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A NOTE ON THE MACKEY TOPOLOGY FOR $(C^b(X)^*, C^b(X))$

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Let τ denote a completely regular Hausdorff topology on the point set X, let $C^b(X)$ denote the continuous, bounded real-valued functions on X and let $C^b(X)^*$ denote its Banach dual. If each point of X is identified with the evaluation functional at the point, then X may be treated as a subset of $C^b(X)^*$. The restriction to X of the Mackey topology for the pair $(C^b(X)^*, C^b(X))$ will be denoted by $\mu(\tau)$. The purpose of the paper is to study the topology $\mu(\tau)$ and its relation to τ. (Obviously, $\mu(\tau)$ is finer than τ.) It is proved that $\tau = \mu(\tau)$ if and only if τ is discrete. It is shown that $\mu(\tau)$ is always totally disconnected and that if τ is first countable, then $\mu(\tau)$ is discrete. An example is given to show that $\mu(\tau)$ is not discrete in general.

Finally a few results are proved about the stability of the class of spaces for which $\mu(\tau)$ is discrete. (This class is strictly larger than the class of first countable spaces.)

1. The topology $\mu(\tau)$.

PROPOSITION 1.1. *Let τ be compact Hausdorff. Every zero set in (X, τ) is open in $\mu(\tau)$.*

Proof. Let F be a zero set in (X, τ) and let $f \in C(X)$ be such that $F = \{x \in X: f(x) = 0\}$. For each natural number n, define $U_n = \{x \in X: |f(x)| < 1/n\}$. Let (g_n) be a sequence of functions from $C(X)$ satisfying the following:

(1) $0 \leqq g_n \leqq 1$, for all $n \in N$,

(2) $g_1(x) = \begin{cases} 1, & x \in X - U_2 \\ 0, & x \in F \end{cases}$

(3) $g_n(x) = \begin{cases} 1, & x \in \bar{U}_n - U_{n+1}, \text{ for } n = 2, 3, \cdots \\ 0, & F \cup (X - U_{n-1}). \end{cases}$

Since the sequence (g_n) is uniformly bounded and converges pointwise to zero, it converges weakly to zero. Hence the set $A = \{g_n: n \in N\}$ is weakly relatively compact, and so A^{00} is a weakly compact convex, balanced set by Krein's Theorem ([5], p. 325). Hence A^0 is a neighborhood of zero for the Mackey topology for the pair $(C^b(X)^*, C^b(X))$. (See [5], Theorem 2.) Let U be an open neighborhood of zero contained in $1/2A^0$. Then $W = \bigcup\{x + U: x \in F\}$ is an open set for the Mackey topology so that $W \cap X$ is open in $\mu(\tau)$. But, as is easily verified, $W \cap X = F$. The proof is complete.

PROPOSITION 1.2. *Let τ be a completely regular Hausdorff topology on X. Then every zero set in (X, τ) is open in $\mu(\tau)$.*

Proof. Let $(\beta X, \beta)$ be the Stone-Cêch compactification of X. For $f \in C^b(X)$, let $T(f)$ denote its unique continuous extension to βX. Then T is a linear isometry of $C^b(X)$ onto $C(\beta X)$. Hence T is a topological isomorphism for the weak topologies. Hence it follows that $\mu(\tau)$ is the relative topology of $\mu(\beta)$ on X. (Here X is assumed to be a subset of βX.) Since every zero set in X is the trace of a zero set in βX, the result follows from Proposition 1.1.

THEOREM 1.3. *Let τ be a completely regular Hausdorff topology on X. If τ is first countable, then $\mu(\tau)$ is discrete.*

Proof. If τ is first countable, then every point in X is a zero set. It then follows from Proposition 1.2 that the points of X are open in $\mu(\tau)$. The proof is complete.

THEOREM 1.4. *For any completely regular Hausdorff topology τ on X, $\mu(\tau)$ is totally disconnected.*

Proof. Every point X is the intersection of the zero sets in τ which contain it. Since the zero sets in τ are closed-open in $\mu(\tau)$, it follows that the points of X are the components of X for $\mu(\tau)$. The proof is complete.

The statements and proofs of several of the results to follow are heavily dependent on the theory of ordinal and cardinal numbers. For a discussion of the general theory as it is used in the paper, the reader is referred to [2], Chapter II. The following specific comments, however, seem to be in order. The class $\mathscr{O}$ of *ordinal numbers* satisfies the following conditions and is a subclass of any class which satisfies these conditions: (1) $\varnothing \in \mathscr{O}$, (2) if $\alpha \in \mathscr{O}$, then $\alpha^+ = \alpha \cup \{\alpha\} \in \mathscr{O}$ and (3) if $A \subset \mathscr{O}$ and if A is set, then $\bigcup\{\alpha\colon \alpha \in A\} \in \mathscr{O}$. The class of ordinal numbers is well-ordered by inclusion. As a matter of notation, the Greek letters $\alpha, \beta, \gamma, \delta$ will be used to represent ordinals unless otherwise specified.

