

FIXED POINT AND MIN-MAX THEOREMS

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The unifying idea in this paper is the existence and uniqueness of cohomology homomorphisms in an admissible range of dimensions, induced by suitable restricted upper semi-continuous homotopies. One consequence of this underlying homotopy result is a fixed point theorem for multiple valued self maps of an n -ball which allows nonacyclic images of points. Another consequence is a min max theorem for a continuous real valued function on a product of finite dimensional compact convex bodies where the usual min and max sections are no longer required to be either convex or acyclic.

An appraisal of the exact contributions in the areas considered and the different approach begun in this article is in point. Thus the algebraic topological development of the fixed point theory of set valued maps has been restricted almost entirely to restraining conditions of convex or certainly acyclic point images following the basic paper of Eilenberg and Montgomery [2]. Presentations of a Lefschetz number or numbers algorithm do exist [3], [4] where the acyclicity conditions have been waived, but effectively assume families of homology group homomorphisms in all dimensions and constitute formal but hardly applicable existence theorems. A similar situation maintains for the so-called min max or saddle point theorems which are the backbone of game theory. Indeed for these only the convex or possibly acyclic point images seem to have been considered.

In another study [1] the writer has obtained results which inter alia include set valued fixed point theorems for which the image sets, $f(x)$, need not be acyclic. However these latter results as they stand are not applicable to the min max case. In the present paper by reason of compactness assumptions somewhat stronger fixed point conclusions are available which are adequate for the formulation of min max results.

1. Preliminaries. We assume X and Y are paracompact. Let f be an upper semi continuous or usc surjection taking *compact sets* of X into *compact sets* of Y . The graph of f is

$$\Gamma(f) = \{(x, y) \mid y \in f(x)\} \subset X \times Y.$$

Denote by p_1 and by p_2 the projections of $\Gamma(f)$ onto X and onto Y respectively. Denote point set boundaries by a dot, thus \dot{X} . We understand the *reduced Alexander cohomology groups over the rationals*. In order to make contact with the notation in our antecedent papers we use

D^{n+1} and \dot{D}^{n+1} indiscriminately to denote the closed simplex and its boundary or the unit ball and the n -sphere. The set valued transformation f of X to Y where X and Y are paracompact is upper semi-continuous use if the antecedent of a closed set is closed and because of complete regularity this is equivalent to the assertion of a closed graph for f . We write $f^*(m) = (p_1(m)^*)^{-1}p_2^*(m)$ for the induced homomorphism $H^m(Y) \rightarrow H^m(X)$ if $p_1(m)^*$ is an isomorphism on $H^m(X)$ to $H^m(\Gamma(f))$.

Let the *singular set* S be defined as the union of the singular components μ_r with

$$(1.1) \quad \begin{aligned} \mu_r &= \{x \mid H^r f(x) \not\approx 0\} \\ d_r &= \dim \mu_r \end{aligned}$$

where $\dim \mu_r$ is the maximum covering dimension of sets $A \subset X$, closed in Y and contained in μ_r . For some assigned q ,

$$(1.2) \quad \bar{p} = 1 + \sup_{\mu_r \neq \emptyset, r < q} (r + d_r).$$

For the case of a use *homotopy* h , X is replaced by $X \times I$.

Specifically

$$h: X \times I \longrightarrow Y$$

with $I = \{s \mid 0 \leq s \leq 1\}$ and $h(x, s)$ is closed. The graph of h is

$$\Gamma(h) = \{(x, s, y) \mid y \in h(x, s)\} \subset X \times I \times Y$$

and is often written

$$\Gamma(h) = \{x, s, h(x, s)\}.$$

The projections of $\Gamma(h)$ on $X \times I$ and on Y are denoted by P_1 and p_2 respectively. Moreover

$$(1.3) \quad \begin{aligned} V_r &= \{(x, s) \mid H^r h(x, s) \not\approx 0\} \\ \delta_r &= \dim V_r. \end{aligned}$$

$$(1.4) \quad \Pi = 1 + \sup_{\substack{V_r \neq \emptyset \\ r < q}} (r + \delta_r).$$

