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SOME RESULTS ON LACUNARY WALSH SERIES

JOHN COURY

It is known that if a lacunary trigonometric series converges to 0 on a set of positive measure, then the series vanishes identically. In the present paper, the following analogue for the Walsh system is proved: a lacunary Walsh series which converges to 0 almost everywhere is identically zero.

In particular, let $S(x) = \sum_1^\infty c_k w_{n_k}(x)$ be a lacunary Walsh series, with $n_{k+1}/n_k \geqq q > 1$ for each k. We prove that if $S(x)$ converges to 0 on a set of positive measure, or on a set of the second category having the property of Baire, then the series is a finite sum. If $S(x)$ converges to 0 on a set of sufficiently large measure (the measure depending only on the degree of lacunarity q), then $S(x)$ is identically zero. Hence we prove that the only lacunary Walsh series converging to 0 almost everywhere is the identically zero series. Finally, sufficient conditions are given for a set to be a set of uniqueness for lacunary Walsh series.

1. Preliminaries.

If $x \in [0, 1[$ has the dyadic development $\sum_{k=1}^\infty x_k 2^{-k}$, where x_k is 0 or 1, then the $(k-1)$st Rademacher function r_{k-1} evaluated at x has the value $(-1)^{x_k}$. (For dyadic rationals in $[0, 1[$, which have two such expansions, we agree to take the finite development.) If we write a positive integer n as $2^{n_1} + 2^{n_2} + \cdots + 2^{n_v}$, where $n_1 > n_2 > \cdots > n_v \geqq 0$, then the nth Walsh function is given by $w_n = r_{n_1} \cdot r_{n_2} \cdot \cdots \cdot r_{n_v}$ (following Paley's modification). Define $w_0(x) \equiv 1$; then the functions $\{w_n\}_{n=0}^\infty$ form a complete orthonormal set on $[0, 1[$. For $n \geqq 1$, w_n is plainly discontinuous.

Fine [2] has shown that the Walsh functions may be identified with the full character group $\{w_n^*\}$ of 2^ω, where 2^ω denotes the countable product of the two-element group $\{0, 1\}$. For $\boldsymbol{x} = (x_1, x_2, \cdots)$ in 2^ω, define $w_n^*(\boldsymbol{x}) = r_{n_1}^*(\boldsymbol{x}) \cdot \cdots \cdot r_{n_v}^*(\boldsymbol{x})$, where $n = 2^{n_1} + \cdots 2^{n_v}$ and $r_{k-1}^*(\boldsymbol{x}) = (-1)^{x_k}$. To simplify notation, we shall suppose henceforth that $r_{k-1}^*(\boldsymbol{x}) = (-1)^{x_{k-1}}$, where $k \geqq 2$.

Let $\phi(\boldsymbol{x}) = \sum_{k=1}^\infty x_k 2^{-k}$, where $\boldsymbol{x} = (x_1, x_2, \cdots) \in 2^\omega$. Then $\phi(\boldsymbol{x})$ is a continuous measure-preserving map of 2^ω onto $[0, 1[$ but is not injective: a dyadic rational in $[0, 1[$ is the image of two points in 2^ω. If $\mathscr{F}$ denotes the set of sequences in 2^ω that are eventually 1, then

$\phi(x)$ is one-to-one on the complement of $\mathscr{F}$ and $\phi(x)$ maps $\mathscr{F}$ onto the dyadic rationals of $[0, 1[$. This 'exceptional' set $\mathscr{F}$ corresponds to the infinite expansion of dyadic rationals; $\mathscr{F}$ is plainly countable and hence has Haar measure 0.

With a Walsh series $\sum c_n w_n(x)$ on $[0, 1[$, we may associate the corresponding 'Walsh' series $\sum c_n w_n^*(\boldsymbol{x})$ on 2^ω, where $\boldsymbol{x}$ is such that $\phi(\boldsymbol{x}) = x$. Clearly, if we neglect the points of $\mathscr{F}$, the former series converges to a value c when and only when the latter series also converges to c.

