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THE MULTIDIMENSIONAL CONTENT OF THE FRUSTUM OF THE SIMPLEX

GIULIO VARSI

The content of the intersection of a simplex with a semispace is computed by means of a dissection technique. An efficient algorithm, suitable for automatic calculation, is given. For an n-dimensional space, the algorithm needs only $n-1$ storage location at most, and requires $\sim n^2$ operations.

Introduction. The simplex S considered is the $(n-1)$-dimensional polytope defined as the convex hull of its n vertices V_j in R^n, the Euclidean space in n dimensions:

$$V_j \equiv \{v_{j1}, v_{j2}, \cdots, v_{jn}\} ,$$

where

$$v_{ji} = \delta_{ji} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \qquad j, i = 1, 2, \cdots, n .$$

This choice of geometry is convenient in certain applications of the algorithm that arise in statistical mechanics and allocation theory, but does not result in a loss of generality (see Appendix).

The frustum F_1 of S is defined as the nonempty intersection of S with the semispace σ:

$$\sigma \equiv \{x_1, x_2, \cdots, x_n \mid \sum p_i x_i \leqq G\} ,$$

where the real numbers p_i are the coefficients that characterize the hyperplane γ, boundary of the semispace.

With a minor loss of generality, that will be removed subsequently, it will be assumed that

$$(1) \qquad \begin{array}{ll} p_1 < p_2 < \cdots < p_n . & \\ p_i \neq G & i = 1, 2, \cdots, n . \end{array}$$

It can then be immediately verified that the condition for $S \cap \sigma$ to be nonempty is

$$p_1 \leqq G \leqq p_n .$$

The content $C[F_1]$ of the frustum can be represented as

$$\iint \cdots \int_X dV^{n-1}$$

where the region X is defined by the following constraints:

$$0 \leqq x_i,\ i = 1, 2, \cdots, n , \tag{2}$$

$$\sum x_i = 1 , \tag{3}$$

$$\sum p_i x_i \leqq G . \tag{4}$$

The last inequality can also be written as:

$$\sum m_i x_i \leqq 0 , \tag{4'}$$

where:

$$m_i = p_i - G ,$$

and the condition for $S \cap \sigma$ to be nonempty is:

$$m_1 \leqq 0 \leqq m_n .$$

The form (4′) will be used throughout the paper.

It is often convenient to measure $C[F_1]$ in units of $C[S]$, i.e., to express $C[F_1]$ as a fraction of $C[S]$. This latter quantity can be readily evaluated by projecting S onto the $x_j = 0$ hyperplane and by multiplying the content of the projection by the secant of the angle formed by S with that hyperplane. It results [1]:

$$C[S] = \frac{\sqrt{n}}{(n-1)!} .$$

The symbol ρ' represents the ratio $C[F_1]/C[S]$.

Properties of the frustum. The hyperplane γ defined by

$$\sum m_i x_i = 0 \tag{5}$$

dissects the simplex S into two frusta F_1 and F_2 so that, for any $x \in S$, if $x \in F_1$, then $\sum m_i x_i \leqq 0$; if $x \in F_2$, then $\sum m_i x_i \geqq 0$.

One property of a simplex is that any one of its faces contains all the vertices but one [1]; ϕ_l denotes that $(n-2)$-face of S which does not contain V_l, and E_{ij} denotes the edge that connects V_i and V_j. If the hyperplane γ cuts S, it will partition the vertices V_l into two nonempty sets Z_1 and Z_2, corresponding to the frusta F_1 and F_2. One can distinguish two cases:

(1) Either Z_1 or Z_2 contains only one vertex. Let Z_1 contain only V_1, then every face ϕ_l $(l \neq 1)$ will have vertices on both sides of γ and, therefore, will be cut by γ; F_1 will then have a portion of all the ϕ_l $(l \neq 1)$ $(n-2)$-faces of S plus the $(n-2)$-face lying on γ, or a total of n $(n-2)$-faces. Furthermore, F_1 will have as vertices the intersections of all the edges $E_{1l} (l \neq 1)$ with γ plus V_1 itself. Therefore, F_1 is a simplex and its content $C[F_1]$ can be evaluated

immediately by means of the well-known determinant. One also has $C[F_2] = C[S] - C[F_1]$.