The class $\mathscr{C}$ of *cardinal numbers* is then a subclass of $\mathscr{O}$. An ordinal α is an element of $\mathscr{C}$ if and only if it is the least element in the set of ordinals which are equipotent with it. Let $\aleph_0$ denote the smallest infinite cardinal. If $\aleph^*$ is any infinite cardinal, then the set $\{\aleph \in \mathscr{C}\colon \aleph_0 \leqq \aleph < \aleph^*\}$ is a well-ordered set; and hence there is a unique ordinal α such that this set is order isomorphic to $\{\beta \in \mathscr{O}\colon \beta < \alpha\}$. The ordinal $\aleph^*$ is then denoted by $\aleph_\alpha$. In this way each ordinal α is assigned a unique infinite cardinal $\aleph_\alpha$, and each infinite cardinal is of the form $\aleph_\alpha$ for a unique ordinal α. (This relation between α and $\aleph_\alpha$ is essential in the formulation of Proposition 1.7 below.) If $\aleph_\alpha$ is to be regarded as an ordinal, we will write it as ω_α.

LEMMA 1.5. *Let α be an ordinal and let $\mathscr{S} = \{S_\beta: \beta < \alpha\}$ be a family of subsets of X which satisfy the following conditions*:

(1) *for each $\beta < \alpha$, S_β is closed-open for τ,*

(2) *for each $\gamma < \beta < \alpha$, $S_\beta \subset S_\gamma$,*

(3) *if β is a limit ordinal, then $S_\beta = \bigcap\{S_\gamma: \gamma < \beta\}$. Then $\bigcap \mathscr{S}$ is open in $\mu(\tau)$.*

Proof. Assume without loss of generality that α is a limit ordinal and that $S_0 = X$. For each $\beta < \alpha$ define $f_\beta = \mathscr{X}_{S_\beta} - \mathscr{X}_{S_{\beta+1}}$ where $\mathscr{X}_S$ denotes the characteristic function of the set S. Since S_β is closed-open for each $\beta < \alpha$, it follows that the set $A = \{0\} \cup \{f_\beta: \beta < \alpha\} \subset C^b(X)$. Furthermore, A is relatively weakly compact as will now be shown. If T is the linear transformation which maps each f in $C^b(X)$ to its unique continuous extension Tf in $C(\beta X)$, then T is a topological isomorphism for the weak topologies. Hence it is enough to show that $T[A]$ is relatively weakly compact, or, equivalently by the Eberlein-Ŝmulian theorem ([5], p. 313), that $T[A]$ is weakly sequentially compact. Hence let (Tf_{β_n}) be a sequence in $T[A]$; and, by passing to a subsequence if necessary, assume that $\beta_n \neq \beta_m$ for all $n, m \in N$. (If this were not possible, (Tf_{β_n}) would contain a constant subsequence which, of course, would be weakly convergent.) Since $\{f_\beta: \beta < \alpha\}$ is a set of disjoint idempotents (i.e., $f_\beta^2 = f_\beta$ and $f_\beta f_\gamma = 0$ for $\beta \neq \gamma$) in $C^b(X)$ and since T is a ring isomorphism, $T[A]$ is a set of disjoint idempotents in $C(\beta X)$. From this it is clear that (Tf_{β_n}) converges pointwise to zero. Since the sequence (Tf_{β_n}) is uniformly bounded, it follows that (Tf_{β_n}) converges weakly to zero. Thus $T[A]$ is weakly sequentially compact.

By Krein's theorem ([5], p. 325), the bipolar A^{00} of A is weakly compact so that A^0 is a Mackey neighborhood of zero. Let U be an open Mackey neighborhood of zero such that $U \subset 1/2A^0$, and define $W = \bigcup\{x + U: x \in \bigcap \mathscr{S}\}$. Then W is open in the Mackey topology and so $W \cap X$ is $\mu(\tau)$ open.

It will now be shown that $W \cap X = \bigcap \mathscr{S}$, and the proof will be complete. It is clear that $\bigcap \mathscr{S} \subset W \cap X$. Now assume that $y \in X - \bigcap \mathscr{S}$. Let β_0 be the least ordinal such that $y \notin S_{\beta_0}$. Then β_0 is not a limit ordinal since otherwise the assumption that $S_{\beta_0} = \bigcap\{S_\beta: \beta < \beta_0\}$ would imply that $y \notin S_\beta$ for some $\beta < \beta_0$. Hence $y \in S_{\beta_0 - 1} - S_{\beta_0}$ so that $f_{\beta_0 - 1}(y) = 1$. It then follows that $y \notin W$. Indeed, if $y \in W$, then

$y \in x + U$ for some $x \in \bigcap \mathscr{S}$. But then $y - x \in 1/2A^0$ so that for all $\beta < \alpha$, $|f_\beta(y)| = |f_\beta(y) - f_\beta(x)| \leqq 1/2$. This implies that $f_\beta(y) = 0$ for all $\beta < \alpha$ which contradicts the fact that $f_{\beta_0 - 1}(y) = 1$. The proof is complete.

