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WILD ARCS IN THREE-SPACE

1: FAMILIES OF FOX-ARTIN ARCS

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Roughly speaking, a Fox-Artin arc is an arc which is tame modulo one endpoint at which it has penetration index three, and which may be constructed in the way that the examples of R. H. Fox and E. Artin were constructed in their classical paper of 1948.

For each oriented Fox-Artin arc, there is an associated infinite sequence of oriented prime 2-component links, which is an invariant of the local embedding type of the arc in R^3. Using existence results from link theory, this result yields the corollary: If M is a 3-manifold and p a point in the interior of M, then there exists an uncountable family of locally non-invertible Fox-Artin arcs in M, which are wild at p.

Later papers will be concerned with developing invariants of the oriented local embedding type of an arc k_n which is tame modulo one endpoint, at which it has penetration index $2n+1$.

0. Introduction. The results of this paper originated in attempts to answer the following two questions:

1. Do there exist uncountably many arcs in Euclidean 3-space R^3, which are tame modulo one endpoint? ([1], p. 33. Such arcs are called "nearly polyhedral" in [1].)

2. Are the arcs 1.1, 1.1*, 1.3 of [3] amphicheiral or invertible? (Problem 17 of [2].)

In 1961, R. H. Fox and O. G. Harrold [4] succeeded in completely classifying the Wilder arcs of penetration index 2, and the existence of uncountably many non-invertible Wilder arcs follows immediately from their classification. In 1963, an affirmative answer to question 1 was announced by Giffen [5] (a more detailed development of Giffen's ideas is given in [13]); however, Giffen developed no new invariant of local embedding type, so there was still no way of distinguishing nearly polyhedral arcs of the same penetration index. While S. J. Lomonaco ([8], 1967) succeeded in algebraically distinguishing the local types of arcs at one interior wild point, there was no theory of local embedding type for nearly polyhedral arcs.

To the best of the present author's knowledge, the invariants of local type of an oriented nearly polyhedral arc, developed in this paper and in [9], are the only ones available (other than penetration index invariants) that will differentiate one nearly polyhedral arc type from another. Use of these invariants will give a detailed affirmative answer to question 1 (see § 2, and p. 91 of [9]) and a partial answer to question 2.

The author wishes to thank Professor N. Smythe for his helpful advice in the writing of his Ph. D. thesis [9], presented to the University of New South Wales. The development of the invariants of [9] will be the subject of later papers in this series.

1. Preliminaries. Let X be a set—we use $\mathrm{Bd}\, X$, $\mathrm{Cl}\, X$, and $\mathrm{Int}\, X$ to denote the boundary, closure, and interior respectively of X. X has $N(X)$ elements. If k is an oriented arc in R^3 (the orientation of R^3 is fixed), and X is an oriented surface, $\nu(k \cap X)$ is the algebraic intersection number of k with X.

An oriented link $L = l_1 \cup l_2$ of two components in R^3 is *splittable* if there exists a 2-sphere $S \subset R^3$, such that l_1 and l_2 lie in different components of $R^3 - S$; S is said to split L. If no such 2-sphere exists, L is unsplittable. If L' is another link in R^3 which is F-isotopic to L (for the definition of F-isotopy, see either [14] or [10]), then L' is splittable iff L' is splittable, by Theorem 1 of [14].

The main theorem of [6] states that if L is unsplittable, L has a unique factorisation into prime links; by Theorem 1 of the same paper, at most one of these prime links is a 2-component link L^*. (The other factors are 1-component links, i.e. knots, which are factors of the knot types of the components of L.) We call L^* the (oriented) *prime hub* of L, and note that L and L^* are F-isotopic.

Let k be an oriented arc in R^3, with endpoints p and q. Let E_1 and E_2 be tame closed 3-cell neighbourhoods of p, with $N(k \cap Bd\, E_i) = 3$. If $E_2 \subset \mathrm{Int}\, E_1$, the set $k \cap \mathrm{Cl}\,(E_1 - E_2)$ is either (i) three arcs, each with one endpoint on $\mathrm{Bd}\, E_1$ and the other on $\mathrm{Bd}\, E_2$, or (ii) one arc γ_1 connecting $\mathrm{Bd}\, E_1$ and $\mathrm{Bd}\, E_2$, one arc α_1 whose endpoints both lie on $\mathrm{Bd}\, E_1$, and one arc β_1 with both endpoints on $\mathrm{Bd}\, E_2$.

In case (i), we say E_1 and E_2 are *k-similar* and write $E_1 \sim E_2$, without distinguishing E_1 and E_2. We note that "$\sim$" is not transitive, but if E_1, E_2, E_3 are tame closed 3-cell neighbourhoods of p with $E_{i+1} \subset \mathrm{Int}\, E_i$, and $E_1 \sim E_2$, $E_2 \sim E_3$, then $E_1 \sim E_3$. The use of the symbol $\sim$ for a relation which is not an equivalence relation is unfortunate but seems unavoidable.