(2) Both Z_1 and Z_2 contain at least two of the vertices V_l. Then every face ϕ_l will have vertices on both sides of γ: γ cuts every face. Both F_1 and F_2 contain a portion (which shall again be called ϕ_l) of each face of S, plus the common face lying on γ (which shall again be called γ) for a total of $n + 1$ $(n - 2)$-faces. Neither F_1 nor F_2 is a simplex. If Z_1 and Z_2 contain, respectively, z_1 and z_2 $(z_1 + z_2 = n)$ vertices V_l, then F_1 contains all the vertices in Z_1 plus all the intersections of γ with the edges E connecting elements of Z_1 with elements of Z_2, for a total of $(z_1 \times z_2) + z_1 = (z_2 + 1)z_1$ vertices.

LEMMA. *The frustum F_1 has the following property: any one of its vertices W belongs to all the $(n - 2)$-faces except two.*

Proof. If W coincides with one of the V, say V_l, then W belongs to all the ϕ except ϕ_l and does not belong to γ. If W does not coincide with any V then it belongs to γ, and it must belong to the edge connecting two of the V; let it be E_{jk}. Since $V_j \in \phi_h (h \neq j)$ and $V_k \in \phi_h (h \neq k)$, then $E_{jk} \in \phi_h (h \neq j, k)$ and, since $W \in E_{jk}$, then $W \in \phi_h (h \neq j, k)$.

Therefore, F_1 can be dissected in $(z_2 + 1)z_1$ different ways into two pyramids,[1] which are characterized by selecting a vertex W and the two faces that do not contain it.

Dissection technique. In this section, the algorithm is described in detail. In essence, the frustum is dissected into two polytopes, which are shown to be pyramids; the bases of these pyramids have the same property as the frustum: namely, each basis can be dissected into two pyramids of one dimension less than the previous ones. The two-fold dissection can be repeated on the new bases until a simplex is obtained. It may appear that the number of two-dimensional pyramids (or triangles) required should be of the order of 2^{n-2} and therefore exceed the present computing capabilities for even relatively modest values of n. In fact, very substantial simplifications are possible so that the number of operations required is of the order of n^2.

With no further loss of generality, it will be assumed that

$$(6) \qquad \begin{aligned} m_i &< 0 \qquad i = 1, \cdots, J . \\ m_i &> 0 \qquad i = J + 1, \cdots, n . \end{aligned}$$

[1] A q-dimensional pyramid is defined as a polytope with nonnull q-dimensional content, such that all its vertices but one belong to a $(q - 1)$-flat, called the basis of the pyramid. A $(q - 1)$-flat is the set of all points in $(q + k)$ space that can be represented as linear combinations of q linearly independent points $(k \geqq 0)$.

The hyperplane defined by equation (3) and the faces of S and of F_1 that belong to it are designated as η; the hyperplane defined by equation (5) and the faces belonging to it are designated as γ as in the previous section. Furthermore, c_i indicates the coordinate hyperplane defined by $x_i = 0$. The intersection of two or more hyperplanes is indicated as a list; $\eta c_1 c_7 \gamma$ indicates the $(n-4)$-flat obtained by intersecting the two coordinate hyperplanes c_1 and c_7 with η and γ.

From the sign of $\delta = \{\sum m_i x_i\}$ it will be noted that all V_i, for which $1 \leqq i \leqq J$, lie on one side of γ, and all those V_i for which $J < i \leqq n$ lie on the other side [2].

A vertex of F_1 is indicated as W_i if it coincides with the vertex V_i of the simplex S, or as W_{lk} if it is the intersection of γ with the edge E_{lk}. (Since $W_{lk} \equiv W_{kl}$, the convention will be followed that $l < k$.) It will be noted that, for any vertex W_{lk}, the following inequalities obtain:

$$1 \leqq l \leqq J < k \leqq n ,$$

because V_l and V_k have to be on opposite sides of γ for E_{lk} to intersect γ.