THEOREM 1.6. *Let τ be a completely regular Hausdorff topology on X. Then $\tau = \mu(\tau)$ if and only if τ is discrete.*

Proof. Since $\tau \subset \mu(\tau)$ is always valid, it is clear that $\tau = \mu(\tau)$ if τ is discrete. Hence assume thar $\tau = \mu(\tau)$. It will be shown that every closed set for τ is also open for τ. Since τ is completely regular Hausdorff, every τ-closed subset of X is the intersection of the zero sets which contain it. If F is a closed set in τ, let $\iota(F)$ be the least cardinal for which there is a family of zero sets which has that cardinal number and which is such that the intersection over the family is F. The proof will be by induction on $\iota(F)$. If $\iota(F) = 1$, then F is a zero set; and hence open in $\mu(\tau)$ by Proposition 1.2. Since $\tau = \mu(\tau)$, F is open in τ. Now assume for some cardinal $\aleph$ that if F is a τ-closed set with $\iota(F) < \aleph$, then F is τ-open. Let F be a τ-closed set with $\iota(F) = \aleph$. Let α be the least ordinal whose cardinal is $\aleph$, and let $\{F_\beta : \beta < \alpha\}$ be a family of τ-closed zero sets such that $F = \bigcap \{F_\beta : \beta < \alpha\}$. Define $S_0 = X$ and for each ordinal β with $0 < \beta < \alpha$, define $S_\beta = \bigcap \{F_\gamma : \gamma < \beta\}$. Note that for each $\beta < \alpha$, S_β is τ-closed and $\iota(S_\beta) < \aleph$ so that S_β is closed-open for τ by the induction assumption. Also if β is a limit ordinal, then $S_\beta = \bigcap \{S_\gamma : \gamma < \beta\}$. Indeed, it is clear that $S_\beta \subset S_\gamma$ for $\gamma < \beta$ so that $S_\beta \subset \bigcap \{S_\gamma : \gamma < \beta\}$. Now let $x \in S_\gamma$ for all $r < \beta$. Now take $\gamma_0 < \beta$ arbitrarily. Then $\gamma_0 + 1 < \beta$ since β is a limit ordinal. Hence $x \in S_{\gamma_0+1} \subset F_{\gamma_0}$. Thus $x \in \bigcap \{F_{\gamma_0} : \gamma_0 < \beta\} = S_\beta$.

The family $\{S_\beta : \beta < \alpha\}$ thus satisfies the conditions of Lemma 1.5 so that $F = \bigcap \{F_\beta : \beta < \alpha\} = \bigcap \{S_\beta : \beta < \alpha\}$ is open in $\mu(\tau)$. But $\tau = \mu(\tau)$ so that F is open in τ and the proof is complete.

Define $\mu^0(\tau) = \tau$, and if α is not a limit ordinal, define $\mu^\alpha(\tau) = \mu(\mu^{\alpha-1}(\tau))$. If α is a limit ordinal, let $\mu^\alpha(\tau)$ be the topology generated by the base $\bigcup \{\mu^\beta(\tau) : \beta < \alpha\}$. Note that $\mu^\alpha(\tau) \subset \mu^\beta(\tau)$ whenever $\alpha \leqq \beta$. Since X is a set, there is clearly an ordinal α of cardinal less than $2^{2^\aleph}$ such that $\mu^\alpha(\tau) = \mu^{\alpha+1}(\tau)$. But then by Theorem 1.6, it follows that $\mu^\alpha(\tau)$ is discrete so that $\mu^\beta(\tau)$ is discrete for all $\beta \geqq \alpha$. Let $\alpha(\tau)$ denote the least ordinal with the property that $\mu^{\alpha(\tau)}(\tau)$ is discrete. Then the next theorem gives an upper bound for the ordinal $\alpha(\tau)$. First we prove the following. (See the discussion preceding Lemma 1.5 for the definition of $\aleph_\alpha$.)

PROPOSITION 1.7. *Let the set X and the topology τ be fixed, and let α be an ordinal. If F is a τ-closed subset of X and if there is* *a family* $\mathscr{F}$ *of* τ*-zero sets such that* $F = \bigcap\{Z: Z \in \mathscr{F}\}$ *and such that the cardinal of* $\mathscr{F}$ *is less than or equal to* $\aleph_\alpha$, *then* F *is open in* $\mu^{\alpha+1}(\tau)$.