In case (ii), we write $E_1 \succ E_2$ if the pair (α_1, β_1) is unsplittable in the sense of [10]; that is, there exist oriented non-singular arcs $\alpha' \subset \mathrm{Bd}\, E_1$ and $\beta' \subset \mathrm{Bd}\, E_2$ such that the link $L = (\alpha_1 \cup \alpha') \cup (\beta_1 \cup \beta')$ is unsplittable. (Hereafter, we shall always assume that the arcs α' and β' are chosen to make L a consistently oriented link.) Let $L(E_1, E_2)$ denote the prime hub of L.

Unless otherwise stated, all our "3-cells" will be tame closed 3-cell neighbourhoods of p, each meeting k on its boundary in exactly three points. We fix a 3-cell E_0, which will contain all other 3-cells in its interior. When we write $E_2 \subset E_1$, it is implicit that $E_2 \subset \operatorname{Int} E_1$.

We make the following observations:

A: "$\sim$" is not a transitive relation.

B: If E_1, E_2, and C are 3-cells with $E_1 \supset C \supset E_2$, and $E_1 \sim C$, then $C \succ E_2$ iff $E_1 \succ E_2$. In either case, $L(C, E_2) = L(E_1, E_2)$.

B^*: If $C \sim E_2$, then $C \prec E_1$ iff $E_2 \prec E_1$, and $L(E_1, C) = L(E_1, E_2)$ in either case.

We shall only prove that $C \succ E_2$ if $E_1 \succ E_2$ and $E_1 \sim C$, and that then $L(C, E_2) = L(E_1, E_2)$.

Let α_1, α', β_1, β', and γ_1 be chosen as above, so that $L = (\alpha_1 \cup \alpha') \cup (\beta_1 \cup \beta')$. Let α_C be an arc on $\operatorname{Bd} C - \gamma_1$ which connects the two points of $\alpha_1 \cap \operatorname{Bd} C$, and let Γ be a regular neighbourhood of γ_1 in $E_1 - (C \cup \alpha_1)$. See Figure 1.

Denote by L_C the link $[\alpha_C \cup (\alpha_1 \cap C)] \cup (\beta_1 \cup \beta')$. The cube $Q = \operatorname{Cl}(E_1 - C \cup \Gamma)$ is an "admissible cube" for the link L, and by [6], p. 284, L is the link-product of L_C and the knot formed by the arcs α' and α_C and the two arcs of $\alpha_1 \cup (E_1 - C)$. This implies (i) L and L_C have the same prime hub, i.e. $L(C, E_2) = L(E_1, E_2)$, and (ii) L_C and L are F-isotopic. Since L is unsplittable, L_C is unsplittable and $C \succ E_2$.

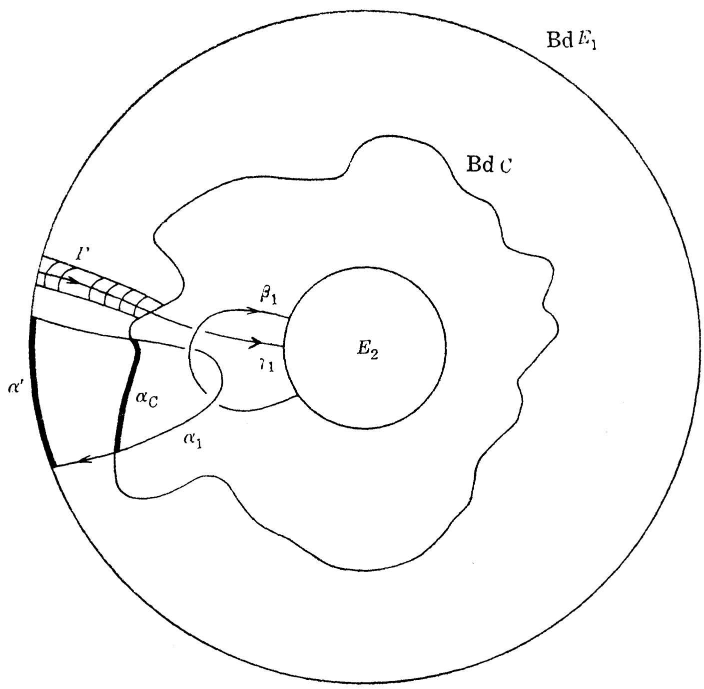

FIGURE 1

2. Statement of the theorem.

DEFINITION. An oriented arc k in R^3, with endpoints p and q, is a *Fox-Artin arc* if $k - p$ is tame, and there exists a sequence $\mathscr{E}: E_0 \succ E_1 \succ E_2 \succ \cdots$ of tame closed 3-cell neighbourhoods of p, with $\cap E_i = \{p\}$. $\mathscr{E}$ is a *Fox-Artin sequence* for k.

By [10], the Fox-Artin arcs are wild, with penetration index three at p. Example 1.2 of [3] is a Fox-Artin arc.

Before stating the main theorem of this paper, we must recall two definitions.

Two sequences $\{a_i\}$ and $\{b_j\}$ are *cofinal* if there exist indices M and N such that $a_{M+r} = b_{N+r}$ for all $r = 0, 1, 2, \cdots$.