The coordinates of W_{lk} can be found immediately by noting that a point x describes the segment $[V_l, V_k]$ if

$$x = \lambda V_l + (1 - \lambda) V_k ,$$

where $0 \leqq \lambda \leqq 1$. If $(l, k/i)$ indicates the ith coordinate of W_{lk}, then

$$\begin{aligned}(l, k/l) &= (m_k)/(m_k - m_l) ,\\ (l, k/k) &= (-m_l)/(m_k - m_l) ,\\ (l, k/i) &= 0 \text{ for } i \neq l, k ,\end{aligned}$$

and it is immediately possible to verify that $W_{lk}(= \{(l, k/1), (l, k/2), \cdots, (l, k/n)\})$ belongs to E_{lk} and to γ:

$$\sum m_i(l, k/i) = m_l(m_k)/(m_k - m_l) + m_k(-m_l)/(m_k - m_l) = 0 .$$

The total number of vertices of F_1 is $N = J + J(n - J) = J(n - J + 1)$, as shown in Table I. The n $(n-2)$-faces of simplex S can now be expressed as the intersections of hyperplane η with the coordinate hyperplanes: $\eta c_1, \eta c_2, \cdots, \eta c_n$.

THEOREM. *The frustum F_1 can be dissected into two pyramids and, furthermore, the basis of each pyramid can also be dissected into two pyramids; the process can be iterated until the dissection yields simplices.*

Proof. The proof will be given by construction and the conditions under which the dissection yields simplices will also be shown. In the nontrivial case (2) of the preceding section, $1 < J < n - 1$, F_1 has the appropriate portions of $\eta c_1, \eta c_2, \cdots, \eta c_n$ and the appropriate portion of γ. It is convenient to dissect F_1 into two pyramids by selecting the apex on γ according to the previous Lemma. Let it be $W_{l_1 k_1}$. Since a vertex W_{lk} belongs to ηc_i if its ith coordinate is zero, then $W_{l_1 k_1}$ belongs to all the faces of F_1 except ηc_{l_1} and ηc_{k_1}. These will be the bases of the two pyramids $\pi^{n-1}(l_1)$ and $\pi^{n-1}(k_1)$; the bases will be designated $\beta^{n-2}(l_1)$ and $\beta^{n-2}(k_1)$. In order to proceed with the dissection, the $(n-3)$-faces of the two bases have to be identified. Consider $\beta^{n-2}(l_1)$. Its faces are the intersections of ηc_{l_1} with the other faces of F_1: $\eta c_{l_1} c_{l_2}$, where $l_2 = 1, 2, \cdots, l_1 - 1, l_1 + 1, \cdots, n$. Is the intersection $\eta c_{l_1} c_{l_2}$ a proper intersection, in the sense that it belongs to F_1? Yes, because $W_{ih} \in \eta c_{l_1} c_{l_2}$, if $i, h \neq l_1, l_2$. Analogously, $\eta c_{l_1} \gamma$ is a proper intersection, and since l_2 can take any one of $n - 1$ values, then $\beta^{n-2}(l_1)$ has $n - 1 + 1 = n$ $(n-3)$-faces. Again, $\beta^{n-2}(l_1)$ has one face more that the simplex of the same $(n-2)$ number of dimensions.

Any one of the vertices of $\beta^{n-2}(l_1)$ can be selected to serve as the apex of the pyramidal dissection, for example, $W_{l_2 k_2}$, where $1 \leqq l_2 \leqq J < k_2 \leqq n$ and $l_2 \neq l_1$. (The last condition is necessary because $W_{l_1 h} (h = J + 1, \cdots, n)$ does not belong to $\beta^{n-2}(l_1)$.)

TABLE I. VERTICES OF F_1

	Vertex	Coordinate 1	2	3	J	$J+1$	.....	n
J	W_1	1	0	0	0	0	.....	0
	W_2	0	1	0	0	0	.....	0
	.	.	.	. .....	.	.	.....	.
	.	.	.	. .....	.	.	.....	
	W_J	0	0	0	1	0	.....	0
J { $n-J$	$W_{1,J+1}$	$\frac{m_{J+1}}{m_{J+1} - m_1}$	0	0	0	$\frac{m_1}{m_{J+1} - m_1}$	.....	0
	.	.	.	. .....	.	.	.....	.
	.	.	.	. .....	.	.	.....	.
	$W_{1,n}$	$\frac{m_n}{m_n - m_1}$	0	0	0	0	.....	$\frac{-m_1}{m_n - m_1}$
$n-J$	$W_{2,J+1}$	0	$\frac{m_{J+1}}{m_{J+1} - m_2}$	0	0	$\frac{-m_2}{m_{J+1} - m_2}$	.....	0
	.	.	.	. .....	.	0	.....	.
	.	.	.	. .....	.	0	.....	.
	.	.	.	. .....	.	.	.....	.
	$W_{J,n}$	0	0	0	$\frac{m_n}{m_n - m_J}$	0	.....	$\frac{-m_J}{m_n - m_J}$

The two new bases are those faces of $\beta^{n-2}(l_1)$ that do not contain

$W_{l_2k_2}$. These faces are $\eta c_{l_1}c_{l_2}$ and $\eta c_{l_1k_2}$. The shorter notation $\beta^{n-3}(l_1, l_2)$ and $\beta^{n-3}(l_1, k_2)$ will be employed.