We shall also use the notation \bar{p} and Π in the sequel for integers at least as large as those defined in (1.2) and (1.4). If $P_1^*(m)$ is an isomorphism on $H^m(X \times I)$ to $H^m(\Gamma(h))$, then we write

$$(1.5) \quad h(m)^* = (P_1^*(m))^{-1}p_2^*(m): H^m(Y) \longrightarrow H^m(X \times I).$$

Let f and g be use transformations on X to Y . Then using the notations (1.3) and (1.4),

DEFINITION 1.6. $f \underset{p \parallel q}{\sim} g$ if there is a usc transformation $h: Y \times I \rightarrow Y$ said to describe the homotopy with $V_r = \emptyset, p \leq r \leq q$ and $p \leq \Pi < q$ where $f = h(\cdot, 0)$ and $g = h(\cdot, 1)$. If $q = \infty$ we write $f \underset{p \parallel}{\sim} g$. (Unless p is the lowest dimension for which $V_r = \emptyset$ in the range pq, Π could be inferior to p , but we ignore this situation.)

2. Homotopy. Our results implicitly involve analysis of usc homotopies through their graphs. We demonstrate a key homotopy theorem. Transformations can be viewed as the special case of independence of the parameter $s, s \in I$. However, it is to be observed that if $(x_0, 0)$ is a singular point then (x_0, s) is singular, $s \in I$, for the case $h(x, s)$ is independent of s and so d_r of (1.1) as related to $f = h(\cdot, 0)$ is $\delta_r - 1$.

THEOREM 2.1. If $f \underset{p \parallel q}{\sim} g$ and h describes this homotopy, with f, g , and h usc, then if $q \geq \Pi + 2, h^*(m)$ exists and $f^*(m) = g^*(m)$ for $\Pi + 1 \leq m < q$.

This result is similar to a previously obtained conclusion under somewhat different hypotheses [1] and in the case of greatest interest to us here, namely that of compact metric spaces it is more general than that of [1].

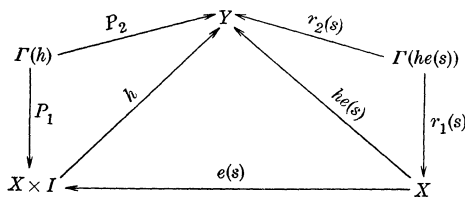
Our first concern is the closure of P_1 . The demonstration is essentially that occurring in the course of the proof of Theorem 10.7 of [1]. Accordingly [5] and (1.5) now ensure that $P_1^*(m)$ is an isomorphism and that $h^*(m)$ exists for $\Pi + 1 \leq m \leq q$. Let $e(s), s \in I$, be the family of maps on X to $X \times I$ defined by

$$e(s)(x) = x \times s .$$

Consider the set valued map

$$he(s): X \longrightarrow Y .$$

This is usc as a composition of a continuous single valued map and a usc map. The basic diagram of commutative triangles is



where $he(s) = h(x, s)$,

$$\Gamma(he(s)) = \{(x, y) \mid y \in h(x, s), s \text{ fixed}\}$$

and

$$\begin{aligned} r_1(s)(x, y) &= x \\ r_2(s)(x, y) &= y. \end{aligned}$$

Again as a consequence of [5] $r_1(s)^*(m)$ is an isomorphism for $\bar{r} + 1 \leq m \leq q$ where $\bar{r} < \Pi$ (1.2). Hence with $e(s)^*$ written for $e(s)^*(m)$

$$(he(s))^*(m) = (r_1(s))^{*-1}(m)r_2(s)^*(m) \quad \Pi + 1 \leq m \leq q$$

and

$$e(s)^*h^*(m) = (he(s))^*(m).$$

In particular

$$\begin{aligned} e(0)^*h^*(m) &= (he(0))^*(m) = f(m)^* \\ e(1)^*h^*(m) &= (he(1))^*(m) = g(m)^*. \end{aligned}$$

