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ON E-COMPACT SPACES AND GENERALIZATIONS OF PERFECT MAPPINGS

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The inverse image preservation problem of R-compact (realcompact) spaces has been studied by R. Blair, N. Dykes, and T. Isiwata. In this paper their results are drawn together, and the inverse images of E-compact spaces under certain kinds of mappings are studied. Actually, a more general question, concerning the notion of E-perfect mappings, is considered. (The inverse image of an E-compact space under an E-perfect mapping is E-compact.) Classes of hereditarily E-compact spaces and their inverse images under certain mappings are also studied.

1. Preliminaries. Throughout this paper spaces are assumed to be Hausdorff and mappings are continuous onto functions. The reader is referred to [9] for basic ideas of E-compact spaces. For convenience we review the terminology and notations. Given two spaces X and E, $C(X, E)$ denotes the set of all continuous functions from X into E. A space X is said to be *E-completely regular* (*E-compact*) provided that X is homeomorphic to a subspace (respectively, closed subspace) of a product E^m for some cardinal m. X is said to be *hereditarily E-compact* provided that every subspace of X is E-compact. A subset A of a space X is said to be *E-embedded* in X provided that every continuous function $f: A \to E$ admits a continuous extention $f^*: X \to E$, and A is an *E-closed subset* of X provided that for some positive integer n there exists a closed subset T of E^n and a continuous function $f: X \to E^n$ such that $A = f^{-1}[T]$. Following Frolik [3], a mapping $\varphi: X \to Y$ (where X and Y are completely regular spaces) is called a *Z-mapping* provided that the image of every zero-set in X is closed in Y, and following Isiwata [7], φ is called a *WZ-mapping* provided that $\mathrm{Cl}_{\beta X}\, \varphi^{-1}(y) = \Phi^{-1}(y)$ for every y in Y, where βX and βY denote the Stone-Čech compactifications of X and Y, respectively, and Φ denotes the Stone extension of φ from βX into βY. A mapping is called *perfect* (*proper*) provided that it is continuous, closed and the inverse images of singletons are compact. It is well known that if X and Y are completely regular spaces, then a mapping $\varphi: X \to Y$ is perfect iff $\Phi[\beta X - X] \subseteqq \beta Y - Y$.

It follows from Theorem 4.14 of [9] that given two E-completely regular spaces X, Y and a continuous function $\varphi: X \to Y$, there exist E-compact extensions $\beta_E X$, $\beta_E Y$ of X, Y, respectively, and a continuous extension $\Phi_E: \beta_E X \to \beta_E Y$ of φ. In the sequel, we shall always use Φ_E to denote such an extension of φ.

Generalizing the notions of Z-mapping, WZ-mapping and perfect mapping, we define the following

DEFINITION 1.1. Let X, Y be E-completely regular spaces, and $\varphi: X \to Y$ be a mapping.

(a) φ is *E-closed* provided that φ maps each E-closed subset of X to a closed subset of Y.

(b) φ is *weakly E-closed* provided that $\mathrm{Cl}_{\beta_E X}\varphi^{-1}(y) = \Phi_E^{-1}(y)$ for each y in Y.

(c) φ is *E-perfect* provided that $\Phi_E[\beta_E X - X] \subseteqq \beta_E Y - Y$.

REMARK. Let I and R denote the spaces of $[0, 1]$ and of all real numbers, respectively. Then the concept of I-closed (weakly I-closed, I-perfect) mapping coincides with that of Z- (WZ-, perfect, respectively) mapping, and the concept of R-perfect mapping coincides with that of real-proper mapping [1].

PROPOSITION 1.2. *A closed mapping is E-closed.*

PROPOSITION 1.3. *If E is a regular space, then an E-closed mapping is weakly E-closed.*

We need the following lemma to prove Proposition 1.3.

LEMMA 1.4. *If E is regular and X is an E-completely regular space, then for each closed subset F of X and each point p in $X - F$, there exists an E-closed subset A of X such that $p \in \mathrm{Int}\, A$ and $A \cap F = \varnothing$* ($\mathrm{Int}\, A$ *denotes the interior of* A).