Thus, since $\mathscr{F}$ is of the first category and has Haar measure zero in 2^ω, it suffices to prove the theorems stated in the introduction on the group 2^ω, using lacunary series of the form $S^*(\boldsymbol{x}) = \sum c_k w_{n_k}^*(\boldsymbol{x})$. The advantage is that the functions w_n^* are continuous with respect to the usual (product) topology of 2^ω; however, all of the results of §§ 2 and 3 are valid for lacunary Walsh series defined on the interval $[0, 1[$. Henceforth we write $S(\boldsymbol{x})$ for $S^*(\boldsymbol{x})$ and w_n for w_n^*.

2. **The main results.** A Walsh series $S(\boldsymbol{x}) = \sum_{k=1}^\infty c_k w_{n_k}(\boldsymbol{x})$ is called *q-lacunary* if $n_{k+1}/n_k \geqq q > 1$ for all k, where q is the supremum of all such numbers.

LEMMA. *Let $S(\boldsymbol{x})$ be a q-lacunary Walsh series, with $q \geqq 2$. Suppose that $S(\boldsymbol{x})$ converges to* 0 *(or is constant) on a set E of Haar measure exceeding* 1/2. *Then $S(\boldsymbol{x})$ vanishes identically.*

Proof. Since $q \geqq 2$, each n_k contains a power of 2 greater than each power of 2 in n_{k-1}. Thus the series $S(\boldsymbol{x})$ is simply a series $\sum c_k X_k$, where the X_k are independent random variables of the Bernoulli type. This latter series may in turn be viewed as a Rademacher series $R(\boldsymbol{x}) = \sum_{k=1}^\infty c_k r_k(\boldsymbol{x})$. Thus, as far as measure is concerned, the series $R(\boldsymbol{x})$ and $S(\boldsymbol{x})$ have the same properties, and so it suffices to prove the lemma for Rademacher series.

For an arbitrary positive integer N, let $\boldsymbol{u}_N$ denote the point in 2^ω with entry 1 in the Nth coordinate and 0's elsewhere. Since adding $\boldsymbol{u}_N$ to a point $\boldsymbol{x} \in 2^\omega$ affects only the Nth coordinate of $\boldsymbol{x}$, it follows that $r_k(\boldsymbol{x} + \boldsymbol{u}_N) = r_k(\boldsymbol{x})$ for every $k \neq N$, and $r_N(\boldsymbol{x} + \boldsymbol{u}_N) = -r_N(\boldsymbol{x})$.

Let μ denote (normalized) Haar measure on 2^ω. Since $\mu(E) > 1/2$, the set $E \cap (E + \boldsymbol{u}_N)$ has positive measure for every N, where $E + \boldsymbol{u}_N = \{\boldsymbol{x} + \boldsymbol{u}_N : \boldsymbol{x} \in E\}$. Thus there exists $z \in E$ such that $z + \boldsymbol{u}_N \in E$. If R_n denotes the nth partial sum of R, we have

$$R_n(z + \boldsymbol{u}_N) - R_n(z) = 0 \qquad \text{for} \quad n < N,$$

and

$$R_n(z + \boldsymbol{u}_N) - R_n(z) = -2c_N r_N(z) \qquad \text{for} \quad n \geqq N.$$

Passing to the limit as $n \to \infty$, we obtain $-2c_N r_N(z) = 0$, whence

$c_N = 0$. Since N was arbitrary, we conclude that $R(\boldsymbol{x})$ vanishes identically.

It is well-known (see, for example, [1, vol. II, p. 233]) that for a given $q > 1$, any lacunary sequence of integers can be written as the finite union of pairwise disjoint lacunary subsequences each of whose degree of lacunarity is at least q. The least number of q-lacunary subseries, with $q \geqq 2$, into which a given lacunary series can be partitioned will be called the *index of* 2-*lacunarity* of the series. We now generalize the result of the lemma to arbitrary lacunary Walsh series.

THEOREM 1. *Let $S(\boldsymbol{x})$ be a lacunary Walsh series, and let M denote its index of* 2-*lacunarity. Suppose that $S(\boldsymbol{x})$ converges to* 0 (*or is constant*) *on a set E with $\mu(E) > 1 - 1/2^M$. Then $S(\boldsymbol{x})$ is identically zero.*

Proof. By induction on M. For $M = 1$, the result follows from the lemma. Thus assume the theorem is true for an index $M - 1$; write $S(\boldsymbol{x}) = \sum_{i=1}^{M} S_i(\boldsymbol{x})$, where $S_i(\boldsymbol{x}) = \sum_{k=1}^{\infty} c_{i,k} w_{n_{i,k}}(\boldsymbol{x})$ and for each fixed i the sequence $(n_{i,k}\}$ is q-lacunary with $q \geqq 2$.

