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CONCERNING DENTABILITY

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It is shown that c_0 contains a closed and bounded convex body which is dentable but fails to have extreme points. On the other hand, there exists a strictly convex, closed, symmetric, convex body which fails to be dentable. (Thus dentability is, in general, unrelated to extremal structure.)

1. In [2], Rieffel introduced the notion of dentability for a subset K of a Banach space X. Rephrased, it reads:

1.1. K is dentable if, for every $\varepsilon > 0$, there is an $x \in K$ and an $f \in X^*$ such that some hyperplane determined by f separates x from $K_\varepsilon = K \sim \overline{B(x, \varepsilon)}$, where $B(x, \varepsilon)$ is the ball of radius ε about x.

One of the questions asked by Rieffel [Ibid., p. 77] is whether a closed and bounded convex set exists in some Banach space which is dentable but has no strongly exposed points. We answer this question by exhibiting a dentable symmetric closed convex body in c_0 which has no extreme points at all. To further show that the connection between dentability and extreme structure can be quite tenuous, we also exhibit in c_0 a strictly convex body which (in spite of the fact that each boundary part is exposed) is not dentable.

Another question of Rieffel, namely, whether each weakly compact subset of a Banach space is dentable has recently been answered in the affirmative by Troyanski [3]. The example of the unit ball in the conjugate Banach space m is used by us (Proposition 3) to show that, in contrast to the above, a weak*-compact set need not be dentable.

2. Dentability properties of certain subsets of c_0 and m.

PROPOSITION 1. *There is a dentable closed and bounded convex body in c_0 which has no extreme point.*

Proof. For $n = 1, 2, \cdots$ set $B_n = B((2 - 2^{1-n})e_n, 2^{1-n})$, where $e_n = \{x_i\} \in c_0$ with $x_n = 1$, $x_i = 0$ for $i \neq n$. Let $C_n = (-B_n) \cup B_n$ and $C = \overline{co}\,(\bigcup_{n=1}^\infty C_n)$. We claim that C has the desired properties.

(i) C has no extreme points.

Suppose, for a contradiction, that C has an extreme point

$$y = (y_1, y_2, \cdots)\,.$$

Clearly, $\|y\| > 1$ (since $\bar{C}_1$ contains the unit ball) and without restriction of generality we may assume that $\|y\| = y_k$ for some k. Let $\{u^{(m)}\}$ be a sequence in $co\{\bigcup_{n=1}^{\infty} C_n\}$ converging to y with

$$(1) \qquad \|u^{(m)} - y\| < \min(y_k - 1, 2^{-k-2}) \qquad (m = 1, 2, \cdots) .$$

Write

$$(2) \qquad u^{(m)} = \sum_{i=1}^{l} \lambda_i u^{(mi)}$$

with $u^{(mi)} \in C_i$, $\lambda_i \geqq 0$ $(i = 1, 2, \cdots, l)$, and $\sum_{i=1}^{l} \lambda_i = 1$. It is clear from the definition of the B_i that, for $i > k$, $u_k^{(mi)} \leqq 2^{1-i} \leqq 2^{-k}$, where $u_k^{(mi)}$ is the kth coordinate of $u^{(mi)}$.

Thus, by (1),

$$1 < u_k^{(m)} = \sum_{i=1}^{k} \lambda_i u_k^{(mi)} + \sum_{i=k+1}^{l} \lambda_i u_k^{(mi)} \leqq 2 \sum_{i=1}^{i=k} \lambda_i + 2^{-k}\left(1 - \sum_{i=1}^{k} \lambda_i\right) .$$

It follows that

$$(3) \qquad \sum_{i=1}^{k} \lambda_i > \frac{1 - 2^{-k}}{2 - 2^{-k}} > \frac{1}{2} - \frac{1}{2^{k+1}} \geqq \frac{1}{4} .$$

Now let j be a positive integer with the property that $|y_j| < 2^{-k-3}$. To show that y, contrary to assumption, cannot be an extreme point, we exhibit two points $\bar{y}$ and $\underline{y}$ in C such that $\bar{y}_j > y_j > y_{-j}$ with all other coordinates of these points equal. To this end define $\{\bar{u}^{(m)}\}$ and $\{\underline{u}^{(m)}\}$ as follows.

Using (2), set

$$\bar{u}_n^{(mi)} = \underline{u}_n^{(mi)} = u_n^{(mi)}$$

for $m = 1, 2, \cdots, j$; $n \neq j$, $i = 1, 2, \cdots, l$;

$$\bar{u}_j^{(mi)} = -\underline{u}_j^{(mi)} = \begin{cases} 2^{-k} & \text{for } i \leqq k \\ 0 & \text{for } i > k \end{cases} .$$

It follows from (3) that

$$\bar{u}_j^{(m)} = -\underline{u}_j^{(m)} \geqq 2^{-k-2} .$$

Thus, $\bar{u}_j^{(m)} \geqq y_j + 2^{-k-3}$ and $\underline{u}_j^{(m)} \leqq y_j - 2^{-k-3}$. It is now obvious that $\{\bar{u}^{(m)}\}$ and $\{\underline{u}^{(m)}\}$ converge to points $\bar{y}$ and $\underline{y}$, respectively, having the desired properties. This completes the proof that C has no extreme points.

(ii) C is dentable.

Let $\varepsilon > 0$ be given and choose n so that $2^{2-n} < \varepsilon$. We show that $\overline{co}(C \sim B)$ wehre $B = B(2e_n, \varepsilon)$ does not contain $2e_n \in C$.