Two arcs k_1 and k_2 are of the same *oriented local type* at points p_1 and p_2 if there exist neighbourhoods U_i of p_i (which inherit their orientation from R^3), and an orientation-preserving homeomorphism of U_1 to U_2 which takes $(U_1 \cap k_1, p_1)$ onto $(U_2 \cap k_2, p_2)$ ([8], p. 323).

THEOREM. *Let k_1 and k_2 be Fox-Artin arcs with wild endpoints p_1 and p_2, and Fox-Artin sequences*

$$\mathscr{E}_1: E_0 \succ E_1 \succ E_2 \succ \cdots, \mathscr{E}_2: B_0 \succ B_1 \succ B_2 \succ \cdots$$

for k_1 and k_2 respectively. Then if k_1 and k_2 have the same oriented local type at p_1 and p_2, the sequences of oriented prime 2-component links $L(\mathscr{E}_1) = \{L(E_i, E_{i+1})\}$ and *$L(\mathscr{E}_2) = \{L(B_j, B_{j+1})\}$ are cofinal.*

COROLLARY 1. *There exists an uncountable family of locally non-invertible Fox-Artin arcs in R^3, and an uncountable family of non-amphicheiral such arcs.*

Proof 1. It is shown in [11] that there exists an infinite family of non-invertible prime 2-component links, and the links L_n, $n = 1, 2, 3, \cdots$ of [7] form an infinite family of non-amphicheiral prime links.

Let k be a Fox-Artin arc, and $\mathscr{E}$ one of its Fox-Artin sequences. Let $\lambda_1, \lambda_2, \lambda_3, \cdots$ be an ordering of the prime links which have property P, where P connotes non-invertibility or non-amphicheirality. Obtain a binary number $\alpha(k)$ by the rule: the ith digit of $\alpha(k)$ is 1 if the prime λ_i occurs infinitely often in the sequence $L(\mathscr{E})$, and is 0 otherwise. Since all the binary numbers may be obtained in this way, the result follows.

Figure 2(a) shows a Fox-Artin arc, which may be constructed from an infinite family of cylinders of the type shown in Figure 2(a) (cf. [3] or [1] for details). For each i, the link $L(E_i, E_{i+1})$ is the link

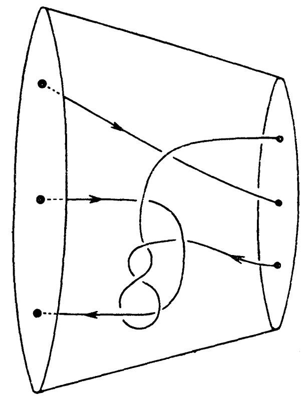

FIGURE 2 (a)

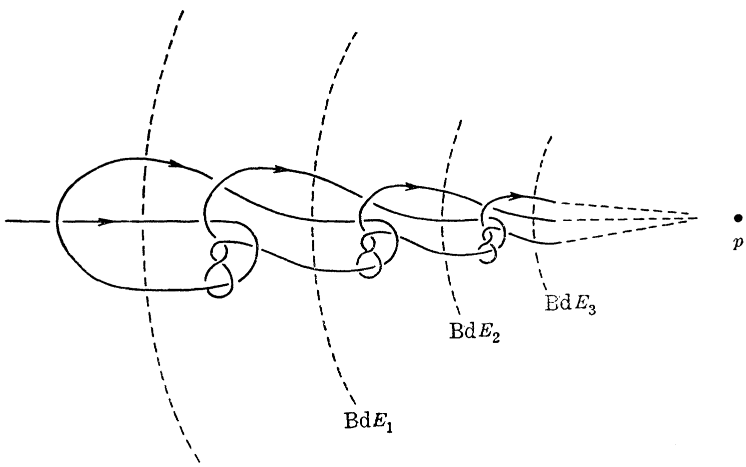

FIGURE 2 (b)

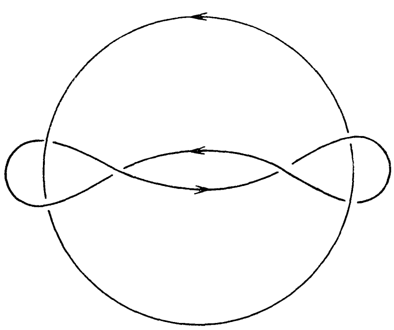

FIGURE 3

of Figure 3, which is known to be non-amphicheiral ([7], p. 653), so the arc of Figure 2 (b) is non-amphicheiral.

COROLLARY 2. *Let M be a 3-manifold, and p a point in the interior of M. If M is non-orientable, there exist uncountably many* *locally non-invertible Fox-Artin arcs which are wild at p. If M is oriented, there exist uncountably many locally non-invertible Fox-Artin arcs which are wild at p, and uncountably many non-amphicheiral such arcs.*

Using the results of C. D. Sikkema in [12], we obtain

COROLLARY 3. *There exist uncountably many crumpled* 3*-cubes.*

A *crumpled n-cube* is a topological space which is homeomorphic to a closed complementary domain of an $(n-1)$-sphere in S^n. An *n-psuedo-half-space* M^n is an n-manifold with boundary, whose boundary is homeomorphic to R^{n-1} and whose interior is homeomorphic to R^n. For $n \neq 3$, every n-psuedo-half-space is homeomorphic to the closed half-space R^n_+ ([12], p. 399, p. 411). However, for $n = 3$, Theorem 8 of [12] yields.