Proof. Since X is E-completely regular, by [9; Theorem 3.8], there exists a continuous function f from X into E^n for some finite n such that $f(p) \notin \mathrm{Cl}_{E^n} f[F]$. Since E^n is regular, there exist disjoint open neighborhoods U, V of $f(p)$, $\mathrm{Cl}_{E^n} f[F]$, respectively. Let $A = f^{-1}[E^n - V]$. Clearly, $p \in \mathrm{Int}\, A$ and $A \cap F = \varnothing$.

Proof of Proposition 1.3. Let X, Y be E-completely regular spaces and $\varphi: X \to Y$ be an E-closed mapping. Assume that φ is not weakly E-closed. Then there exists a point y in Y and a point p in $\Phi_E^{-1}(y) - \mathrm{Cl}_{\beta_E X}\varphi^{-1}(y)$. By Lemma 1.4, there exists an E-closed subset A of $\beta_E X$ such that $p \in \mathrm{Int}\, A$ and $A \cap \mathrm{Cl}_{\beta_E X}\varphi^{-1}(y) = \varnothing$. Let $M = A \cap X$. Then M is an E-closed subset of X, and hence $\varphi[M]$ is closed in Y. Now $M \cap \varphi^{-1}(y) = \varnothing$, hence $y \notin \varphi[M]$. On the other hand, $p \in \mathrm{Int}\, A \subseteqq \mathrm{Cl}_{\beta_E X}\, \mathrm{Int}\, A = \mathrm{Cl}_{\beta_E X}\, [\mathrm{Int}\, A \cap X] \subseteqq \mathrm{Cl}_{\beta_E X}\, [A \cap X] = \mathrm{Cl}_{\beta_E X}\, M$, hence $y = \Phi_E(p) \in \Phi_E[\mathrm{Cl}_{\beta_E X}\, M] \subseteqq \mathrm{Cl}_{\beta_E Y}\, \Phi_E[M] = \mathrm{Cl}_{\beta_E Y}\, \varphi[M]$. This implies that $y \in \mathrm{Cl}_{\beta_E Y}\, \varphi[M] \cap Y = \mathrm{Cl}_Y\, \varphi[M] = \varphi[M]$ which is a contradiction.

2. E-perfect mappings. The consideration of E-perfect mappings is motivated by the following obvious results.

PROPOSITION 2.1. *Let X, Y be E-completely regular spaces. If Y is E-compact and if there exists an E-perfect mapping from X onto Y, then X is E-compact.*

For a space E we shall let $\mathfrak{C}(E)$ ($\mathfrak{K}(E)$) denote the class of all E-completely regular (respectively, E-compact) spaces. The following theorem is due to Mrówka [9; 4.1].

THEOREM 2.2. *Let E_1, E_2 be two spaces with $\mathfrak{C}(E_1) = \mathfrak{C}(E_2)$. Then $\mathfrak{K}(E_1) \subseteqq \mathfrak{K}(E_2)$ iff $\beta_{E_2} X \subset_{\text{ext}} \beta_{E_1} X$ for each $X \in \mathfrak{C}(E_1)$,* i.e., *there exists a homeomorphism h from $\beta_{E_2} X$ into $\beta_{E_1} X$ such that $h(p) = p$ for each p in X.*

In the following we shall always assume that E_1, E_2 be spaces with $\mathfrak{C}(E_1) = \mathfrak{C}(E_2)$ and $\mathfrak{K}(E_1) \subseteqq \mathfrak{K}(E_2)$. We are ready to show the following

THEOREM 2.3. *Let X, Y be two E_1-completely regular spaces, $\varphi: X \rightarrow Y$ a weakly E_1-closed mapping. Then φ is E_2-perfect iff $\varphi^{-1}(y)$ is closed in $\beta_{E_2} X$ for each y in Y.*

Furthermore, if Y is E_2-compact, then φ is E_2-perfect iff X is E_2-compact.

Proof. Necessity. Since $\Phi_{E_2}[\beta_{E_2} X - X] \subseteqq \beta_{E_2} Y - Y$, for each y in Y, we have $\varphi^{-1}(y) = \Phi_{E_2}{}^{-1}(y)$ which is closed in $\beta_{E_2} X$.

