]. Write

$$e(x) = e^{-\pi(x,x)} \qquad (x \in M)\,.$$

Then e lies in $\mathscr{S}(M)$ and, moreover, $\hat{e} = e$. Consider the integral

$$\int_G (\Delta^{[n/2]}e)(x)\,|x|^s d_G(x)$$

which defines $\zeta(\Delta^{[n/2]}e, \Phi, s)$: This integral converges absolutely in the region $\mathrm{Re}\,(s) > n - 1$, and, moreover, the holomorphic function $\zeta(\Delta^{[n/2]}e, \Phi, s)$ so defined can be meromorphically continued to the whole s-plane and is equal to

$$(2\pi\sqrt{-1})^{2n[n/2]}\pi^{2^{-1}(\Sigma_i s_i - n(2[n/2]+s))}\prod_{i=1}^{n}\prod_{k=1}^{[n/2]}\left(\frac{1 + s_i - s}{2} + k - 1\right)\Gamma\left(\frac{s - s_i}{2}\right),$$

Γ the classical Gamma function. [Here $s_1, \cdots, s_n$ are the parameters of Φ.] On the other hand, consider the integral

$$\int_G (\Delta^{[n/2]}e)^{\Lambda}(x)\overline{\Phi(x)}\,|x|^s d_G(x)$$

which defines $\zeta((\Delta^{[n/2]}e)^{\Lambda}, \bar{\Phi}, s)$: This integral converges absolutely in the region $\mathrm{Re}\,(s) > n - 1$, and, moreover, the holomorphic function $\zeta((\Delta^{[n/2]}e)^{\Lambda}, \bar{\Phi}, s)$ so defined can be meromorphically continued to the whole s-plane and is equal to

$$(2\pi\sqrt{-1})^{2n[n/2]}\pi^{2^{-1}(\Sigma_i\bar{s}_i - n(2[n/2]+s))}\prod_{i=1}^{n}\prod_{k=1}^{[n/2]}\left(\frac{s-\bar{s}_i}{2}+k-1\right)\Gamma\left(\frac{s-\bar{s}_i}{2}\right),$$

$\bar{s}_1, \cdots, \bar{s}_n$ being the parameters of $\bar{\Phi}$.

Bearing in mind the above relations, it is then a simple matter to compute the scalar $C(s)$ which appears in Lemma 3 ($n-1<\mathrm{Re}(s)<2[n/2]+1$). Thus we must have

$$\zeta(\Delta^{[n/2]}e, \Phi, s) = C(s)\zeta((\Delta^{[n/2]}e)^{\Lambda}, \bar{\Phi}, n-s)$$

for all s in the strip $n-1<\mathrm{Re}(s)<2[n/2]+1$. This being so, if we now take into account the fact that

$$\bar{s}_i = n-1-s_{\sigma(i)} \qquad (1 \leqq i \leqq n)$$

for some permutation σ of the set $(1, \cdots, n)$ (which follows from the relation $\bar{\Phi}(x)=\Phi(x^{-1})$ $(x\in G)$) and then utilize the explicit formulae given above, we immediately deduce that

$$C(s) = \gamma(s_1, \cdots, s_n)\prod_{i=1}^{n}\Gamma_R(s-s_i) \ (n-1<\mathrm{Re}(s)<2[n/2]+1)$$

where, for complex z,

$$\Gamma_R(z) = \pi^{1/2-z}\frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)}.$$

Here, for simplicity, we have written

$$\gamma(s_1, \cdots, s_n) = \pi^{-\mathrm{Re}\,(\Sigma_i s_i)+2^{-1}(n^2-n)}.$$

We shall now identify the scalar $c(s)$ which appears in the Corollary to Lemma 2 with the scalar $C(s)$ above.

LEMMA 4. *Let $f\in I_c^\infty(G)$—then the zeta function $\zeta(f, \Phi, s)$ of f with respect to Φ has the value*

$$\left\{\gamma(s_1, \cdots, s_n)\prod_{i=1}^{n}\Gamma_R(s-s_i)\right\}\zeta(\hat{f}, \bar{\Phi}, n-s)$$

for all s such that $\mathrm{Re}\,(s)<1$.