COROLLARY 4. *There exists an uncountable family of topologically different* 3*-psuedo-half-spaces.*

We conclude our list of corollaries with a partial answer to question 2 of §0, namely.

COROLLARY 5. *The arcs* 1.1, 1.1*, *and* 1.2 *of* [3] *are not amphicheiral.*

Proof 5. We prove that the arc 1.2 (whose projection is shown in Figure 7 of [3]) is non-amphicheiral. Denote this arc by k, and its mirror image by k'. k is constructed from cylinders of the type shown in Figure 1 of [3], while k' may be constructed from cylinders which are the mirror images of those used to construct k. A glance at Figure 1 of [3] shows that the link obtained by identifying r_- and s_-, and t_+ with r_+, has linking number $+1$, while the corresponding link in the mirror image of this cylinder has linking number -1. For k, then, the sequence of oriented prime 2-component links is a sequence of simply linked circles with linking number $+1$, whereas the sequence of oriented prime 2-component links associated with k' is a sequence of simply linked circles with linking number -1. k and k' (the mirror image of k) therefore cannot have the same oriented local type, so k cannot be amphicheiral.

3. The proof of the theorem. The proof consists of a sequence of manipulatory lemmas. In each, k is a fixed Fox-Artin arc, with fixed Fox-Artin sequence $\mathscr{E}: E_0 \succ E_1 \succ E_2 \succ \cdots$. α_i and β_i are the arcs of $k \cap \mathrm{Cl}\,(E_i - E_{i+1})$ whose endpoints both lie on $\mathrm{Bd}\, E_i$, and on

$\operatorname{Bd} E_{i+1}$ respectively. γ_i is the unique subarc of k which connects $\operatorname{Bd} E_i$ and $\operatorname{Bd} E_{i+1}$. $L_i = L(E_i, E_{i+1})$ denotes the prime hub of the link $(\alpha_i \cup \alpha) \cup (\beta_i \cup \beta)$, where α and β are suitably chosen arcs on $\operatorname{Bd} E_i$ and $\operatorname{Bd} E_{i+1}$ respectively.

LEMMA 1. *Let C be a 3-cell with $E_i \supset C \supset E_{i+1}$, some i. Then either $C \sim E_i$, or $C \sim E_{i+1}$.*

Proof. Suppose $\alpha_i \subset \operatorname{Int}(E_i - C)$—we assert that $\beta_i \cap \operatorname{Bd} C \neq \varnothing$ and that $C \sim E_{i+1}$.

So suppose $\beta_i \cap \operatorname{Bd} C = \varnothing$, i.e. that $\beta_i \subset \operatorname{Int} C$. Then $\operatorname{Bd} C$ is a 2-sphere that splits the pair (α_i, β_i), contradicting the hypothesis that $E_i \succ E_{i+1}$. Hence $\beta_i \cap \operatorname{Bd} C \neq \varnothing$, and β_i therefore meets $\operatorname{Bd} C$ in precisely two points, for $N(k \cap \operatorname{Bd} C) = 3$ and $\nu(\beta_i \cap \operatorname{Bd} C) = 0$. Also, γ_i must meet $\operatorname{Bd} C$ in only one point.

To prove $C \sim E_{i+1}$, assume to the contrary, and that α is a subarc of k in $C - E_{i+1}$ whose endpoints both lie on $\operatorname{Bd} C$. If α joins the points of $\beta_i \cap \operatorname{Bd} C$, then $\alpha \cup (\beta_i \cap \operatorname{Cl}(E_i - C))$ is a simple closed curve in the arc k, which is impossible. One of the endpoints of α must therefore be the point $\gamma_i \cap \operatorname{Bd} C$ – so α must be a subarc of γ_i, else k intersects itself at $\gamma_i \cap \operatorname{Bd} C$. But then γ_i meets $\operatorname{Bd} C$ in at least two points, for

$$2 = N(\alpha \cap \operatorname{Bd} C) \leqq N(\gamma_i \cap \operatorname{Bd} C)$$

since $\alpha \subset \gamma_i$. But γ_i meets $\operatorname{Bd} C$ in only one point, so $\gamma_i \cap \operatorname{Bd} C$ cannot be an endpoint of α.

Hence no such subarc α of k exists, and $k \cap \operatorname{Cl}(C - E_{i+1})$ must therefore consist of three arcs connecting $\operatorname{Bd} C$ and $\operatorname{Bd} E_{i+1}$. Thus $C \sim \mathrm{E}_{i+1}$.

Similarly, $C \sim E_i$ if $\beta_i \subset \operatorname{Int} \mathrm{C}$.