Sufficiency. To show that $\Phi_{E_2}[\beta_{E_2} X - X] \subseteqq \beta_{E_2} Y - Y$, it suffices to show that for any $z \in \beta_{E_2} X$, if $\Phi_{E_2}(z) \in Y$, then $z \in X$. So let $\Phi_{E_2}(z) = y \in Y$. Since φ is weakly E_1-closed, we have

$$\begin{aligned} z \in \Phi_{E_2}{}^{-1}(y) &= \Phi_{E_1}{}^{-1}(y) \cap \beta_{E_2} X = \mathrm{Cl}_{\beta_{E_1} X} \varphi^{-1}(y) \cap \beta_{E_2} X \\ &= \mathrm{Cl}_{\beta_{E_2} X} \varphi^{-1}(y) = \varphi^{-1}(y) \subseteqq X . \end{aligned}$$

Now assume that Y is E_2-compact. If φ is E_2-perfect, by Proposition 2.1, X is E_2-compact. Conversely, if X is E_2-compact, then $\beta_{E_2} X = X$. Thus, $\mathrm{Cl}_{\beta_{E_2} X} \varphi^{-1}(y) = \mathrm{Cl}_X \varphi^{-1}(y) = \varphi^{-1}(y)$ for each y in Y. Hence $\varphi^{-1}(y)$ is closed in $\beta_{E_2} X$ for each y in Y. This shows that φ is E_2-perfect.

Before we give applications of Theorem 2.3, we first show the following

LEMMA 2.4. *Let X, Y be two E-completely regular spaces, $\varphi: X \rightarrow Y$*

a mapping and y an arbitrary point in Y. If $\varphi^{-1}(y)$ is E-compact and E-embedded in X, then $\varphi^{-1}(y)$ is closed in $\beta_E X$.

Proof. Since $\varphi^{-1}(y)$ is E-compact, $\beta_E\varphi^{-1}(y) = \varphi^{-1}(y)$. Consequently, it suffices to show that $\mathrm{Cl}_{\beta_E X}\varphi^{-1}(y) = \beta_E\varphi^{-1}(y)$. First, $\mathrm{Cl}_{\beta_E X}\varphi^{-1}(y)$, being a closed subset of the E-compact space $\beta_E X$, is E-compact. Also, since $\varphi^{-1}(y)$ is E-embedded in X, it is also E-embedded in $\beta_E X$, hence it is E-embedded in $\mathrm{Cl}_{\beta_E X}\varphi^{-1}(y)$. Thus, by Theorem 4.14 (b) of [9], $\mathrm{Cl}_{\beta_E X}\varphi^{-1}(y) = \beta_E\varphi^{-1}(y)$.

As an immediate consequence of Theorem 2.3 and Lemma 2.4 we obtain

THEOREM 2.5. *Let X, Y be two E_1-completely regular spaces, $\varphi: X \to Y$ a weakly E_1-closed mapping. If $\varphi^{-1}(y)$ is E_2-compact and E_2-embedded in X for each y in Y, then φ is E_2-perfect.*

We now turn to the R-compact (realcompact) case. Throughout the remainder of this section spaces are assumed to be Hausdorff and completely regular.

Let $E_1 = I$ and $E_2 = R$ in Theorems 2.3 and 2.5, we obtain

COROLLARY 2.6. *Let $\varphi: X \to Y$ be a WZ-mapping. Then we have*

(a) *φ is R-perfect iff $\varphi^{-1}(y)$ is closed in the Hewitt realcompactification νX of X for each y in Y.*

(b) *Let Y be R-compact. Then φ is R-perfect iff X is R-compact.*

(c) *If $\varphi^{-1}(y)$ is R-compact and R-embedded in X for each y in Y, then φ is R-perfect.*

A subset X_0 of a space X is said to be z-embedded in X provided that for every zero-set Z in X_0 there exists a zero-set Z' in X such that $Z = Z' \cap X_0$, and X_0 is said to be *R^*-embedded* in X provided that every bounded continuous real-valued function on X_0 admits a continuous real-valued extension to X. It is easy to show that every R-embedded subset of X is R^*-embedded in X and every R^*-embedded subset in X is z-embedded in X. Conversely, it was shown in [5] that every z-embedded subset X_0 of X which is completely separated from every zero-set disjoint from it is R-embedded. Furthermore, it was shown in [6] that every Lindelöf space X_0 in X is z-embedded in X. We have the following

