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CENTRAL 2-SYLOW INTERSECTIONS

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Let G be a finite group. A subgroup D of G is called a 2-*Sylow intersection* if there exist distinct Sylow 2-subgroups S_1 and S_2 of G such that $D = S_1 \cap S_2$. An involution of G is called *central* if it is contained in a center of a Sylow 2-subgroup of G. A 2-Sylow intersection is called *central* if it contains a central involution. The aim of this work is to determine all non-abelian simple groups G which satisfy the following condition
B: the 2-rank of all central 2-Sylow intersections is not higher than 1, under the additional assumption that the centralizer of a central involution of G is solvable.

In 1964, M. Suzuki [5] determined all simple groups with all 2-Sylow intersections being trivial (i.e. of rank 0). Using a recent fusion theorem by E. Shult [3, p. 62] the author proved [4] that no additional simple groups are involved if Suzuki's condition is weakened to read: all central 2-Sylow intersections are trivial (i.e. no central involution is contained in a 2-Sylow intersection).

This paper is a step toward the characterization of all simple groups G which satisfy Condition B (in short $G \in B$). We will prove the following

THEOREM. *Let G be a non-abelian simple group. Suppose that $G \in B$ and the centralizer of a central involution z in G is solvable. Then G is isomorphic to one of the following groups*:

(i) $\mathrm{PSL}(2, q)$, $q = 2^n > 2$;
(ii) $\mathrm{Sz}(q)$, $q = 2^n \geqq 8$;
(iii) $\mathrm{PSU}(3, q)$, $q = 2^n > 2$ *and*
(iv) $\mathrm{PSL}(2, q)$, $q \equiv 3$ or $5 \pmod 8$, $q > 5$.

A finite group G is of 2-rank n if an elementary abelian 2-subgroup of G of maximal order contains 2^n elements. The 2-length of G is denoted by $1_2(G)$. The maximal power of 2 dividing $|G|$ is denoted by $|G|_2$. An involution z of G is called isolated if it belongs to a Sylow 2-subgroup S of G and $z^g \in S$ implies $z^g = z$. The maximal normal subgroup of G of odd order is denoted by $0(G)$. Finally the groups Q_8, S_3 and S_4 are the ordinary quarternion group, the symmetric group on 3 letters and the symmetric group on 4 letters, respectively.

2. Properties of groups satisfying Condition B.

LEMMA 1. *Let* $G \in B, H \subseteqq G$.

(i) *If* $|H|_2 = |G|_2$ *then* $H \in B$.
(ii) *If* $H \triangleleft G$ *and* $|G/H|_2 = |G|_2$ *then* $G/H \in B$.

Proof. (i) is obvious. If H is a normal subgroup of G of odd order, then the S_2-subgroups of $\bar{G} = G/H$ are of the form $SH/H = \bar{S} \cong S$, where S is an S_2-subgroup of G. Let S_1 and S_2 be S_2-subgroups of G such that $\bar{S}_1 \cap \bar{S}_2$ is a central 2-Sylow intersection of 2-rank at least 2. Since H is of odd order, there exists a 2-subgroup D of G, such that $\bar{S}_1 \cap \bar{S}_2 = DH/H = \bar{D} \cong D$. It is clear that there exist $h_1, h_2 \in H$ such that $D \subseteqq S_1^{h_1} \cap S_2^{h_2}$. If zH is a central involution of $SH/H, z \in S$, then $[z, s] \in S \cap H = 1$ for all $s \in S$, hence $z \in Z(S)$. Thus D contains a central involution of G and as $G \in B$ and the 2-rank of D is at least 2, it follows that $S_1^{h_1} = S_2^{h_2}, \bar{S}_1 = \bar{S}_2$ and $\bar{D}$ is not a 2-Sylow intersection of G. Thus $\bar{G} \in B$.

LEMMA 2. *Let* $G \in B, H \subseteqq G$ *and suppose that the following assumptions hold*:
(i) H *is solvable;*
(ii) $|H|_2 = |G|_2$ *and*
(iii) $0_2(H)$ *contains a central involution of* G.
Then $1_2(H) = 1$, *unless* $0_2(\bar{H}) \cong Q_8$ *and* $\bar{H}/0_2(\bar{H}) \cong S_3$, *where* $\bar{H} = H/0(H)$.

Proof. By Lemma 1 H and $\bar{H}$ satisfy Condition B and $0_2(\bar{H})$ obviously contains a central involution of $\bar{H}$. If $0_2(\bar{H})$ is cyclic or generalized quaternion (but not ordinary quaternion), then $\mathrm{Aut}(0_2(\bar{H}))$ is a 2-group and therefore $\bar{H}/C(0_2(\bar{H}))$ is a 2-group. As $\bar{H}$ is solvable, $C(0_2(\bar{H})) \subseteqq 0_2(\bar{H})$ and consequently $\bar{H}$ is a 2-group, hence $1_2(H) = 1$.

If $0_2(\bar{H})$ is of 2-rank at least 2, then $\bar{H} \in B$ forces $\bar{H}$ to be 2-closed, hence $1_2(H) = 1$.