The cases $\alpha_i \subset E_i - C$ and $\beta_i \subset \operatorname{Int} C$ are the only two cases possible; it is impossible that both $\alpha_i \cap \operatorname{Bd} C$ and $\beta_i \cap \operatorname{Bd} C$ be nonempty together. For α_i and β_i are disjoint, and both have algebraic intersection number zero with $\operatorname{Bd} C$, so

$$N(k \cap \operatorname{Bd} C) \geqq N(\alpha_i \cap \operatorname{Bd} C) + N(\beta_i \cap \operatorname{Bd} C) \geqq 4$$

because α_i and β_i must both meet $\operatorname{Bd} C$ in at least two points. So if $N(k \cap \operatorname{Bd} C) = 3$, one and only one of the sets $\alpha_i \cap \operatorname{Bd} C$, $\beta_i \cap \operatorname{Bd} C$ is empty.

LEMMA 2. *If C is a 3-cell, $C \sim E_i$, $E_{i-1} \supset C \supset E_{i+1}$, then*

$$\mathscr{E}'\colon E_0 \succ E_1 \succ \cdots \succ E_{i-1} \succ C \succ E_{i+1} \succ \cdots$$

is a Fox-Artin sequence for k, and $L(\mathscr{E}') = L(\mathscr{E})$.

Proof. This is a straightforward application of the results B and B^* of §1.

If $C \supset E_i$, then $C \prec E_{i-1}$ and $L(E_{i-1}, C) = L(E_{i-1}, E_i)$, by B^*. Also, $E_i \succ E_{i+1}$ so $C \succ E_{i+1}$ and $L(C, E_{i+1}) = L(E_i, E_{i+1})$, by B. This is sufficient to show that $\mathscr{E}'$ is a Fox-Artin sequence for k.

If $C \subset E_i$, the proof is similar.

Let $E_{i+1} \in \mathscr{E}$. A *containing sequence for E_{i+1}, of length s in E_0*, is a 3-cell sequence

$$E_0 = B_0 \succ B_1 \succ B_2 \succ \cdots \succ B_s \succ B_{s+1} = E_{i+1} .$$

The *associated sequence of prime links* is the sequence $\{L(B_i, B_{i+1})\}$.

LEMMA 3. *Any two containing sequences for E_{i+1} in E_0 have the same length* (length i), *and the associated sequences of prime links are identical. Thus any two Fox-Artin sequences $\mathscr{E}$ and $\mathscr{E}'$ which start at E_0 and have E_{i+1} as a common term also have the first $i + 1$ terms of $L(\mathscr{E})$ and $L(\mathscr{E}')$ in common.*

Proof. Let $\mathscr{E}_{i+1}$ denote the containing sequence

$$E_0 \succ E_1 \succ \cdots \succ E_i \succ E_{i+1}$$

for E_{i+1} in E_0, let

$$E_0 \succ B_1 \succ \cdots \succ B_s \succ E_{i+1}$$

be another containing sequence for E_{i+1}, and assume $s > i$. Let $\mathscr{B}$ be the class of all containing sequences for E_{i+1}, of length s in E_0, whose associated sequences of prime links are identical with the sequence $\{L(B_j, B_{j+1})\}$. $\mathscr{B}$ is not empty by hypothesis, so there exists a sequence

$$E_0 = C_0 \succ C_1 \succ C_2 \succ \cdots \succ C_s \succ C_{s+1} = E_{i+1}$$

in $\mathscr{B}$ with the properties:

(i) each 2-sphere $\mathrm{Bd}\, C_j$ is in general position with respect to the 2-spheres $\mathrm{Bd}\, E_1, \cdots, \mathrm{Bd}\, E_i$, and

(ii) the number of elements of the family

$$C(\mathscr{E}_{i+1}) = \bigcup_{j=1}^{s} \bigcup_{h=1}^{i} \mathrm{Bd}\, C_j \cap \mathrm{Bd}\, E_h$$

of intersection curves is minimal in $\mathscr{B}$, and none of these intersection curves meets k.

Note that by (i), each intersection curve is a simple closed curve $\sigma \subset \mathrm{Bd}\, C_j \cap \mathrm{Bd}\, E_h$, which bounds discs on both $\mathrm{Bd}\, C_j$ and $\mathrm{Bd}\ E_h$.

Our first step is to show $C(\mathscr{E}_{i+1}) = \emptyset$.

For some l and h, let $\sigma \subset \mathrm{Bd}\, C_l \cap \mathrm{Bd}\, E_h$ be an intersection curve. σ separates $\mathrm{Bd}\, E_h$ into two discs, D_1 and D_2, and one of these discs contains at most one point of k, because $N(k \cap \mathrm{Bd}\, E_h) = 3$. Let this disc be $D(\sigma)$.