It is standard that $e(s)^*$ is independent of s on $H^*(X \times I)$ to $H^*(X)$. Accordingly

$$f(m)^* = g(m)^* \quad \Pi + 1 \leq m \leq q.$$

3. Fixed points. The following results stated for compact metric spaces are true generalizations of corresponding results in [1]. They are perhaps better evaluated in context with the following example: Consider the unit ball D^{n+1} of even dimension. If $x \neq 0$, the ray through x intersects the boundary S^n in the point x' . Define $g(x)$ as x' and $g(0)$ as S^n . Let r be a fixed point free, self mapping of S^n . Then $f = rg$ is on D^{n+1} to D^{n+1} and is usc. The singular set consists of the one point 0 and $f(0)$ has co-dimension 1. Evidently f has no fixed point. (Compare a somewhat similar example in [6] where f is both usc and lsc and yet has no fixed point.) It was the fruitless search for an example of a fixed point free usc self map f of D^{n+1} with $f(x)$ of large co-dimension for $x \in S$, that led to the main theorems in this paper.

LEMMA 3.1. *There is no set valued usc transformation f of the $n + 1$ disk D^{n+1} onto S^n if (a) $f(x) \cap -x = \emptyset$ (b) μ_r is empty for $r \geq n - 2$ and (c) $\bar{p} \leq n - 2$ (1.2).*

Let i be the inclusion $S^n = \dot{D}^{n+1} \rightarrow D^{n+1}$ and denote by g the transformation of S^n to S^n obtained by following i with f . We shall show g is $n - 2$ $n - 1$ homotopic to the identity. Let d indicate arc length. Since f and therefore g are usc, (3.1a) guarantees that

$$\inf d(g(x), -x) \geq \varepsilon > 0 .$$

Define $C(x)$ as the closed subset of S^n

$$C(x) = \{y \mid d(y, -x) \leq \varepsilon\} .$$

Any $p \neq -x$ is on some semi great circle $T(x)$ through x and $-x$ with $d(p, -x) = t$. Let $h(x, p, s)$ $s \in I$ be the point on $T(x)$ of arc length $s + (1 - s)t$ from $-x$. A triangle inequality argument shows that $h(\)$ is continuous on

$$(S^n \times S^n - \mathbf{U}(x \times C(x)) \times I \text{ to } S^n .$$

We assert

$$h'(x, s) = h(x, g(x), s)$$

is usc. Indeed with the notation $w \in h'(x, s)$, suppose $w^n \in h'(x^n, s^n)$ where

$$\begin{aligned} x^n &\longrightarrow \bar{x} \\ s^n &\longrightarrow \bar{s} \\ w^n &\longrightarrow \bar{w} . \end{aligned}$$

Then for some $z^n \in g(x^n)$, $w^n \in h(x^n, z^n, s^n)$. By compactness some subsequence again denoted by $\{z^n\}$ converges. Since g is usc,

$$z^n \longrightarrow \bar{z} \in g(\bar{x}) .$$

The continuity of h implies $h(x^n, z^n, s^n)$ converges to $h(\bar{x}, \bar{z}, \bar{s})$, that is to say

$$w^n \longrightarrow \bar{w} = h(\bar{x}, \bar{z}, \bar{s}) \subset h(\bar{x}, g(\bar{x}), \bar{s}) = h'(\bar{x}, \bar{s}) .$$

Evidently for fixed x and all $s < 1$, the sets $h'(x, s)$ are homeomorphs (and for $s = 1$, there are no singular sets since $h'(1)$ is the identity map). Accordingly, (cf. 1.3), $V_r \subset \mu_r \times [0, 1]$ and

$$(3.1.1) \quad \hat{\delta}_r \leq d_r + 1$$

while V_r is empty for $r \geq n - 2$. Hence by Hypothesis (c) and (3.1.1), $\Pi \leq n - 1$ so $g_{n-2, n-1} \mathbf{1}$. The hypotheses of Theorem 2.1 are therefore satisfied whence

$$(3.1.2) \quad 0 \neq 1(n)^* = g(n)^* = (if)^*(n) .$$

We now establish the existence of $f^*(n)$. Take $X = D^{n+1}$, $Y = S^n$ and

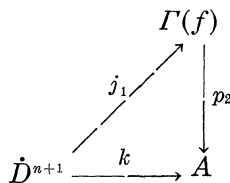
$$p_1^{-1}x = x \times f(x) \in I'(f) .$$