LEMMA 2.7. *Let $\varphi: X \to Y$ be a Z-mapping, y an arbitrary point of Y. If $\varphi^{-1}(y)$ is z-embedded in X, then $\varphi^{-1}(y)$ is R-embedded in X.*

Proof. It suffices to show that $\varphi^{-1}(y)$ is completely separated from every zero-set disjoint from it. Let Z be such a zero-set. Then $y \notin \varphi[Z]$ and $\varphi[Z]$ is closed in Y. Hence there exists an $f \in C(Y, R)$ such that $f(y) = 0$ and $f[\varphi[Z]] = 1$. Therefore, $f \circ \varphi \in C(X, R)$ and $(f \circ \varphi)[Z] = 1$, $(f \circ \varphi)[\varphi^{-1}(y)] = 0$.

With the above information the following corollary is easily obtained.

COROLLARY 2.8. *Let $\varphi: X \rightarrow Y$ be a Z-mapping. If one of the following conditions holds, then φ is R-perfect.*

(a) *$\varphi^{-1}(y)$ is R-compact and z-embedded in X for each y in Y.*

(b) *$\varphi^{-1}(y)$ is R-compact and R^*-embedded in X for each y in Y.*

(c) *X is normal and $\varphi^{-1}(y)$ is R-compact for each y in Y.*

(d) *$\varphi^{-1}(y)$ is Lindelöf for each y in Y.*

REMARK. It should be pointed out that somewhat more restricted forms of Corollary 2.8 can be found in [1], [2] and [7]. In particular, (a) can be found in [1], (b) in [7], (c) in [1], [7] and (d) in [1] and [2].

3. **Hereditarily E-compact spaces.** In this section we give several characterizations of certain classes of hereditarily E-compact spaces. As by-products of the characterizations, sufficient conditions for the preservation of inverse images of hereditarily E-compact spaces are derived. The space E in this section will be assumed to have a continuous binary operation θ and two fixed distinct points e_0 and e_1 satisfying the following conditions:

(i) $e\theta e_0 = e_0$, $e\theta e_1 = e$ for each e in E.

(ii) For every closed subset A of E^n(n is a finite positive integer) and every point p in $E^n - A$, there exists an $f \in C(E^n, E)$ such that $f[A] = e_0$ and $f(p) = e_1$.

(iii) For every two disjoint closed subsets A, B of E, there exists a $g \in C(E, E)$ such that $g[A] = e_0$ and $g[B] = e_1$.

It is easy to see that a space E which satisfies (ii) (iii) is regular (respectively, normal).

In the sequel all spaces are assumed to be E-completely regular. For convenience we state two lemmas from [10] which are needed for the proof of Theorem 3.3. We note that conditions (i), (ii) and (iii) of the space E are essential for the proof of these lemmas.

LEMMA 3.1. *In an E-completely regular space, the union of a compact subspace with an E-compact subspace is E-compact.*

LEMMA 3.2. *If X is an E-completely regular space which is the union of finitely many E-compact subspaces, each of which is E-embedded in X except at most one, then X is E-compact.*

We are now ready to prove the main theorem of this section.

THEOREM 3.3. *The following conditions on an E-completely regular space Y are equivalent.*

(1) *Y is hereditarily E-compact.*

(2) *$Y - \{y\}$ is E-compact for each y in Y.*

(3) *For every space X, if there exists a mapping $\varphi: X \to Y$ such that $\varphi^{-1}(y)$ is compact for each y in Y, then X is E-compact.*

(4) *For every space X, if there exists a one-to-one mapping $\varphi: X \to Y$, then X is E-compact.*

(5) *For every space X, if there exists a mapping $\varphi: X \to Y$ such that $\varphi^{-1}(y)$ can be expressed as the union of finitely many E-compact, E-embedded subspaces of X for each y in Y, then X is E-compact.*

(6) *For every space X, if there exists a mapping $\varphi: X \to Y$ such that $\varphi^{-1}(y)$ is an E-compact, E-embedded subspace of X for each y in Y, then X is E-compact.*

Proof. It is obvious that (1) implies (2), (3) implies (4) and (5) implies (6).