Proof. The proof is by transfinite induction. The statement is true for $\alpha = 0$ by Proposition 1.2 and the fact that the intersection of a countable set of zero sets is a zero set. Now assume that the result is true for all $\alpha' < \alpha$. Let F be τ-closed and let $\mathscr{F}$ be a family of zero sets such that $F = \bigcap\{Z: Z \in \mathscr{F}\}$ and such that the cardinal number of $\mathscr{F}$ is $\aleph_\alpha$. Let $\mathscr{F} = \{Z_\beta: \beta < \omega_\alpha\}$ be an enumeration of $\mathscr{F}$. (Recall that $\omega_\alpha = \aleph_\alpha$. See the discussion preceding Lemma 1.5.) For each $\beta < \omega_\alpha$, define $S_\beta = \bigcap\{Z_\gamma: \gamma < \beta\}$. Then the cardinal of β is $\aleph_{\alpha'}$ for some $\alpha' < \alpha$. Hence by the induction assumption, S_β is closed-open in $\mu^{\alpha'+1}(\tau)$. Hence S_β is closed-open in $\mu^\alpha(\tau)$ for all $\beta < \omega_\alpha$. Furthermore, it can be shown in the same manner as in the proof of Theorem 1.6 that $S_\beta = \bigcap\{S_\gamma: \gamma < \beta\}$ for each limit ordinal $\beta < \omega_\alpha$. Hence by Lemma 1.5, it follows that $F = \bigcap\{S_\beta: \beta < \omega_\alpha\}$ is open in $\mu^{\alpha+1}(\tau)$. The proof is complete.

DEFINITION. Let (X, τ) be a completely regular Hausdorff space. For each point $x \in X$, the *index* $i(x)$ of the point x is the least cardinal number for which there is a family $\mathscr{F}$ of τ-zero sets with that cardinal such that $\{x\} = \bigcap\{Z: Z \in \mathscr{F}\}$. Let $\beta(\tau)$ be the least ordinal in the set $\{\beta: \forall x \in X, i(x) \leqq \aleph_\beta \text{ and } \aleph_\beta \leqq 2^{2^{\aleph}}\}$. Then the ordinal $\beta(\tau)$ is called the *local index* of τ.

It is clear that the local index is a topological invariant and that the local index of a first countable space is 0. The following theorem gives an upper bound for $\alpha(\tau)$ as promised above.

THEOREM 1.8. *Let* τ *be a completely regular topology on* X, *and let* $\beta(\tau)$ *be the local index of* τ. *Then* $\alpha(\tau) \leqq \beta(\tau) + 1$. (*That is,* $\mu^{\beta(\tau)+1}(\tau)$ *is discrete.*)

Proof. By Proposition 1.7, $\{x\}$ is open in $\mu^{\beta(\tau)+1}(\tau)$ for each $x \in X$.

In general, if $\beta(\tau)$ is the local index of τ then $\beta(\tau) + 1$ is strictly larger than $\alpha(\tau)$. Indeed, if (X, τ) is the Stone-Cêch compactification of the first uncountable ordinal with its order topology, then $\alpha(\tau) = \beta(\tau) = 1$. The author, however, would venture to conjecture that the bound $\beta(\tau) + 1$ is best possible in the sense that given an ordinal β, then there is a space (X, τ) with $\beta = \beta(\tau)$ and $\alpha(\tau) = \beta(\tau) + 1$. A possible candidate for this space is $[0, 1]^{\omega_\beta}$ with its product topology. It has local index β, although it is not clear what $\alpha(\tau)$ is for this space. (See Corollary 2.5, however.)

2. A space for which $\mu(\tau)$ is not discrete. It is natural to conjecture that $\mu(\tau)$ is always discrete. The goal of the present section is to show that this is not the case (Corollary 2.5). We begin by stating the following definition and theorem which may be found in [3, p. 269].

DEFINITION. Let X be a compact Hausdorff space. A set A of continuous real-valued functions on X is *quasi-equicontinuous* if for every convergent net (x_i), $i \in I$ on X with $x_i \to x$, for every positive number ε and for every $i_0 \in I$, there are $i_1, \cdots, i_n \in I$ with $i_0 \leqq i_k$ for $k = 1, 2, \cdots, n$ such that for every $f \in A$,

$$\min_{k=1,\cdots,n} |f(x_{i_k}) - f(x)| < \varepsilon .$$

THEOREM 2.1. *Let X be a compact Hausdorff space, and let A be a set of continuous real-valued functions on X. Then A is relatively weakly compact if and only if A is uniformly bounded and quasi-equicontinuous.*

In what follows, it will be necessary to have certain definitions and notations which we will now formulate. Let $\{X_i: i \in I\}$ be a family of topological spaces. For each set $J \subset I$, X_J will denote the product $\prod\{X_i: i \in J\}$ with the product topology. The projection from X_I onto X_J will be denoted by P_J.

DEFINITION. Let $X = \prod\{X_i: i \in I\}$. A function $f \in C(X)$ is *supported by the coordinates* $J \subset I$ if there is an $f' \in C(X_J)$ such that $f = f' \circ P_J$. A set $A \subset C(X)$ is supported by coordinates $J \subset I$ if f is supported by J whenever $f \in A$.