For a positive integer N, write $N = 2^{N_1} + \cdots + 2^{N_v}$, with $N_1 > N_2 > \cdots > N_v \geqq 0$. Then for $\boldsymbol{x} \in 2^\omega$, we shall say that the coordinates of $\boldsymbol{x}$ which "correspond to N" are the N_ith entries, $i = 1, 2, \cdots, v$.

We may suppose that $n_{M,1} < n_{1,1}$. Since $n_{1,1}$ has a power of 2 that $n_{M,1}$ does not, choose z_1 for $n_{1,1}$ so that in those coordinates of z_1 corresponding to $n_{M,1}$ there are an odd number of 1's, and in the coordinates corresponding to every $n_{1,k}$, $k \geqq 1$, there are an even number of 1's. For arbitrary $\boldsymbol{x} \in 2^\omega$, we have

$$\begin{aligned} S(\boldsymbol{x} + z_1) - S(\boldsymbol{x}) &= \sum_{k=1}^{\infty} c_{1,k}[w_{n_{1,k}}(z_1) - 1] w_{n_{1,k}}(\boldsymbol{x}) \\ &\quad + \sum_{i=2}^{M} \sum_{k=1}^{\infty} c_{i,k}[w_{n_{i,k}}(\boldsymbol{x} + z) - w_{n_{i,k}}(\boldsymbol{x})] \\ &= \sum_{i=2}^{M} \sum_{k=1}^{\infty} c_{i,k}[w_{n_{i,k}}(z_1) - 1] w_{n_{i,k}}(\boldsymbol{x}) , \end{aligned}$$

since $w_{n_{1,k}}(z_1) = 1$ for every $k \geqq 1$. The left-side of the equation is 0 on the set $E \cap (E + z_1)$; since $\mu(E \cap (E + z_1)) > 1 - 1/2^{M-1}$, it follows from the induction hypothesis that $c_{i,k}[w_{n_{i,k}}(z_1) - 1] = 0$ for each $i \geqq 2$ and all k. In particular, taking $i = M$ and $k = 1$, we have $-2c_{M,1} = 0$, since $w_{n_{M,1}}(z_1) = -1$. Thus $c_{M,1} = 0$. Similarly, we prove that $c_{1,1} = c_{2,1} = \cdots = c_{M-1,1} = 0$.

Now suppose that we have shown $c_{i,1} = c_{i,2} = \cdots = c_{i,p-1} = 0$ for $i = 1, 2, \cdots, M$, and consider $c_{M,p}$. Let $n_{i,p}$ $(i \neq M)$ be such that

$n_{M,p} < n_{i,p}$. Choose z_p for $n_{i,p}$ so that in the coordinates of z_p corresponding to $n_{M,p}$ there are an odd number of 1's, and an even number of 1's in the coordinates which correspond to each $n_{i,j} (j \geqq p)$. Then, as above, we conclude that $c_{M,p} = 0$. Thus $c_{i,k} = 0$ for each $i = 1, 2, \cdots, M$ and every $k \geqq 1$, and hence the given series $S(\boldsymbol{x})$ vanishes identically.

COROLLARY. *A lacunary Walsh series which converges to* 0 (*or is constant*) *almost everywhere vanishes identically.*

This latter result is the Walsh analogue of the following important theorem for trigonometric series: a lacunary trigonometric series converging to 0 on a set of positive measure is the identically zero series ([8, vol. I, p. 206]). The statement in the corollary is the best possible in that convergence a.e. cannot be replaced by convergence on a set of measure less than 1. This follows at once from the fact that in $[0, 1[$(or 2^ω), the Walsh-Fourier series of the characteristic function of a dyadic interval (basic open set) has only finitely many nonzero coefficients ([7, p. 288]).

The trigonometric result does, however, have the following counterpart in the Walsh system.