(2) implies (3). Let X and φ satisfy the assumptions of (3). It follows from (2) and Lemma 3.1 that Y is E-compact. Hence φ admits a continuous extension $\Phi_E: \beta_E X \to Y$. Now consider any point y in Y. By 4.9 of [9], the set $X_0 = \Phi_E^{-1}[Y - \{y\}]$ is E-compact, hence by Lemma 3.1 again, the set $X_0 \cup \varphi^{-1}(y)$ is also E-compact. Since $X \subseteqq X_0 \cup \varphi^{-1}(y) \subseteqq \beta_E X$, we have $X_0 \cup \varphi^{-1}(y) = \beta_E X$. In other words, Φ_E maps no point of $\beta_E X - X$ to y. As this holds for every y in Y, we have $\beta_E X - X = \emptyset$. This shows that X is E-compact.

(4) implies (1). Let F be any subspace of Y. Enlarge the topology of Y by making both F and $Y - F$ open. The new space X thus defined is E-completely regular and the relative topology on F is the same in X as in Y. Since the identity function from X onto Y is continuous, (4) implies that X is E-compact. Therefore F, which is a closed subset of X, is also E-compact.

(2) implies (5). This is analogous to the proof of (2) implies (3). Here instead of using Lemma 3.1, we apply Lemma 3.2 to show that $X_0 \cup \varphi^{-1}(y)$ is E-compact.

(6) implies (4). If φ is a one-to-one mapping, then for each y in Y, $\varphi^{-1}(y)$, which is a singleton, is clearly E-compact and E-embedded in X.

As an immediate consequence of Theorem 3.3, we have

COROLLARY 3.4. *Let Y be a hereditarily E-compact space. If there exists a mapping $\varphi: X \to Y$ which satisfies one of the following conditions, then X is hereditarily E-compact.*

(1) *For each y in Y, $\varphi^{-1}(y)$ is finite.*

(2) *φ is one-to-one.*

(3) *For each y in Y, $\varphi^{-1}(y)$ can be expressed as follows: $\varphi^{-1}(y) = F_1 \cup \cdots \cup F_n$ (n is a finite positive integer) where F_i is hereditarily E-compact and each subspace of F_i is E-embedded in X for $i = 1, \cdots, n$.*

(4) *For each y in Y, $\varphi^{-1}(y)$ is hereditarily E-compact and each subspace of $\varphi^{-1}(y)$ is E-embedded in X.*

REMARK. It is obvious that the space R and N (N denotes the discrete space of nonnegative integers) satisfy conditions (i), (ii) and (iii). Therefore, all results in this section hold true for hereditarily R-compact and hereditarily N-compact spaces. In fact, for $E = R$, the equivalence of (1), (2), (3) and (4) of Theorem 3.3 was proved by Gillman and Jerison [4, p. 122]; the equivalence of (1) and (6) was proved by Blair [1, 3.1]; Corollary 3.4(2) was proved by Shirota [11, Theorem 6] and Corollary 3.4(4) was proved by Blair [1, 3.2]. For $E = N$, Corollary 3.4(2) was proved by Mrówka [8, p. 599].

To see that our results in this section are applicable to other classes of hereditarily E-compact spaces, we state the following

THEOREM 3.5. *All results in this section hold true if E is an arbitrary 0-dimensional chain.*

This theorem follows immediately from the following lemmas whose proofs can be found in [10].

LEMMA 3.6. *Every 0-dimensional chain which has first and last elements satisfies conditions* (i), (ii) *and* (iii) *of the space E.*

LEMMA 3.7. *Let X_0 be an E-embedded subspace of a spce X, and E' be a space homeomorphic to a subspace of E^m for some cardinal m. If E' is a retract of E^m, then X_0 is E'-embedded in X.*

LEMMA 3.8. *For every 0-dimensional chain E, there exists a 0-dimensional chain E' which has first and last elements such that*

(1) $\mathfrak{R}(E) = \mathfrak{R}(E')$,

(2) *E' is a retract of E^2 (hence every E-embedded subspace of a space X is E'-embedded in X).*

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