In general, the μ-dimensional base of a $(\mu + 1)$-dimensional pyramid will be indicated by $\beta^{\mu}(h_1, h_2, \cdots, h_{n-\mu-1})$. Its $\mu + 2$ $(\mu - 1)$-dimensional faces are:

$$\eta c_{h_1}c_{h_2} \cdots c_{h_{n-\mu-1}}c_i ,$$

and

$$\eta c_{h_1}c_{h_2} \cdots c_{h_{n-\mu-1}}\gamma .$$

where $i \neq h_1, h_2, \cdots, h_{n-\mu-1}$.

The argument $h_1, \cdots, h_{n-\mu-1}$ is a combination without repetition of $n - \mu - 1$ among the n numbers $1, 2, \cdots, n$.

The next dissection is accomplished by selecting a vertex W_{lk} with the usual conditions on l and k with the further limitation

$$l, k \neq h_i \qquad i = 1, \cdots, n - \mu - 1 .$$

The new bases are $\beta^{\mu-1}(h_1, \cdots, h_{n-\mu-1}, l)$ and $\beta^{\mu-1}(h_1, \cdots, h_{n-\mu-1}, k)$.

It is convenient to represent the pyramidal dissection geometrically with a tree. Each node of the tree represents a pyramid and two branches issue from it.

It has been noted that one of the pyramids may in fact be a simplex; therefore, the dissection process ends on that branch. This occurs where one of the intersections of the base of the pyramid with the remaining coordinate hyperplanes or with γ is improper. A generic intersection $I \equiv \eta c_{h_1} \cdots c_{h_l} \cdots c_{h_\nu}$ (where $h_\nu = 0$ indicates γ) is improper if it does not contain any one of the W_{lk} or W_i. This happens in either of the following two cases:

(1) If for each $i \leqq J$ there exists an $h_l \ni h_l = i$. Since all the vertices of F_1 have a nonzero coordinate, the cardinal number i of which is less than or equal to J, no vertex can have zero coordinate for all $i \leqq J$, and therefore no vertex can belong to I.

(2) If $h_l = 0$ for some l and for each $i \geqq J + 1$ there exists an $h_l \ni h_l = i$. Since all the vertices that belong to γ have a nonzero coordinate the cardinal number i of which is greater than $J + 1$, no vertex can belong to I.

A base $\beta^{n-\nu-1}$ that forms an improper intersection has only $n - \nu$ $(n - \nu - 2)$-faces, rather than $n - \nu + 1$ and is, therefore, a simplex. This occurs if the argument list of $\beta^{n-\nu-1}$ contains $J - 1$ indices smaller than or equal to J (one of the intersections of $\beta^{n-\nu-1}$ will fall under Case 1) or if it contains all the $n - J$ indices greater than J (Case 2).

The conditions stated above can be verified also by considering the number of vertices of a base $\beta^{n-\nu-1}(h_1, h_2, \cdots, h_\nu)$. Since the order of the arguments in the list is irrelevant, we may assume that they are ordered as follows:

$$(7) \qquad h_1 < h_2 < h_3 < \cdots < h_q \leqq J < \cdots < h_\nu .$$

By inspection of Table I one can verify that from the total number $N = J(n - J + 1)$ of vertices of F_1, $(n - J + 1)$ are excluded for each of the $h_i \leqq J$ and $(J - q)$ are excluded for each one of the $h_i \geqq J + 1$. The number N_β of vertices belonging to that base is

$$\begin{aligned} N_\beta &= J(n - J + 1) - (n - J + 1)q - (J - q)(\nu - q) \\ &= (J - q)(n + 1 - J - \nu + q) . \end{aligned}$$

Since the base $\beta^{n-\nu-1}$ is of dimension $n - \nu - 1$, it is a simplex if it has $N_\beta = (n - \nu - 1) + 1 = n - \nu$. This condition, a second-degree equation in q, has two solutions: $q_1 = J - 1$ and $q_2 = J + \nu - n$. The first condition is clearly equivalent to the one enunciated previously that led to case (1); the second can be rewritten as $\nu - q_2 = n - J$ and can be shown to coincide with the second condition given above. The proof of the statement of the theorem is thus complete.