THEOREM 2. *A lacunary Walsh series which converges to* 0 (*or is constant*) *on a set of positive measure has only finitely many nonzero terms.*

Proof. The function $\pi: \boldsymbol{x} \to \mu(E \cap (E + \boldsymbol{x}))$ is continuous ([3, (20. 17)]). Hence if $\boldsymbol{x}$ is 'close' to 0 (that is, if $\boldsymbol{x}$ has sufficiently many 0's initially), then $\mu(E \cap (E + \boldsymbol{x})) > 0$. Let N be the smallest integer for which $E \cap (E + \boldsymbol{x})$ has positive measure whenever $\boldsymbol{x}$ has its first N coordinates equal to 0.

Let M be the index of 2-lacunarity of the series $S(\boldsymbol{x})$; we prove the theorem by induction on M. If $M = 1$, let R be the largest power of 2 appearing in the base 2 development of $n_1, n_2, \cdots, n_N$. Define $\boldsymbol{s}_{N+1}$ as follows: let $\boldsymbol{s}_{N+1}$ have 0's in every coordinate up to and including the Rth coordinate; an odd number of 1's in the coordinates which correspond to n_{N+1}; an even number of 1's in the coordinates corresponding to each $n_j, j > N + 1$; and 0's in the coordinates not otherwise determined. Then we have

$$S(\boldsymbol{x} + \boldsymbol{s}_{N+1}) - S(\boldsymbol{x}) = -2c_{N+1}w_{n_{N+1}}(\boldsymbol{x})$$

for every $\boldsymbol{x} \in 2^\omega$. Since each of $n_1, \cdots, n_N$ has a power of 2 greater than each power of 2 in its predecessor, it follows that $R \geqq N$, hence $\boldsymbol{s}_{N+1}$ has at least N zeros initially. Thus $E \cap (E + \boldsymbol{s}_{N+1})$ is nonempty. As in the proof of the lemma, it can then be shown that $c_{N+1} = 0$.

Proceeding by induction, we conclude that $c_n = 0$ for every $n > N$.

We retain the notation in the proof of Theorem 1 and suppose the theorem is true for an index $M-1$; that is, if the given series is the sum of $M-1$ q-lacunary series ($q \geqq 2$) and converges to 0 on a set of positive measure, then $c_{i,j} = 0$ for each i and all $j > N$, where N is as in the first paragraph. Consider now the series $S(\boldsymbol{x}) = S_1(\boldsymbol{x}) + \cdots + S_M(\boldsymbol{x})$. Let R be the largest power of 2 appearing in the base 2 expansion of $n_{M,1}, n_{M,2}, \cdots, n_{M,N}$. Choose $n_{i,J} (i \neq M)$ so that $n_{i,J} > n_{M,N+1}$ and $n_{i,J}$ contains a power of 2 greater than each power of 2 in $n_{M,N+1}$ (this is always possible unless each subsequence $\{n_{i,k}\}$ is finite, in which case we are done). Define $\boldsymbol{s}_J$ for $n_{i,J}$ as follows: let $\boldsymbol{s}_J$ have 0's in every coordinate up to and including the Rth; let $\boldsymbol{s}_J$ have an odd number of 1's in the coordinates corresponding to $n_{M,N+1}$; let $\boldsymbol{s}_J$ have an even numer of 1's in the coordinates which correspond to each $n_{i,j}$, for every $j \geqq J$. Assign 0 to those coordinates not already determined. Then for each $\boldsymbol{x} \in 2^{\omega}$,

$$S(\boldsymbol{x} + \boldsymbol{s}_J) - S(\boldsymbol{x}) = \sum_{k=1}^{J-1} c_{i,k}[w_{n_{i,k}}(\boldsymbol{s}_J) - 1]w_{n_{i,k}}(\boldsymbol{x}) + \sum_{\substack{r=1 \\ r \neq i}}^{M} [S_r(\boldsymbol{x} + \boldsymbol{s}_J) - S_r(\boldsymbol{x})] ,$$

since $w_{n_{i,k}}(\boldsymbol{s}_J) = 1$ for $k \geqq J$. Because $\boldsymbol{s}_J$ has at least N zeros initially, $E \cap (E + \boldsymbol{s}_J)$ has positive measure and so the left-side of this equation vanishes for $\boldsymbol{x}$ in a set of positive measure. Also, since the first sum on the right-side is finite, it is necessarily constant on sets of positive measure. Thus,