LEMMA 2.2. *Let $\{X_i: i \in I\}$ be a family of compact Hausdorff spaces, and let $X = \prod\{X_i: i \in I\}$. Assume that $A \subset C(X)$ is relatively weakly compact and contains no constant functions. For each $f \in A$, let f be supported by the coordinates $I(f) \subset I$. If $\mathscr{S} = \{I(f): f \in A\}$ is a disjoint family of sets* (i.e., *for $f_1, f_2 \in A$, either $I(f_1) = I(f_2)$ or $I(f_1) \cap I(f_2) = \varnothing$*), *then $\mathscr{S}$ is at most countable.*

Proof. Without loss of generality, assume that if $f_1, f_2 \in A$ and $f_1 \neq f_2$, then $I(f_1) \cap I(f_2) = \varnothing$. Also assume that A (and hence $\mathscr{S}$) is not countable. For each $f \in A$, there is $f' \in C(X_{I(f)})$ such that $f = f' \circ P_{I(f)}$. Since f is not constant, there are points $x_f^{(1)}, x_f^{(2)} \in X_{I(f)}$ such that $f'(x_f^{(1)}) \neq f'(x_f^{(2)})$. Since A is uncountable, there is a positive number ε and an uncountable subset $B \subset A$ of cardinal $\aleph_1$ such that $\varepsilon < |f'(x_f^{(1)}) - f'(x_f^{(2)})|$ for all $f \in B$.

Let $\{f_\alpha : \alpha < \omega_1\}$ be an enumeration of B. (Of course, ω_1 is the first uncountable ordinal.) Let $J = I - \bigcup\{I(f_\beta) : \beta < \omega_1\}$, and for each $i \in J$, let $x_i \in X_i$ be fixed. Now define a net (x_α), $\alpha \in \omega_1$ on X as follows:

$$x_\alpha(i) = \begin{cases} P_{\{i\}}(x_{f_\beta}^{(1)}) , & \text{if } \beta < \alpha \text{ and } i \in I(f_\beta) \\ P_{\{i\}}(x_{f_\beta}^{(2)}) , & \text{if } \alpha \leqq \beta \text{ and } i \in I(f_\beta) \\ x_i , & \text{if } i \in J . \end{cases}$$

Since $\{I(f_\beta) : \beta < \omega_1\}$ is a disjoint family, x_α is well-defined. It is clear that the net (x_α), $\alpha \in \omega_1$ converges in X (i.e., converges pointwise) to the point x where,

$$x(i) = \begin{cases} P_{\{i\}}(x_{f_\beta}^{(1)}) , & \text{if } i \in \bigcup\{I(f_\beta) : \beta < \omega_1\} \\ x_i , & \text{if } i \in J . \end{cases}$$

Since B is relatively weakly compact, it is quasi-equicontinuous by Theorem 2.1. Hence there are ordinals $\alpha_1, \alpha_2, \cdots, \alpha_n < \omega_1$ (take $i_0 = 0$ in the definition of quasi-equicontinuity) such that for all $f \in B$,

$$(*) \qquad \min_{k=1,\cdots,n} |f(x_{\alpha_k}) - f(x)| < \varepsilon .$$

Now let $\beta < \omega_1$ be such that $\alpha_k < \beta$ for $k = 1, \cdots, n$. Then,

$$f_\beta(x_{\alpha_k}) = f'_\beta \circ P_{I(f_\beta)}(x_{\alpha_k}) = f'_\beta(x_{f_\beta}^{(2)})$$

and

$$f_\beta(x) = f'_\beta \circ P_{I(f_\beta)}(x) = f'_\beta(x_{f_\beta}^{(1)}) .$$

Hence for all $k = 1, 2, \cdots, n$,

$$|f_\beta(x_{\alpha_k}) - f_\beta(x)| = |f'_\beta(x_{f_\beta}^{(2)}) - f'_\beta(x_{f_\beta}^{(1)})| > \varepsilon .$$

This contradicts (*) and the proof is complete.

DEFINITION. A family $\{X_i : i \in I\}$ of topological spaces has the *countable support property* if for each $f \in C(X)$ where $X = \prod\{X_i : i \in I\}$, there is a set $J \subset I$ which is at most countable such that f is supported by J.

The following theorem is a special case of some results of Engelking [4] which are given in a much more general context. The theorem below may be proved with an application of the Stone-Wierstrass theorem.

THEOREM 2.3. *Let $\{X_i: i \in I\}$ be a family of compact Hausdorff spaces. Then the family has the countable support property.*

The following theorem is of interest in its own right since it gives some information about the weakly compact subsets of $C^b(X)$ when X if a product of compact metric spaces.