The proof that p_1 is closed is of the same type as that for the closure of P_1 . Hence p_1 satisfies the conditions in [5] with $\bar{p} \leq n - 1$. Accordingly $p_1^*(m)$ is an isomorphism for $m > n - 1$. Therefore $(if)^*(n) = f^*(n)i^*(n)$. Since $i^*(n) = 0$ this is out of accord with (3.1.2).

If f is usc on X to X , \bar{x} is a *fixed* point if $\bar{x} \in f(\bar{x})$.

LEMMA 3.2. *There is no set valued usc transformation f on D^{n+1} to $\dot{D}^{n+1} \times I$ with $f(x) = x, x \in \dot{D}^{n+1}$ and $\bar{p} \leq n - 1$.*

The argument is similar to that for the latter part of the proof of (3.1) deriving from (3.1.2). Thus let p_1 and p_2 be the usual projections on $\Gamma(f)$ to D^{n+1} and to $\dot{D}^{n+1} \times I$ respectively and write $A = \text{Im } p_2$. Then by interpreting (2.1) for transformations (or by first noting as in [1, Theorem 10.7] that p_1 is closed and then appealing to [5]) we infer $p_1^*(n)$ is the trivial isomorphism. The purported commutativity in



where k is inclusion and $j = p_1 j_1$, with j inclusion of \dot{D}^{n+1} in D^{n+1} , yields

$$k^*(n) = j_1^*(n) p_2^*(n) .$$

Since $p_1^*(n)$ is trivial so is $j_1^*(n)$ while $k^*(n)$ is obviously a non trivial epimorphism.

THEOREM 3.3. *Let f be a usc transformation on D^{n+1} to D^{n+1} . Let S be the singular set with $\dim S = d$. Let $\mu_r = \emptyset$ for $r \geq n - 1 - d = \bar{r}$. Then f has a fixed point.*

Suppose there is no fixed point. The cone $C^+(x)$ is generated by lines from points of $f(x)$ through x and intersects D^{n+1} in a set $F(x)$. Write $d(x)$ for $d(x, \dot{D}^{n+1})$. Identify \dot{D}^{n+1} with $\dot{D}^{n+1} \times 0$ and denote a generic point by x'' . Define the homotopy on D^{n+1} to the annulus $\dot{D}^{n+1} \times I$ by

$$\begin{aligned}
 (3.3.1) \quad h(x, s) &= \bigcup_{x' \in f(x)} z(x, x')((1 - s)/\|z\| + s) \\
 z(x, x') &= x''(x, x') + d(x)(x - x') .
 \end{aligned}$$

where x'', x' and x are collinear, the addition is, of course, vector addition, and $h(x, s), s \neq 0, x \in \dot{D}^{n+1}$, is a homeomorph of $f(x)$ in $C^+(x)$.

We show h is usc on $D^{n+1} \times I$ to $S^n \times I$ and so in particular $\Gamma(h)$ is compact. For simplicity we assume s fixed. Thus suppose $x_n \rightarrow \bar{x}$ and $z_n \rightarrow \bar{z}$. Continuity guarantees $dx_n \rightarrow d(\bar{x})$. By (3.3.1) x_n'' and x_n' are determined by z_n . Suppose $x_n'' \rightarrow \bar{x}'', x_n' \rightarrow \bar{x}'$. The usc property for f guarantees $\bar{x}' \in f(\bar{x})$. The demonstration of (10.9) in [1] establishes that F is usc so $\bar{x}'' \in F(\bar{x})$. Hence by (3.3.1)

$$\bar{z} = \bar{x}'' + d(\bar{x})(x - \bar{x}')$$

or h is usc for each fixed s . It is now trivial to show that h remains usc when s varies.