Suppose there exists an index j and an intersection curve $\alpha \subset \mathrm{Bd}\, C_j \cap \mathrm{Int}\, D(\sigma)$, which bounds a disc $\mathrm{D} \subset \mathrm{Int}\, D(\sigma)$ containing no other intersection curves. α also separates $\mathrm{Bd}\, C_j$ into two discs; one of these, say D', together with D is the boundary of a 3-cell S which does not contain C_{j+1}. Let N be a closed regular neighbourhood of S. We write $C'_j = C_j \cup N$ if $D \subset \mathrm{Cl}\,(E_0 - C_j)$, and $C'_j = \mathrm{Cl}\,(C_j - N)$ if $D \subset C_j$.

Then C'_j is a tame closed 3-cell neighbourhood of p, with $C_{j-1} \supset C'_j \supset C_{j+1}$; we will show

(a) $N(k \cap \mathrm{Bd}\, C'_j) = N(k \cap \mathrm{Bd}\, C_j) = 3$, and

(b) $C'_j \prec C_{j-1}$, $C'_j \succ C_{j+1}$,

$$L(C_{j-1}, C'_j) = L(C_{j-1}, C_j)\ ,$$

and

$$L(C'_j, C_{j+1}) = L(C_j, C_{j+1})\ .$$

The sequence

$$E_0 \succ C_1 \succ \cdots \succ C_{j-1} \succ C'_j \succ C_{j+1} \succ \cdots \succ C_s \succ E_{i+1}$$

will then be a sequence in $\mathscr{B}$ by Lemma 2; and we note that

$$\mathrm{Bd}\, C'_j \cap \mathrm{Bd}\, E_l \subset \mathrm{Bd}\, C_j \cap \mathrm{Bd}\, E_l\ ,$$

particularly if $l = h$, when

$$\mathrm{Bd}\, C'_j \cap \mathrm{Bd}\, E_h \subset \mathrm{Bd}\, C_j \cap \mathrm{Bd}\, E_h - \{\alpha\}\ .$$

But this new sequence in $\mathscr{B}$ has at least one less intersection curve in $C(\mathscr{E}_{i+1})$ than our original sequence, contradicting the minimality hypothesis (ii). Hence if (a) and (b) hold, we must conclude that $C(\mathscr{E}_{i+1})$ is empty.

(a) k meets $\mathrm{Bd}\, C'_j$ in at least three points, for k has penetration index three at p. So $N(k \cap \mathrm{Bd}\, C'_j) \geqq N(k \cap \mathrm{Bd}\, C_j)$, i.e. $N(k \cap D) \geqq N(k \cap D')$. But

$$N(k \cap D') \leqq N(k \cap D) \leqq N(k \cap D(\sigma)) \leqq 1\ ;$$

so certainly $k \cap D'$ is empty if $k \cap D = \emptyset$, and k meets D' in precisely one point if $N(k \cap D) = 1$, because $\nu(k \cap \mathrm{Bd}\, S) = 0$. In both cases, then, $N(k \cap D) = N(k \cap D')$ and $N(k \cap \mathrm{Bd}\, C_j') = 3$.

(b) If $k \cap D = \varnothing$, this is trivial. If $k \cap D$ and $k \cap D'$ are both one-point sets, $k \cap S$ must consist of one arc γ joining D and D', so C_j' and C_j are k-similar. Then it follows from Lemma 1 that $C_j' \prec C_{j-1}$ and $C_j' \succ C_{j+1}$, and from B and B^* that

$$L(C_{j-1}, C_j') = L(C_{j-1}, C_j)$$

and

$$L(C_j', C_{j+1}) = L(C_j, C_{j+1}) .$$

We conclude that $C(\mathscr{E}_{i+1}) = \varnothing$; that is, that $\mathrm{Bd}\, C_j \cap \mathrm{Bd}\, E_h = \varnothing$ for all indices $j = 1, \cdots, s$ and $h = 1, \cdots, i$. Our next aim is to show that the assumption $s > i$ leads to a contradiction; and to this end, we let $n(h)$ denote the number of 2-spheres $\mathrm{Bd}\, C_j$ contained in $\mathrm{Int}\,(E_h - E_{h+1})$, $h = 0, \cdots, i$.

That $n(0)$ and $n(i)$ are both at most 1 follows from Lemma 1. Indeed, suppose $\mathrm{Bd}\, C_{s-1} \cup \mathrm{Bd}\, C_s \subset \mathrm{Int}\,(E_i - E_{i+1})$. Then $C_s \sim E_i$ because $C_s \succ E_{i+1}$, so there is no subarc of k in $\mathrm{Cl}\,(E_i - C_s)$ which has both endpoints lying on $\mathrm{Bd}\, C_s$. Since $E_i \supset C_{s-1} \supset C_s$, it is therefore impossible that $C_{s-1} \succ C_s$. So C_{s-1} must contain E_i in its interior, and $n(i) \leqq 1$. Similarly $n(0) \leqq 1$.

Suppose $n(h) = r$; that is, that the 2-spheres $\mathrm{Bd}\, C_j, \cdots, \mathrm{Bd}\, C_{j+r-1}$ all lie in $\mathrm{Int}\,(E_h - E_{h+1})$. If C_j and E_h are not k-similar, then $C_j \sim E_{h+1}$ by Lemma 1. Again, this means that there is no subarc of k in $C_j - E_{h+1}$ which has both its endpoints on $\mathrm{Bd}\, C_j$, so it is impossible that $C_j \succ C_{j+1}$ if $C_j \supset C_{j+1} \supset E_{h+1}$. So $r = 1$ if $C_j \sim E_{h+1}$.