THEOREM 2.4. *Let $\{X_i: i \in I\}$ be a family of compact metric spaces and let $X = \prod \{X_i: i \in I\}$. If A is a relatively weakly compact subset of $C(X)$, then there is a set $J \subset I$ of cardinal at most $\aleph_1$ which supports A.*

Proof. There is no loss in generality if we take $I = \omega$ (where ω is an ordinal) and if we assume that A contains no constant functions. Assume that A is not supported by any set of coordinates of cardinal at most $\aleph_1$. (In particular it follows that $\omega_1 < \omega$.) By Theorem 2.3 and the axiom of choice, there is for each $f \in A$ a non-empty set $I(f) \subset \omega$ such that $I(f)$ is at most countable and such that f is supported by $I(f)$.

Let $\mathscr{S} = \{I(f): f \in A\}$. (If $\mathscr{S}' \subset \mathscr{S}$, let $\bigcup \mathscr{S}' = \bigcup\{I(f): I(f) \in \mathscr{S}'\}$.). We will define by transfinite induction a set $\mathscr{S}_\alpha \subset \mathscr{S}$ for each ordinal $\alpha < \omega_1$ such that if $E_\alpha = \bigcup\{\bigcup \mathscr{S}_\beta: \beta < \alpha\}$, $\mathscr{E}_\alpha = \{I(f): f \in A$ and f is not supported on $E_\alpha\}$ and $\mathscr{S}_\alpha^* = \{I(f) - E_\alpha: I(f) \in \mathscr{S}_\alpha\}$, then the following conditions hold:

(1) $\mathscr{S}_\alpha \neq \varnothing$,

(2) $\mathscr{S}_\alpha \subset \mathscr{E}_\alpha$,

(3) E_α is at most countable,

(4) $\mathscr{S}_\alpha^*$ is a maximal disjoint subset of $\{I(f) - E_\alpha: I(f) \in \mathscr{E}_\alpha\}$,

(5) $\mathscr{S}_\alpha^*$ is at most countable.

To begin, let $\mathscr{S}_0$ be a maximal disjoint subset of $\mathscr{S}$. (Such sets exist by Zorn's lemma.) Then $E_0 = \varnothing$ $\mathscr{E}_0 = \mathscr{S}$ and $\mathscr{S}_0^* = \mathscr{S}_0$ so that Conditions (1)—(5) all hold. Of course, $\mathscr{S}_0$ is at most countable by Lemma 2.2. Now let $\alpha < \omega_1$ be fixed, and assume that $\{\mathscr{S}_\beta: \beta < \alpha\}$ is a family of subsets of $\mathscr{S}$ such that Conditions (1)—(5) are satisfied for each $\mathscr{S}_\beta$ with $\beta < \alpha$. Now define $E_\alpha = \bigcup\{\bigcup \mathscr{S}_\beta: \beta < \alpha\}$ and $\mathscr{E}_\alpha = \{I(f): f \in A$ and f is not supported by $E_\alpha\}$. Let $\mathscr{S}_\alpha'$ be a maximal disjoint subfamily of $\{I(f) - E_\alpha: I(f) \in \mathscr{E}_\alpha\}$. (Such sets exist by Zorn's lemma.) Finally define $\mathscr{S}_\alpha = \{I(f) \in \mathscr{E}_\alpha: I(f) - E_\alpha \in \mathscr{S}_\alpha'\}$. Having now defined $\mathscr{S}_\alpha$, we must now verify Conditions (1)—(5). First note that $\mathscr{S}_\alpha^* = \mathscr{S}_\alpha'$. Since $\bigcup \mathscr{S}_\beta \subset E_\beta \cup \bigcup \mathscr{S}_\beta^*$, it follows from Conditions (3) and (5) that $\bigcup \mathscr{S}_\beta$ is at most countable for all $\beta < \alpha$. Since $\alpha < \omega_1$, this means that E_α is at most countable so that Condition (3) holds. Since A is not supported on any set which is at most countable, it follows that $\mathscr{E}_\alpha$ is not empty. Hence $\mathscr{S}_\alpha \neq \varnothing$ so that Condition (1) holds. Condition (2) holds by the definition of $\mathscr{S}_\alpha$. It is clear that $\mathscr{S}_\alpha^* = \mathscr{S}_\alpha'$ so that Condition (4) holds. It is also the case that $\mathscr{S}_\alpha^*$ is at most countable (i.e., that Condition (5) holds). However, the proof of this fact is slightly involved. (We will assume it here and prove it later.) It now follows by transfinite induction that there is a family $\{\mathscr{S}_\alpha : \alpha < \omega_1\}$ of subsets of $\mathscr{S}$ satisfying Conditions (1)—(5) above.