A homotopy argument based on (2.1) would entail a study of S for F . Such complications can be avoided by noting h yields a deformation retraction of $\text{Im } h(, 1)$ onto \dot{D}^{n+1} . Since the retraction induces the identity homomorphism, existence of $h(, 1)^*(n)$ implies that of $h(, 0)^*(n)$ and we then need show $h(1)^*(n) \neq 0$ whence $h(0)^*(n) \neq 0$. Thus just as in (3.2) a contradiction would be reached. However our main conclusion already follows on considering $h(1) = F'$ alone. The singular set for F' is at most that for f . Accordingly \bar{p} for F' is given by

$$\bar{p} \leq 1 + \sup_{r \leq \bar{r}-1} (d_r + r) \leq d + \bar{r} = n - 1.$$

Thus F' satisfies the hypotheses of (3.2) in contradiction to the conclusion of that lemma. Hence $f(x)$ admits a fixed point since $F(x)$ does.

The ball may be replaced by a somewhat more general set. Thus

COROLLARY 3.4. *Let E be a retract of D^{n+1} and suppose f is a usc self map of E . Let its singlar set S_{n-1} be finite and disjoint from \dot{E} . Then f admits a fixed point. If the singular set is not required to be away from \dot{E} there is nevertheless a fixed point if $\mu_r = \emptyset, r \geq n - 2$.*

Denote the retracting function on D^{n+1} to E by r . Then f extends to f' on D^{n+1} to E and therefore to D^{n+1} by

$$(3.4.1) \quad f'(z) = f(rz), z \in D^{n+1}.$$

Let z_n converge to z . Write x_n for rz_n . Since r is continuous x_n converges to x where $x = rz$. Let $w_n \in f'(z_n)$. Then $w_n \in f(x_n) \subset E$. Hence w_n converges to $w \in f(x) = f(rz) = f'(z)$. Thus f' is ucs. The singular set for f' is again S_{n-1} . Hence by (3.3), f' admits a fixed point, z , which obviously must lie in E . Thus z is a fixed point of f .

Suppose

$$S_{n-2} \cap \dot{E} = \{x_i \mid i = 1, \dots, k\}.$$

Then the singular set S' for f' is

$$S_{n-2} \cup \{r^{-1}x_i \mid i = 1, \dots, k\}.$$

Therefore $\dim S' = 1$ and again (3.3) applies.

4. **Min max results.** The results above have immediate application to the central theorem of game theory, namely the saddle point theorem or min-max theorem. We establish a wide generalization which, as in the case of Theorem 3.3, breaks through the usual restriction to point-convex set or point-acyclic set maps.

Let X and Y be compact convex bodies in R^k and R^l respectively and let f be a real valued continuous map on $X \times Y$. A *saddle point* (x^0, y^0) is defined by

$$\text{Min}_{y \in Y} f(x^0, y) = f(x^0, y^0) = \text{Max}_{x \in X} f(x, y^0).$$

Let

$$M(y) = \{x \mid f(x, y) = \text{Max}_{x \in X} f(x, y)\} \subset X$$

$$N(x) = \{y \mid f(x, y) = \text{Min}_{y \in Y} f(x, y)\} \subset Y.$$

Thus M is a set valued function on Y to X and N is a set valued function on X to Y . Define the set valued function g on $X \times Y$ to $X \times Y$ by

$$g(x, y) = M(y) \times N(x).$$

Since $X \times Y$ can be identified with D^{k+l} the analogues of the transformations introduced in § 3 are immediate. However for clarity of exposition we state the definitions. We shall sometimes write z for the point (x, y) . Thus let dz be $d(z, (X \times Y)^\bullet)$ and write $C(z)^+$ for the forward cone with base $g(z)(z \in g(z))$, passing through $z = (x, y)$. Then $G(z)$ is $C(z)^+ \cap (X \times Y)^\bullet$. Each generator of $C(z)^+$ meets $(X \times Y)^\bullet$ in a unique point $z'' \equiv z''(z; z'), z' \in g(z)$

i.e.
$$G(x, y) = \bigcup_{x', y' \in g(x, y)} z''(x, y; x', y').$$

Define

$$G'(z) = \bigcup_{z' \in g(z)} (z''(z, z') + d(z)(z - z'))$$

where $z = (x, y)$ and $z' = (x', y')$ and vector difference is intended in the last parentheses.