If $C_j \sim E_h$, then $E_h \succ C_{j+1}$ (by the remark B in §1) because $C_j \succ C_{j+1}$. By Lemma 1, C_{j+1} and E_{h+1} must be k-similar. It is then impossible that $C_{j+1} \succ C_{j+2}$ if $\mathrm{Bd}\, C_{j+2} \subset \mathrm{Int}\,(C_{j+1} - E_{h+1})$, so $C_{j+2} \subset E_{h+1}$. Hence $r = 2$ if $C_j \sim E_h$.

Thus $n(h) = r \leqq 2$ for all h. Suppose $n(h) = 2$ and $n(l) = 1$ for all $l < h$. Then $E_{l-1} \supset C_l \supset E_l$ for all l, and $C_1 \sim E_1$ because $C_1 \prec E_0$, by Lemma 1. C_2 and E_1 cannot be k-similar, for then $C_1 \sim C_2$, so $C_2 \sim E_2$ by Lemma 1. Similarly, $C_3 \sim E_3$, $C_4 \sim E_4, \cdots, C_h \sim E_h$: but the preceding paragraphs show $C_{h+1} \sim E_h$ because $n(h) = 2$. Because $C_h \supset E_h \supset C_{h+1}$, C_h and C_{h+1} must be k-similar, which contradicts the hypothesis $C_h \succ C_{h+1}$. Then $n(l)$ must be zero for at least one $l < h$.

Similarly, we may show that if h is the last index such that $n(h) = 2$, then there is an index $l > h$ such that $n(l) = 0$.

Now suppose $n(h) = n(t) = 2$, but $n(r) = 1$ for all $h < r < t$. Let C_l be the 3-cell which is k-similar to E_{h+1}. Then $C_{l+1} \sim E_{h+2}$, so $C_{l+2} \sim E_{h+3}, \cdots$, and $C_{l+t-h-1} \sim E_t$. But because

$$C_{l+t-h-1} \supset E_t \supset C_{l+t-h} \succ C_{l+t-h+1} \supset E_{t+1} ,$$

it follows that $C_{l+t-h} \sim E_t$; but then $C_{l+t-h-1} \sim C_{l+t-h}$, which is impossible. So $n(r)$ must be zero for some r, $h < r < t$.

Thus, in the sequence $n(0), n(1), \cdots, n(i)$, we have shown that there is an 0 preceding the first 2, succeeding the last 2 is a 0, and between any two 2's there is a 0. There are more 0's than 2's therefore. But then

$$s = \sum_{h=0}^{i} n(h) < i + 1 \leqq s$$

which is impossible. So the assumption $s > i$ leads to a contradiction. However if $s < i$, we let $m(j)$ denote the number of 2-spheres $\operatorname{Bd} E_h$ in $\operatorname{Int}(C_j - C_{j+1})$, and deduce, as above, that

$$i = \sum m(j) < s + 1 \leqq i .$$

This forces $s = i$.

We now have two containing sequences for E_{i+1}, of length i in E_0; we wish to show that the associated sequences of prime links are identical.

For each index h, it is impossible that $C_h \succ E_h$ if $C_h \supset E_h$, for then

$$E_0 \succ C_1 \succ C_2 \succ \cdots C_h \succ E_h$$

is a containing sequence for E_h, of length h in E_0, while

$$E_0 \succ E_1 \succ \cdots \succ E_{h-1} \succ E_h$$

is a containing sequence of length $h - 1$ in E_0. But from the part of the lemma already proven, any two containing sequences for E_h must have the same length $- h - 1 -$ in E_0. Therefore C_h and E_h are k-similar if $C_h \supset E_h$; similarly, $C_h \sim E_h$ if $C_h \subset E_h$.

Then to prove $L(C_h, C_{h+1}) = L(E_h, E_{h+1})$, we have several possibilities to consider: we shall only prove that the prime hubs are identical if $E_h \supset C_h \succ C_{h+1} \supset E_{h+1}$. We have

$$L(C_h, C_{h+1}) = L(E_h, C_{h+1}) = L(E_h, E_{h+1})$$

where the equalities follow from the statements B and B^* of §1.

The other possibilities may be considered similarly, and we conclude that $L(C_h, C_{h+1}) = L(E_h, E_{h+1})$ for all h; that is, the associated sequences of prime links are identical.