Let $E = \bigcup\{E_\alpha : \alpha < \omega_1\}$. Then by (3), E has cardinal at most $\aleph_1$. By assumption A is not supported on E so that there is a function $g \in A$ which is not supported on E. In particular, g is not supported on E_α for $\alpha < \omega_1$ so that $I(g) \in \mathscr{E}_\alpha$ for all $\alpha < \omega_1$. By (4) it follows that for each $\alpha < \omega_1$, there is an $f_\alpha \in A$ such that $I(f_\alpha) \in \mathscr{S}_\alpha$ and

$$(*) \qquad I(g) \cap (I(f_\alpha) - E_\alpha) = (I(g) - E_\alpha) \cap (I(f_\alpha) - E_\alpha) \neq \emptyset .$$

Since $I(f_\alpha) \subset E_\beta$ whenever $\alpha < \beta$, it follows that

$$(**) \qquad (I(f_\alpha) - E_\alpha) \cap (I(f_\beta) - E_\beta) = \emptyset$$

for $\alpha \neq \beta$. From (*) and (**) it follows that $I(g)$ has cardinal at least $\aleph_1$ which contradicts the fact that $I(g)$ was chosen to be at most countable. As promised above, we now verify the following:

$\mathscr{S}_\alpha^*$ is at most countable. Let $Y_1 = X_{E_\alpha}$ and $Y_2 = X_{\omega - E_\alpha}$, and indentify X with $Y_1 \times Y_2$ in the obvious way. Since E_α is at most countable and since $X_{\{\alpha\}}$ is a compact metric space for each $\alpha < \omega$, the space Y_1 is a compact metric space. Hence Y_1 is separable. Let $\{y_n : n \in N\}$ be a countable dense subset of Y_1. For $n \in N$, define $A_n = \{f \in A : I(f) \in \mathscr{S}_\alpha$ and f_{y_n} is not constant$\}$, where for each $y \in Y_1$, f_y denotes the function on Y_2 defined by $f_y(x) = f(y, x)$ for $x \in Y_2$. Define $A_0 = \{f \in A : I(f) \in \mathscr{S}_\alpha\}$. Then $A_0 = \bigcup\{A_n : n \in N\}$. Indeed, it is clear by definition that $\bigcup\{A_n : n \in N\} \subset A_0$. On the other hand, let $f \in A - \bigcup\{A_n : n \in N\}$. Then f_{y_n} is constant for all $n \in N$. But since $\{y_n : n \in N\}$ is dense in Y_1 and since f is continuous, it follows that f_y is constant for all $y \in Y_1$. This means that f is supported by the coordinates E_α so that $I(f) \notin \mathscr{E}_\alpha$. Since $\mathscr{S}_\alpha \subset \mathscr{E}_\alpha$, $I(f) \notin \mathscr{S}_\alpha$ and so $f \notin A_0$.

Since $A_0 = \bigcup\{A_n : n \in N\}$, it follows that $\mathscr{S}_\alpha^* = \bigcup_{n=1}^\infty \{I(f) - E_\alpha : f \in A_n\}$. We will now show that $\{I(f) - E_\alpha : f \in A_n\}$ is at most countable for each $n \in \aleph$. Define the space $X_n = \{x \in X : \forall \beta \in E_\alpha, x(\beta) = y_n(\beta)\}$, and let $B_n = \{f|_{X_n} : f \in A_n\}$ where $f|_{X_n}$ denotes the restriction of f to X_n. Then B_n is a relatively weakly compact subset of $C(X_n)$ which contains no constants since f_{y_n} is not constant if $f \in A_n$. Let X_n be identified with X_I in the obvious way where $I = \omega - E_\alpha$; and let T be the norm preserving isomorphism from $C(X_n)$ onto $C(X_I)$

induced by this identification. Then T is weakly continuous so that $T[B_n]$ is a relatively weakly compact subset of $C(X_I)$ which contains no constants. For each $f \in A_n$, $T(f|_{X_n})$ is supported by the coordinates $I(f) - E_\alpha \subset I$. Indeed, this statement is equivalent to saying that if $x_1, x_2 \in X$ are such that $P_{E_\alpha}(x_1) = P_{E_\alpha}(x_2)$ and $x_1(\beta) = x_2(\beta)$ for all $\beta \in I(f) - E_\alpha$, then $f(x_1) = f(x_2)$. But this is true since $I(f)$ supports f. Since $\{I(f) - E_\alpha: f \in A_n\}$ is a disjoint family, it follows from Lemma 2.2 that it is at most countable. The proof is complete.

COROLLARY 2.5. *Let $X = [0, 1]^{\omega_2}$ and let τ be the product topology. Then $\mu(\tau)$ is not discrete.*

Proof. Let x_0 be any point in X. A base for the neighborhood system at x_0 for $\mu(\tau)$ is $\{(x_0 + A^0) \cap X: A \subset C(X)$ is weakly compact$\}$. If $A \subset C(X)$ is a weakly compact set, then by Theorem 2.4, there is a set $J \subset \omega_2$ of cardinal at most $\aleph_1$ on which A depends. Let $x_1 \in X$ be any point such that $x_0(\alpha) = x_1(\alpha)$ for all $\alpha \in J$. Then $x_1 \in (x_0 + A^0) \cap X$. Thus $\{x_0\}$ is not a neighborhood of x_0 so that $\mu(\tau)$ is not discrete. The proof is complete.