In the finite dimensional case the classical min-max theorem states that for $X, Y, M(y)$ and $N(x)$ convex (acyclic) compacta there

is a saddle point. The following theorem initiates a fundamental advance.

THEOREM 4.1. *Suppose the set valued functions M and N defined above are usc. Two singular sets $S(X), S(Y)$ of dimensions $d(X)$ and $d(Y)$ are defined as in (1.1) and for N and for M respectively by*

$$S(X) = \bigcup_r \mu_r(X), S(Y) = \bigcup_r \mu_r(Y) .$$

Assume that for $x \in S(X), y \in S(Y), H^(N(x), Q)(H^*(M(y), Q))$ is finitely generated. Suppose that $\mu_r(X) = \emptyset$ for $r \geq p$ and that $dX \leq k - p - 2$. Suppose too that $\mu_r(Y) = \emptyset$ for $r \geq q$ with $dY \leq l - 2 - q$. Then there is a saddle point.*

Assume for all $x, y \in X, Y$ there is no saddle point i.e. $x, y \notin g(x, y)$. Write

$$\begin{aligned} L(t) &= \{(x, y) \mid H^t G(x, y) \neq \emptyset\} \\ S(X, Y) &= \bigcup_t L(t) \\ d_t &= \dim L(t) \\ d &= \dim S(X, Y) . \end{aligned}$$

Our concern is with an upper bound for \bar{t} deriving from the hypotheses of the theorem. Evidently, g is usc as a product of usc transformations. It can then be established that G' is usc also. Moreover the singular set for g includes that for G' since no point on $(X \times Y)$ can be singular for G' . The Kunnetth theorem applies to give

$$(4.11) \quad H^t(M(y) \times N(x)) \approx \bigoplus_{a+b=t} H^a M(y) \otimes H^b N(x) .$$

The assertion of (3.3) requires only that $\bar{p} \leq n - 1$. If then $k + l$ is taken as $n + 1$ we need merely establish that \bar{t} the correspondent of \bar{p} is at most $n - 1$ where

$$\bar{t} = 1 + \sup_{L(t) \neq \emptyset} (t + d_t) .$$

There are three situations to consider

$$(4.1.2) \quad x \in S(X), y \in S(Y)$$

$$(4.1.3) \quad x \in S(Y), y \in \overline{S(Y)}$$

$$(4.1.4) \quad x \in \overline{S(X)}, y \in S(Y) .$$

For (4.1.2) the sup is taken for t at most $p + q - 2$. Thus

$$\begin{aligned}
 \bar{t} &= 1 + \sup_{\substack{L(t) \neq \emptyset \\ t \leq p+q-2}} (t + d_i) \leq 1 + p + q - 2 + d \\
 (4.1.5) \qquad &\leq p + dX + q + dY - 1 \\
 &= k + l - 5 \\
 &= n - 4.
 \end{aligned}$$

For (4.1.3) an upper bound ensues if $S(Y)^\sim$ is replaced by R^l , thus

$$\begin{aligned}
 \bar{t} &= 1 + \sup_{\substack{L(t) \neq \emptyset \\ t \leq p=1}} (t + d_i) \leq p + dX + l \\
 (4.1.6) \qquad &\leq k + l - 2 \\
 &= n - 1.
 \end{aligned}$$

By symmetry the same bound covers (4.1.4). Hence the upper bound required is given by (4.1.6). Then by (3.3.1) for some \bar{x}, \bar{y}

$$\begin{aligned}
 &\bar{x}, \bar{y} \in g(\bar{x}, \bar{y}) \\
 \text{i.e.} \qquad &\bar{x} \in \text{Max } f(x, \bar{y}) \\
 &\bar{y} \in \text{Min } f(\bar{x}, y)
 \end{aligned}$$

that is to say, \bar{x}, \bar{y} is a saddle point.

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