$$\sum_{\substack{r=1 \\ r \neq i}}^{M} [S_r(\boldsymbol{x} + \boldsymbol{s}_J) - S_r(\boldsymbol{x})] = \sum_{\substack{r=1 \\ r \neq i}}^{M} \sum_{m=1}^{\infty} c_{r,m}[w_{n_{r,m}}(\boldsymbol{s}_J) - 1]w_{n_{r,m}}(\boldsymbol{x})$$

is constant in a set of positive measure and so by the induction hypothesis, $c_{r,m}[w_{n_{r,m}}(\boldsymbol{s}_J) - 1] = 0$ for $r = 1, 2, \cdots, M$, $r \neq i$, and all $m > N$. In particular, for $r = M$ and $m = N+1$, we have $w_{n_{M,N+1}}(\boldsymbol{s}_J) = -1$ and therefore $c_{M,N+1} = 0$. Proceeding by induction, we show that $c_{M,j} = 0$ for every $j > N$. Thus, $S_M(\boldsymbol{x})$ is a finite sum.

Write $S_1(\boldsymbol{x}) + \cdots + S_{M-1}(\boldsymbol{x}) = -S_M(\boldsymbol{x})$; since $S_M(\boldsymbol{x})$ is finite, it assumes constant values on sets of positive measure. The induction hypothesis then implies that each of $S_1(\boldsymbol{x}), \cdots, S_{M-1}(\boldsymbol{x})$ is also a finite sum, and so therefore is $S(\boldsymbol{x})$.

The previous proof uses the hypothesis that E has positive measure only to ensure that the set $E \cap (E + \boldsymbol{x})$ is nonempty for sufficiently 'small' $\boldsymbol{x}$. However, even for E of measure zero, this will still be true if E is of the second category and has the property of Baire (see [5, p. 21]). (A set is said to have the property of Baire if it can be expressed as the symmetric difference of an open set and a set of the first category.) If a set has the property of Baire, so does its complement ([5, p. 19]); it follows that any residual set (i.e., the complement of a first category set) has the property of Baire. With a slight modification to the proof of Theorem 2, we have the following result.

THEOREM 3. *Suppose that E is of the second category and has the property of Baire. Then a lacunary Walsh series which converges to* 0 (*or is constant*) *on E is necessarily a finite sum. In particular, if E is residual, then the series is identically constant.*

Proof. Let s_J be defined as in the previous proof. Then $E \cap (E + s_J)$ is of the second category ([5, p. 21]); since the class of all sets having the property of Baire is a σ-algebra, $E \cap (E + s_J)$ also has the property of Baire. The first sum on the right-side of the equation in the previous proof is constant on dyadic intervals; since $E \cap (E + s_J)$ must meet some one of these intervals in a set of the second category, the series $\sum_{r=1, r \neq i}^{M} [S_r(x + s_J) - S_r(x)]$ is constant on a second category set having the property of Baire. The proof of Theorem 2 may now be applied, with the obvious modification to the induction hypothesis, to show that the given series is a finite sum.

Finally, if E is residual, then E is dense. Because the series is finite and hence continuous, it can assume only one value.

3. Sets of uniqueness. A set C is called a *set of uniqueness* for lacunary Walsh series, or a *Walsh U_L-set*, if the only lacunary series converging to 0 on the complement of C is the identically zero series.

THEOREM 4. *Suppose that C satisfies one of the following*:

(i) *C is a null set;*

(ii) *C is of the first category;*

(iii) *the complement of C is a dense second category set having the property of Baire.*

Then C is a Walsh U_L-set.

Proof. If C satisfies (i), the result follows from the corollary to Theorem 1. Suppose now that (ii) or (iii) holds; in view of Theorem 3, a lacunary Walsh series converging to 0 on the complement of C is a finite sum and so continuous. Because the complement of C is dense, the series must vanish identically.

None of the conditions in the theorem are necessary. That (i) and

(ii) are not necessary follows from the fact that a null set may be of either category and that there exist sets of the first category of any given measure. To show that (iii) is not necessary, we take C to be a residual set of measure zero; then C is a U_L-set in view of the corollary to Theorem 1.

Lastly, we note that a Walsh U_L-set must have empty interior and so, in particular, cannot be open: for otherwise the Walsh-Fourier series of the characteristic function of a dyadic subinterval, a finite series, would converge to 0 on the complement of the set but not vanish identically.

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UNIVERSITY OF BRITISH COLUMBIA