It would be interesting to know if there is a simpler example than the one given above of a space with $\mu(\tau)$ not discrete. Since $[0, 1]^{\omega_0}$ is metrizable, $\mu(\tau)$ for this space is discrete by Theorem 1.3. By Corollary 2.5 and Proposition 3.1 below, $[0, 1]^{\omega}$ is a space with $\mu(\tau)$ not discrete for $\omega_2 \leqq \omega$. The question of whether $[0, 1]^{\omega_1}$ is a space with $\mu(\tau)$ discrete or not remains open.

3. Topologies τ with $\mu(\tau)$ discrete.

A completely regular topological space (X, τ) has *property* (M) if $\mu(\tau)$ is discrete. It is clear that property (M) is a topological invariant. As we have seen, all first countable spaces have property (M). This does not exhaust the class of spaces with property (M), however, as can be seen by applying Lemma 1.5 to the Stone-Cêch compactification of the first uncountable ordinal with its order topology. It would be of some interest to characterize topologically those spaces with property (M). In this section we will prove a few theorems about such spaces.

PROPOSITION 3.1. *If a space has property (M), then every subspace also has property (M).*

Proof. Let (X, τ) have property (M) and let $Y \subset X$ have the relative topology. Let $T: C^b(X) \rightarrow C^b(Y)$ be the restriction mapping. Then T is norm continuous and hence weakly continuous. In particular, if $A \subset C^b(X)$ is weakly compact, convex and balanced, then so is $T[A]$.

Let $y \in Y$ be fixed. Since (X, τ) has property (M), there is a weakly compact, convex and balanced subset $A \subset C^b(X)$ such that $\{y\} = (y + A^0) \cap X$. But then $\{y\} = (y + T[A]^0) \cap Y$ as is easily checked so that $\{y\}$ is open in $\mu(\sigma)$ where σ is the relative topology on Y. The proof is complete.

PROPOSITION 3.2. *Let τ_1 and τ_2 be two completely regular Hausdorff topologies on X, and assume that $\tau_1 \subset \tau_2$. If (X, τ_1) has property (M), then (X, τ_2) has property (M).*

Proof. By hypothesis $C^b(X, \tau_1) \subset C^b(X, \tau_2)$. The identity map T on $C^b(X, \tau_1)$ is norm continuous and hence weakly continuous from $C^b(X, \tau_1)$ into $C^b(X, \tau_2)$. The remainder of the proof is the same as the proof of Proposition 3.1.

THEOREM 3.3. *Let (X_1, τ_1) and (X_2, τ_2) have property (M). If (X, τ) is their Cartesian product, then (X, τ) has property (M).*

Proof. Let $(x_1, x_2) \in X$ be fixed. Since (X_i, τ_i) has property (M) for $i = 1, 2$, there is a weak compact convex balanced set $A_i \subset C^b(X_i)$ such that $\{x_i\} = (x_i + A_i^0) \cap X_i$. Let $P_i: X \rightarrow X_i$ be the projection on the ith coordinate, and define $T_i: C^b(X_i) \rightarrow C^b(X)$ by $T(f) = f \circ P_i$. Then T is norm continuous and so weakly continuous. Set $A = T_1[A_1] \cup T_2[A_2]$. Then A is weakly compact and so its bipolar A^{00} is weakly compact by Krein's theorem. Hence A^0 is a Mackey neighborhood of zero. It will now be shown that $\{(x_1, x_2)\} = ((x_1, x_2) + A^0) \cap X$ which will complete the proof. Let $(y_1, y_2) \in ((x_1, x_2) + A^0) \cap X$. Then $y_i \in (x_i + A_i^0) \cap X_i$ for $i = 1, 2$ so that $y_i = x_i$ for $i = 1, 2$.

COROLLARY. *The product of a finite number of spaces each of which has property (M) also has property (M).*

APPENDIX. We include here a simpler proof of Theorem 1.6 in the special case that τ is compact Hausdorff. The idea was suggested to the author by Professor James Crenshaw.

THEOREM 3′. *Let τ be compact Hausdorff. Then $\mu(\tau) = \tau$ if and only if X is finite.*

Proof. If $f \in C(X)$, then the range of f is finite. Indeed if f has an infinite range, let a be a limit point of the range. Let $U_0 = \{x: f(x) = a\}$, $U_1 = X$ and $U_n = \{x: |f(x) - a| \leqq 1/n\}$ for $n = 2, 3, \cdots$. Then by Proposition 1.1 U_n is closed-open in $\mu(\tau)$ for $n = 0, 1, 2, \cdots$.

Since $\tau = \mu(\tau)$, $\{U_0\} \cup \{U_n - U_{n+1}: n = 1, 2, \cdots\}$ is an τ-open cover of X which has no finite subcover. Hence every function in $C(X)$ has a finite range. But, as is easily shown for any completely regular Hausdorff space, if every function in $C(X)$ has a finite range, then X is a finite set. The proof is complete.

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