LEMMA 4. *Let $\mathscr{E}$ be a Fox-Artin sequence for k, and let $C \subset \operatorname{Int} E_0$ be a 3-cell. Then C has a containing sequence*

$$E_0 = C_0 \succ C_1 \succ C_2 \succ \cdots \succ C_s \succ C = C_{s+1}$$

such that

$$L(C_i, C_{i+1}) = L(E_i, E_{i+1}) , \qquad i = 0, 1, \cdots, s .$$

Proof. There exists an index h such that $E_h \subset \operatorname{Int} C$. Let $\mathscr{B}$ be the class of all containing sequences for E_h in E_0, which therefore have length $h-1$ in E_0, and the associated sequence of prime links for each containing sequence is the first h terms of the sequence $L(\mathscr{E})$, by Lemma 3. $\mathscr{B}$ is not empty. Then there exists one sequence in $\mathscr{B}$, say

$$E_0 \succ C_1 \succ C_2 \succ \cdots \succ C_{h-1} \succ E_h$$

with the properties:

(i) $\operatorname{Bd} C$ is in general position with respect to $\operatorname{Bd} C_1, \operatorname{Bd} C_2, \cdots, \operatorname{Bd} C_{h-1}$, and

(ii) the number of intersection curves $\operatorname{Bd} C \cap \bigcup \operatorname{Bd} C_j$ is minimal in $\mathscr{B}$, and none of these curves meets k.

As in the proof of Lemma 3, we may keep C fixed and modify our sequence in $\mathscr{B}$ to eliminate those intersection curves which bound discs on $\operatorname{Bd} C$ containing at most one point of k, i.e. to eliminate any intersection curves which may occur. Since the number of curves of $\operatorname{Bd} C \cap \bigcup \operatorname{Bd} C_j$ is assumed to be minimal, we conclude that $\operatorname{Bd} C \cap \operatorname{Bd} C_j = \varnothing$ for all $j = 1, 2, \cdots, h-1$.

Then $C_{j+1} \subset C \subset C_j$ for some j, $0 \leqq j < h-1$. From Lemma 1, it follows that either $C \sim C_j$ or $C \sim C_{j+1}$ and $C \prec C_j$. In the first case,

$$E_0 \succ C_1 \succ \cdots \succ C_{j-1} \succ C$$

is a containing sequence for C, while

$$E_0 \succ C_1 \succ \cdots \succ C_{j-1} \succ C_j \succ C$$

is a containing sequence in the second case. Then we can take the s in the statement of this lemma to be either j or $j-1$, and the result follows because

$$L(C_s, C) = L(C_s, C_{s+1}) = L(E_s, E_{s+1})$$

where the equality on the left follows from either statement B or B^*.

We are now ready to prove the theorem.

Since k_1 and k_2 are of the same oriented local type at their respective endpoints, there exist neighbourhoods U_i of p_i (with orientation inherited from R^3), and an orientation-preserving homeomorphism

$$h: (U_1, U_1 \cap k_1, p_1) \rightarrow (U_2, U_2 \cap k_2, p_2)$$

which takes U_1 onto U_2.

Let $\mathscr{E}_1: E_0 \succ E_1 \succ E_2 \succ \cdots$ and $\mathscr{E}_2: B_0 \succ B_1 \succ B_2 \succ \cdots$ be Fox-Artin sequences for k_1 and k_2 respectively, and consider $h(\mathscr{E}_1)$; that is, the sequence

$$h(E_0) \succ h(E_1) \succ h(E_2) \succ \cdots$$

which is a Fox-Artin sequence for k_2. There exists an index t such that $h(E_t) \subset \operatorname{Int} B_0$; by Lemma 4, there exist 3-cells $C_1, \cdots, C_s$ such that

$$B_0 \succ C_1 \succ C_2 \succ \cdots \succ C_s \succ h(E_t) \succ h(E_{t+1}) \succ \cdots \succ h(E_{t+r})$$

is a containing sequence for $h(E_{t+r})$ in B_0. Also, there exists an index $H(r) = H$ and 3-cells $C_{s+r+2}, \cdots, C_{H-1}$ such that

$$h(E_{t+r}) \succ C_{s+r+2} \succ \cdots \succ C_{H-1} \succ B_H$$

is a containing sequence for B_H in $h(E_{t+r})$.

Then we have two containing sequences for B_H in B_0, namely

$$B_0 \succ B_1 \succ B_2 \succ \cdots \succ B_H$$

and

$$\begin{aligned} B_0 \succ C_1 \succ \cdots \succ C_s \succ h(E_t) \succ h(E_{t+1}) \succ \cdots \\ \succ h(E_{t+r}) \succ C_{s+r+2} \succ \cdots \succ C_{H-1} \succ B_H \,. \end{aligned}$$

Then by Lemma 3,

$$\begin{aligned} L(E_t, E_{t+1}) &= L(h(E_t), h(E_{t+1})) = L(B_{s+1}, B_{s+2}) \,, \\ L(E_{t+1}, E_{t+2}) &= L(h(E_{t+1}), h(E_{t+2})) = L(B_{s+2}, B_{s+3}) \,, \\ &\vdots \\ L(E_{t+r-1}, E_{t+r}) &= L(h(E_{t+r-1}), h(E_{t+r})) = L(B_{s+-1}, B_{s+r}) \,. \end{aligned}$$

Setting $r = 1, 2, 3, \cdots$ shows that the sequences $L(\mathscr{E}_1)$ and $L(\mathscr{E}_2)$ are cofinal.

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