Now, the distance of the apex of the pyramid from its base has to be evaluated. In general, this is the distance of W_{lk} from $\beta^\mu(h_1, h_2, \cdots, h_{n-\mu-2}, k)$. This quantity has a somewhat cumbersome expression because it is the distance between a point and a μ-flat in $R^{\mu+2}$. (The vertex W_{lk} and the vertices belonging to β^μ collectively have $n - (n - \mu - 2 + 1) + 1 = \mu + 2$ coordinates different from zero.) The equivalent case in R^3 is the determination of the distance of a point from a straight line lying on one of the coordinate planes. The expression, however, becomes very simple if, as previously suggested, F_1 is projected onto one of the coordinate hyperplanes.

Since the cosines of η are all equal, the relationship $C[F_1] = C[\Phi_1](n)^{1/2}$ holds for Φ_1, the projection of F_1 on any one of the coordinate hyperplanes. The only condition is that neither c_l nor c_k be used in connection with W_{lk}. As will be shown subsequently, it is quite convenient to use c_J as the projection hyperplane for all the pyramids. In the projection plane, the distance between the projections of W_{lk} and $\beta^\mu(h_1, h_2, \cdots, n_{n-\mu-2}, k)$ is simply the kth coordinate of W_{lk}: $(l, k/k) = (-m_l)/(m_k - m_l)$. This statement can be verified by minimizing the square of the distance d between the projections of a generic point x on β^μ and W_{lk} as follows:

$$\begin{aligned} d^2 &= \sum [x_i - (l, k/i)]^2 \\ &= (l, k/k)^2 + \sum{}^* x_i^2 + [x_l - (l, k/l)]^2 , \end{aligned}$$

where $\sum^*$ indicates the sum for $i \neq h_1, h_2, \cdots, h_{n-\mu-2}, k, l, J$, and where the following constraints apply:

$$\Sigma^+ x_i \leqq 1, \quad 0 \leqq x_i \leqq 1 \quad \text{for } i \neq J \,.$$

Here, Σ^+ indicates the sum for $i \neq J$. It can be verified that d^2 is minimum when all its terms but the first are zero:

$$\begin{aligned} d^2 &= (l, k/k)^2 \,, \\ x_i &= 0 \quad \text{for } i \neq l \,, \\ x_l &= (l, k/l) \,. \end{aligned}$$

The content of the pyramid of base β^μ and apex W_{lk} is then computed from the content of its projection β as [2, 3]:

$$(n^{1/2})\, [1/(\mu + 1)]\, |\, (l, k/k)\, |\, C[\beta(h_1, h_2, \cdots, h_{n-\mu-2}, k)] \,.$$

Description of the algorithm. The rules for the construction of the tree representing the dissection can be obtained from the rules given in the previous section. These rules are summarized as follows:

(1) Start with node Φ_1.

(2) From Φ_1 go to the two nodes labeled l_1 and k_1, where

$$1 \leqq l_1 \leqq J; \quad J + 1 \leqq k_1 \leqq n \,.$$

(3) From each node draw two branches to two further nodes and label them l_μ and k_μ, where $1 \leqq l_\mu \leqq J$ and $J + 1 \leqq k_\mu \leqq n$ and where both l_μ and k_μ are different from all the labels affixed to the previous nodes in the path leading from Φ_1 to the nodes labeled, respectively, l_μ or k_μ, and so on.

(4) A path is terminated:

(a) If $J - 1$ of its nodes, including the terminal one, have labels of numerical value $\leqq J$ or

(b) if $n - J$ of its nodes, including the terminal one, have labels of numerical value $\geqq J + 1$.

The construction can be systematized by the addition of the following rule:

(5) The label assigned to each node is the lowest possible, provided that rule 3 also is observed.

Two symbols are associated with each node: One is the label discussed previously, the other is an element of matrix $A = \|a_{r,s}\|$, where $r = J - h_q$ and $s = n + 1 - h_\nu$, h_q and h_ν being defined as in equation (7).

The value of the matrix element is the content of the projection of the base of the pyramid represented by the node. It should be noted that if two nodes are reached by paths containing the same labels, they have the same value; furthermore, the paths exiting from them are identical.