Proof. Suppose first that g is an arbitrary element in $\mathscr{I}(M)$—then, in view of what has been said above, the function $\zeta(\Delta^{[n/2]}g, \Phi, s)$ has a meromorphic continuation to the whole s-plane give by the functional equation

$$\zeta(\Delta^{[n/2]}g, \Phi, s) = \left\{\gamma(s_1, \cdots, s_n) \prod_{i=1}^{n} \Gamma_R(s - s_i)\right\}\zeta((\Delta^{[n/2]}g)^{\Delta}, \bar{\Phi}, n - s) .$$

Specialize now and choose a $g \in I_c^\infty(G)$ having the property that $\zeta(\Delta^{[n/2]}g, \Phi, s)$ is not identically zero—then, thanks to Lemma 2, we have

$$\begin{aligned}\zeta(f, \Phi, s)\zeta((\Delta^{[n/2]}g)^{\Delta}, \bar{\Phi}, n - s) &= \zeta(\hat{f}, \bar{\Phi}, n - s)\zeta((\Delta^{[n/2]}g), \Phi, s) \\ &= \left\{\gamma(s_1, \cdots, s_n) \prod_{i=1}^{n} \Gamma_R(s - s_i)\right\} \\ &\quad \times \zeta(\hat{f}, \bar{\Phi}, n - s)\zeta((\Delta^{[n/2]}g)^{\Delta}, \bar{\Phi}, n - s)\end{aligned}$$

whenever $\mathrm{Re}(s) < 1$. From these considerations, the lemma then follows at once.

3. The meromorphic continuation.

Let us agree to retain the assumptions and notations which were introduced in the preceding section. As has been mentioned at the beginning, the objective of the present note is to establish the following theorem.

THEOREM. *Let $f \in \mathscr{S}(M)$—then the function*

$$\zeta(f, \Phi, s) = \int_G f(x)\Phi(x)|x|^s d_G(x) ,$$

initially defined for $\mathrm{Re}(s) > n - 1$, *can be meromorphically continued to the whole s-plane, and for* $\mathrm{Re}(s) \leqq 0$, *the functional equation*

$$\zeta(f, \Phi, s) = \left\{\gamma(s_1, \cdots, s_n) \prod_{i=1}^{n} \Gamma_R(s - s_i)\right\}\zeta(\hat{f}, \bar{\Phi}, n - s)$$

obtains.

The proof of this theorem is contained in the discussion infra. As will become apparent, there is no loss of generality in assuming that the Haar measure $d_G(x)$ on G has been so chosen that

$$d_G(x) = |x|^{-n} d_M(x) ,$$

$d_M(x)$ being normalized Lebesgue measure on M. (Recall that in §2 we used a slightly different convention.)

To begin with, suppose that f actually lies in $I_c^\infty(G)$—then the validity of our theorem in this case is ensured by Lemma 4. Now suppose only that f belongs to $I_c^\infty(M)$ (the space of K-biinvariant compactly supported C^∞ functions on M). In this situation one may employ an argument due to Stein [7, p. 488–490] which, in brief, goes as follows. First one easily shows that, for some sufficiently large positive integer N, the function f_N defined by the rule

$$f_N(x) = |x|^N f(x) \qquad (x \in M)$$

has the property that there exists a family φ_ε, $\varphi_\varepsilon \in I_c^\infty(G)$ $(\varepsilon > 0)$, such that:

(1) $\varphi_\varepsilon \to f_N$ in $L^1(M)$ as $\varepsilon \to 0$;

(2) $\hat{\varphi}_\varepsilon \to \hat{f}_N$ in $L^1(M)$ as $\varepsilon \to 0$.