Suppose, finally, that $0_2(\bar{H}) \cong Q_8$. Then $\bar{H}/C(0_2(\bar{H}))$ is isomorphic to a subgroup of S_4 and if $\bar{H}$ is not 2-closed then obviously 24 divides the order of $\bar{H}/C(0_2(\bar{H}))$. Thus $\bar{H}/C(0_2(\bar{H})) \cong S_4$ and $\bar{H}/0_2(\bar{H}) \cong S_3$.

LEMMA 3. *Let* $G \in B$ *and suppose that* S *and* S_1 *are* S_2*-subgroups of* G. *Let* $z \in Z(S)$ *be an involution,* $g \in G$, *and suppose that* $z^g \in S_1$. *Then* $z^g \in Z(S_1)$.

Proof. Suppose that z^g is not central in S_1. Then $S_1 \cap C_G(z^g)$ contains z^g and a central involution of S_1. Let T be an S_2-subgroup of $C_G(z^g)$ containing $S_1 \cap C_G(z^g)$; as $C_G(z^g) \supseteqq S^g$, T is an S_2-subgroup of G. Since the 2-rank of $D = S_1 \cap T$ is at least 2 and D contains a central involution of G, it follows from our assumptions that $S_1 = T$, hence $z^g \in Z(S_1)$, a contradiction.

LEMMA 4. *Let* $G \in B$ *and suppose that* $|\Omega_1(Z(S))| = 2$, *where* S *is an* S_2*-subgroup of* G. *Then* $\Omega_1(Z(S)) \subseteqq Z^*(G)$, *where* $Z^*(G)/0(G) = Z(G/0(G))$.

Proof. Let $z \in \Omega_1(Z(S))$; then by Lemma 3 z is an isolated involution in G. It follows then by the Z^*-theorem of Glauberman [2] that $\Omega_1(Z(S)) \subseteqq Z^*(G)$.

LEMMA 5. *Let* $G \in B$, S *be an* S_2*-subgroup of* G *and* $G = 0(G)S$. *Suppose that* $|\Omega_1(Z(S))| > 2$ *and* S *is not normal in* G. *Then the* 2*-rank of* G *is at most* 2.

Proof. Let G be a counterexample of minimal order. Then S contains an elementary abelian subgroup A of order 8 such that $|A: Z(S) \cap A| \leqq 2$. Let $H = 0(G)$ and $C = C_S(H)$. Then $C \lhd S$ and consequently $C \lhd SH = G$. As G is not 2-closed and $G \in B$, we have $A \not\subset C$. Consider AH; A is not normal in AH and $|A \cap A^h| \leqq 2$ for all $h \in H - N(A)$, as otherwise $G \notin B$. Thus AH is a counterexample and by the minimality of G, $G = AH$.

Let P be a Sylow p-subgroup of H, such that $A \subseteqq N(P)$ and $A \not\subset C(P)$; then again by the minimality of G, $G = AP$. As by a theorem of Burnside A does not centralize $P/\Phi(P)$, it follows by Lemma 1 (ii) and the minimality G that $\Phi(P) = 1$, P is elementary abelian. Since A acts on P in a completely reducible way, it follows again by the minimality of G that A acts irreducibly of P and $A/C_A(P)$ acts faithfully and irreducibly on P. Thus $A/C_A(P)$ is a cyclic group and $C_A(P)$ is a normal subgroup of G of 2-rank 2. As $C_A(P)$ contains a central involution and $G \in B$, it follows that G is 2-closed, a contradiction.

3. Proof of the theorem.

Let $H = C_G(z)$. If H is 2-closed then by Lemma 3 z belongs to a unique Sylow 2-subgroup of G. Therefore by Theorem C of [4] G is isomorphic to one of the groups in (i)-(iii).

Suppose now that H is not 2-closed. Let $\bar{H} = H/0(H)$ and suppose that $0_2(\bar{H}) \cong Q_8$ and $\bar{H}/0_2(\bar{H}) \cong S_3$. Then obviously

$$(*) \qquad \text{2-rank } H = \text{2-rank } G = 2.$$

Otherwise it follows by Lemma 2 that $l_2(H) = 1$, hence $0_{2',2}(H) = SL$, where $L = 0_{2'}(H)$ and S is an S_2-subgroup of G. Since H is not 2-closed, S is not normal in $0_{2',2}(H)$. As G is simple, it follows by Lemma 4 that $|\Omega_1(Z(S))| > 2$ and Lemma 5 then yields (*) again.

Thus in all cases 2-rank $G = 2$ and by the classification theorem of Alperin, Brauer and Gorenstein [1] only three types of 2-groups could occur as a Sylow subgroup S of a group not mentioned in (i)-(iv):

(a) dihedral of order 8 at least,

(b) quasi-dihedral, or

(c) wreathed.

In all of these cases $Z(S)$ is cyclic, hence by Lemma 4 G is non-simple, a contradiction. The proof of the theorem is complete.

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