An obvious consequence of rules 4 and 5 is that label J never appears and that the label of a terminal node is either $J-1$ or n. The computation starts from the pyramid whose matrix element has the lowest indices (to the end of this section: $(l, k/i) \equiv |(l, k/i)|$):

$$\begin{aligned} a_{2,2} &= C[\beta(1, 2, \cdots, J-3, J-2, J+1, \cdots, n-2, n-1)] \\ &= (1/2)\{(J-1, n/J-1)C[\beta(1, 2, \cdots, J-2, J+1, \cdots, n-1, J-1)] \\ &\quad + (J-1, n/n)C[\beta(1, 2, \cdots, J-2, J+1, \cdots, n-1, n)]\} \\ &= (1/2)\{(J-1, n/J-1)a_{12} + (J-1, n/n)a_{21}\} . \end{aligned}$$

Next:

$$\begin{aligned} a_{2,3} &= (1/3)\{(J-1, n-1/J-1)a_{1,3} + (J-1, n-1/n-1)a_{2,2}\} , \\ a_{2,4} &= (1/4)\{(J-1, n-2/J-1)a_{1,4} + (J-1, n-2/n-2)a_{2,3}\}, \text{ etc.} \end{aligned}$$

The last element evaluated in this phase of the calculation is

$$\begin{aligned} a_{2,n-J+1} = (n-J+1)^{-1}\{&(J-1, J+1/J-1)a_{1,n-J+1} \\ &+ (J-1, J+1/J+1)a_{2,n-J}\} . \end{aligned}$$

The next phase requires the evaluation of

$$\begin{aligned} C[\beta(1, 2, \cdots, J-3)] &= a_{3,n-J+1} \\ &= (n-J+2)^{-1}\{(J-2, J+1/J-2)a_{2,n-J+1} \\ &\quad + (J-2, J+1/J+1)a_{3,n-J}\} . \end{aligned}$$

While the element $a_{2,n-J+1}$ has just been calculated, $a_{3,n-J}$ is determined as follows:

$$\begin{aligned} a_{3,n-J} &= C[\beta(1, 2, \cdots, J-3, J+1) \\ &= (n-J+1)^{-1}\{(J-2, J+2/J-2)C[\beta(1, 2, \cdots, J-3, J+1, J-2)] \\ &\quad + (J-2, J+2/J+2)C[\beta(1, 2, \cdots, J-3, J+1, J+2)]\} . \end{aligned}$$

The time-saving technique derives from the observation that, in the above expression,

$$\begin{aligned} &C[\beta(1, 2, \cdots, J-3, J+1, J-2)] \\ &= C[\beta(1, 2, \cdots, J-3, J-2, J+1)] = a_{2,n-J} , \end{aligned}$$

which has been computed.

The general term has the form:

$$\begin{aligned} a_{h,k} = \{(h+k-2)\}^{-1}\{&a_{h,k-1}(J-h+1, n-k+2/n-k+2) \\ &+ a_{h-1,k}(J-h+1, n-k+2/J-h+1)\} , \end{aligned}$$

for $2 \leqq h \leqq J$ and $2 \leqq k \leqq n-J+1$.

It can be verified, by evaluation of the appropriate determinant, that the contents of the projections of the simplices represented by the first row and first column of matrix A are:

$$a_{1,k} = \{(k-1)!\}^{-1}\left\{\prod_{0}^{k-2}{}_i (J, n-i/n-i)\right\}$$
$$= \{(k-1)!\}^{-1}\{(J, n/n)(J, n-1/n-1) \cdots (J, n-k+2/n-k+2)\},$$

or, recursively,

$$a_{1,k} = a_{1,k-1}(J, n-k+2/n-k+2)(k-1)^{-1},$$

for $2 \leqq k \leqq n - J + 1$, and

$$a_{h,1} = \{(h-1)!\}^{-1},$$

for $2 \leqq h \leqq J$, while the element $a_{1,1}$ is indeterminate.

The content of the projection of F_1 is determined as:

$$C[\Phi_1] = C[F_1](n)^{-1/2} = a_{J,n-J+1},$$

the matrix element evaluated last.