This being so, put

$$I_\varepsilon(s) = \frac{1}{\Omega(s)} \int_G \varphi_\varepsilon(x)\Phi(x)\,|x|^s d_G(x)$$

where

$$\Omega(s) = \gamma(s_1, \cdots, s_n) \prod_{i=1}^n \Gamma_R(s - s_i) .$$

Now assume that s lies in the strip $0 \leqq \mathrm{Re}(s) \leqq n$. In view of standard properties of the Gamma function, $1/\Omega(s)$ is meromorphic in this strip and is at most of polynomial growth there. Fix a rational function Q, with roots outside the strip, so that

$$\left|\frac{1}{\Omega(s)} \cdot \frac{1}{Q(s)}\right| \leqq 1 \quad \text{and} \quad \left|\frac{1}{Q(s)}\right| \leqq 1$$

in the strip. Then it is clear that $I_\varepsilon(s)/Q(s)$ is bounded in the strip for each ε. Moreover, we have the estimates

$$\left|\frac{I_\varepsilon(s)}{Q(s)}\right|_{\mathrm{Re}\,(s)=n} \leqq \int_M |\varphi_\varepsilon(x)|\,d_M(x) = \|\varphi_\varepsilon\|_1 ,$$

$$\left|\frac{I_\varepsilon(s)}{Q(s)}\right|_{\mathrm{Re}\,(s)=0} \leqq \int_M |\hat{\varphi}_\varepsilon(x)|\,d_M(x) = \|\hat{\varphi}_\varepsilon\|_1 ,$$

whence, by the maximum principle,

$$\left|\frac{I_\varepsilon(s)}{Q(s)}\right| \leqq \mathrm{Max}\,[\|\varphi_\varepsilon\|_1, \|\hat{\varphi}_\varepsilon\|_1] , \qquad 0 \leqq \mathrm{Re}\,(s) \leqq n .$$

In a similar way, one finds that

$$\left|\frac{I_{\varepsilon_1}(s)}{Q(s)} - \frac{I_{\varepsilon_2}(s)}{Q(s)}\right| \leqq \mathrm{Max}\,[\|\varphi_{\varepsilon_1} - \varphi_{\varepsilon_2}\|_1, \|\hat{\varphi}_{\varepsilon_1} - \hat{\varphi}_{\varepsilon_2}\|_1] , \quad 0 \leqq \mathrm{Re}\,(s) \leqq n .$$

Thus $I_\varepsilon(s)/Q(s)$ and so also $I_\varepsilon(s)$ converges uniformly as $\varepsilon \to 0$ on those bounded subsets of the strip which avoid the finitely many poles of $1/\Omega(s)$. But

$$\lim_{\varepsilon \to 0} I_\varepsilon(s) = \frac{1}{\Omega(s)} \int_G f_N(x)\Phi(x)\,|x|^s d_G(x) \quad \text{for} \quad \mathrm{Re}\,(s) = n ,$$

and

$$\lim_{\varepsilon \to 0} I_\varepsilon(s) = \int_G \hat{f}_N(x)\bar{\Phi}(x)\,|x|^{n-s} d_G(x) \quad \text{for} \quad \mathrm{Re}\,(s) = 0\ ,$$

whence the function

$$I(s) = \frac{1}{\Omega(s)} \int_G f_N(x)\Phi(x)\,|x|^s d_G(x)\ ,$$

initially defined for $\mathrm{Re}\,(s) \geqq n$, has a meromorphic continuation into the strip $0 \leqq \mathrm{Re}\,(s) \leqq n$ and its value for $\mathrm{Re}\,(s) = 0$ is given by $\int_G \hat{f}_N(x)\bar{\Phi}(x)\,|x|^{n-s} d_G(x)$; from the latter integral, the meromorphic continuation of $I(s)$ into the remainder ($\mathrm{Re}\,(s) < 0$) of the s-plane is immediate. It remains to pass from f_N to f itself—this, however, is merely an obvious formal computation (cf. Stein [7, p. 490]).

We now know that our theorem is valid for all f lying in $I_c^\infty(M)$. To extend the result to $\mathscr{S}(M)$, one may argue exactly as we did above in the case of $I_c^\infty(M)$. Here, though, the position is somewhat simpler since $I_c^\infty(M)$ is dense in $\mathscr{S}(M)$ ($\mathscr{S}(M)$ being equipped with its usual Frechet space topology); in particular, one does not have to employ the artifice of first working with f_N and then passing to f itself ($f \in \mathscr{S}(M)$)$\cdots$.

The proof of the theorem is therefore complete.

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