Numerical implementation of the algorithm. It will be noted that since the calculation of each element $a_{h,k}$ of matrix A requires knowledge only of the value of the elements immediately above and to the left of it, the actual machine calculation needs to store only one row (or one column) of the matrix. Furthermore, since the quantity ρ' is expressed as $\rho' = C[F_1]/C[S]$, where a factor $n^{1/2}\{(n-1)!\}^{-1}$ appears in the numerator and in the denominator, the calculation can be simplified if one avoids the carrying of the divisor. If, for example, one computes and stores by rows, one obtains:

$$\rho' \leftarrow \alpha_{n-J+1}$$

where a computational rather than mathematical notation has been used and where the result is obtained by the following steps:

(1) $\alpha_1 = 1, \alpha_2 = \alpha_3 = \cdots = \alpha_{n-J+1} = 0.$

(2) Repeat step 3 for $h = 1, 2, 3, \cdots, J$.

(3) $\alpha_k \longleftarrow \alpha_{k-1}\dfrac{|m_{J+1-h}|}{|m_{J+1-h} - m_{n+2-k}|} + \alpha_k\dfrac{|m_{n+2-k}|}{|m_{J+1-h} - m_{n+2-k}|},$

for $k = 2, 3, \cdots, n - J + 1$, where J is defined as in equation (6).

Since α_{n-J+1} as a function of m_i is defined and continuous in the region defined by the following inequalities,

$$m_1 \leqq m_2 \leqq \cdots \leqq m_J \leqq 0 < m_{J+1} \leqq m_{J+2} \leqq \cdots \leqq m_n$$

and since ρ' is also continuous because of its geometrical interpretation, the restrictions instituted before can be removed and the region of applicability of the algorithm extended as shown by the above inequalities.

Except for the first iteration of step 2, the evaluation of the generic α_k involves 6 elementary operations (1 sum, 2 differences, 2 multiplications, and 1 division). Each iteration of step 3, therefore, requires $6(n - J)$ operations, but the first iteration requires only $4(n - J)$. The complete calculation requires M operations:

$$M = 4(n - J) + [6(n - J)] + 1](J - 1) .$$

The maximum value $M_{\max}$ that M can assume for a given n is

$$M_{\max} = (3n^2 - n)/2 - 1 .$$

The storage requirement for the vector α_i is $n - J + 1$ locations.

Appendix. A generic r-dimensional frustum F' given as the intersection of the semispace $\sigma'(\sigma' \equiv \{\xi_1, \xi_2, \cdots, \xi_r \mid \sum_{i=1}^{r} g_i \xi_i \leqq g_0\})$ and the simplex $S'(S' \equiv$ convex hull of P_i, for $i = 1, 2, \cdots, r + 1$, where $P_i \equiv \{q_{i1}, q_{i2}, \cdots, q_{ir}\})$ can be reduced to a frustum F of the specialized form assumed herein by means of the following change of variables:

$$\xi_i = \xi_i^0 + \sum_1^r {}_j \tau_{ij} x_j ,$$

where

$$\xi_i^0 = q_{r+1,i} ,$$
$$\tau_{ij} = q_{ji} - q_{r+1,i} .$$

The semispace σ' transforms into σ' as follows:

$$\sigma \equiv \left\{x_1, x_2, \cdots, x_r \middle| \sum_1^r {}_i g_i \left(\xi_i^0 + \sum_1^r {}_j \tau_{ij} x_j\right) = \sum_1^r {}_i g_i \xi_i^0 + \sum_1^r {}_j x_j \left(\sum_1^r {}_i \tau_{ij} g_i\right) \leqq g_0 \right\} .$$

Using the notation established herein, setting $r + 1 = n$, and using the prime (′) to indicate that the sequence of coefficients p' has to be ordered before applying the algorithm, we obtain

$$p'_j = \sum_1^r {}_i g_i (q_{ji} - q_{r+1,i}) , \qquad j = 1, 2, \cdots, r$$
$$G = g_0 - \sum_1^r {}_i g_i q_{r+1,i} ,$$
$$p'_{r+1} = p'_n = 0 .$$

It can be verified that

$$C[F'] = C[F]\{\det \| \tau_{ij} \|\} = (r!)^{-1} \alpha_{n-J+1} \cdot \det \| \tau_{ij} \| (r + 1)^{1/2}$$

where α_{n-J+1} is computed as shown previously.

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NATIONAL RESEARCH COUNCIL, WASHINGTON, D. C.
Present address: JET PROPULSION LABORATORY, PASADENA, CALIFORNIA