To this end, consider the set $H^{(n)} = \{x \in co\,(\bigcup_{n=1}^{\infty} C_n)\colon x_n \geqq 2 - 2^{-n}\}$. Any member h of $H^{(n)}$ can be represented in the form $h = \sum_{i=1}^{m} \lambda_i x^i$ with $\lambda_i \geqq 0$, $\sum_{i=1}^{m} \lambda_i = 1$ and $x_i \in C_i$, $i = 1, 2, \cdots, m$; $m \geqq n$. Now, by definition, $h_n = \sum_{i=1}^{m} \lambda_i x_n^i \geqq 2 - 2^{-n}$. On the other hand,

$$h_n = \lambda_n x_n^n + \sum_{i \neq m} \lambda_i x_n^i \leqq \lambda_n x_n^n + (1 - \lambda_n)$$
$$= \lambda_n(x_n^n - 1) + 1 \leqq \lambda_n + 1 .$$

It follows that $\lambda_n \geqq 1 - 2^{-n}$. Consequently,

$$\| 2e_n - h \| \leqq 2^{2-n} \quad (h \in H^{(n)}) ,$$

for $|(2e_n)_n - h_n| \leqq |2 - (2 - 2^{-n})| = 2^{-n}$ and, for $k \neq n$,

$$(2e_n - h)_k = |\sum \lambda_i x_k^i| \leqq 1 - \lambda_n \leqq 2^{2-n} .$$

Thus $B(2e_n, \varepsilon)$ contains $H^{(n)}$ and clearly, $\overline{C \sim H^{(n)}}$ is convex with $2e_n \notin \overline{C \sim H^{(n)}}$. We have shown that C is dentable completing thereby the proof of the proposition.

PROPOSITION 2. *In c_0 there exists a symmetric, closed and bounded convex body which is strictly convex and fails to be dentable.*

Proof. Let

$$C = \left\{x \in c_0\colon \ \| x \| + \left(\sum_{n-1}^{\infty} 2^{-n} x_n^2\right)^{1/2} \leqq 1\right\} .$$

It is well-known (cf. [1, p. 362]) that C defines an equivalent strictly convex norm and, therefore, only the nondentability has to be shown. We note that for $x = (x_1, x_2, \cdots, x_n, \cdots) \in \mathrm{bdry} C$, we have $\| x \| \geqq 1/2$ so that for such an x there is an integer m with $|x_m| = \| x \| \geqq 1/2$. Let $1/4 > \varepsilon > 0$ and choose $0 < \delta < \varepsilon/2$ small enough so that $\| x \| = \| x' \| + \delta$ if x' is the vector obtained from x by replacing each coordinate x_i, with $|x_i| = \| x \|$, by $|x_i| - \delta$. Next, let k be large enough so that $|x_k| < \delta$ and

$$\left(\sum_{n \neq k} 2^{-n} x_n^2 + \frac{1}{2^{k+4}}\right)^{1/2} \leqq \left(\sum_{n=1}^{\infty} 2^{-n} x_n^2\right)^{1/2} + \delta .$$

To prove nondentability, it clearly suffices to exhibit $u, v \in C$ such that $\| (u + v)/2 - x \| < \delta$ and $\| u - v \| \geqq 1/2$. To this end, set $u_i = v_i = x_i$ for those $i \neq k$ for which $|x_i| < \| x \|$; $u_k = -v_k = 1/4$; and $u_j = v_j = x_j - \delta\, x_j/|x_j|$, otherwise. Since $\| u \| = \| v \| = \| x \| - \delta$ and

$$\left(\sum_{n=1}^{\infty} 2^{-n} u_n^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} 2^{-n} v_n^2\right)^{1/2} \leqq \left(\sum_{n=1}^{\infty} 2^{-n} x_n^2\right)^{1/2} + \delta ,$$

we have $u, v \in C$. Also, $\|(u+v)/2 - x\| < \delta$, since $|((u+v)/2 - x)_k| = |x_k| < \delta$, and, for all coordinates $j \neq k$ at which u, v and x are distinct, we have $|((u+v)/2 - x)_j| = \delta$. Finally,

$$\|u - v\| = \|u_k - v_k\| = \frac{1}{2}\,.$$

PROPOSITION 3. *The unit ball in m is not dentable.*

Proof. Let $0 < \varepsilon < 1/4$ and $x = (x_1, x_2, \cdots) \in m$ with $\|x\| \leqq 1$. Either (i) there is an integer k with $|x_k| \leqq 1/4$, or (ii) for every index j, $|x_j| > 1/4$.

In case (i), define $\bar{x}$ and $\underline{x}$ by setting

$$\bar{x} = \left(x_1, x_2, \cdots, x_k + \frac{1}{4}, \cdots\right)$$

$$\underline{x} = \left(x_1, x_2, \cdots, x_k - \frac{1}{4}, \cdots\right)$$

so that $(1/2)(\bar{x} + \underline{x}) = x$ and $\|\bar{x} - \underline{x}\| = 1/2 > \varepsilon$.

In case (ii), define

$$x^{(i)} = \left(x_1, x_2, \cdots, x_i - \frac{x_i}{4\,|x_i|}, \cdots\right) \qquad (i = 1, 2, \cdots)\,,$$

so that $\|x - x^{(i)}\| = 1/4$.

Now, $x \in \overline{co}\,\{x^{(i)}: i = 1, 2, \cdots\}$. For,

$$\left(x - \frac{1}{j}\sum_{n=1}^{j} x^{(n)}\right)_k = \begin{cases} 0, & \text{if } k > j \\ \dfrac{1}{j}\left(x_k - \dfrac{x_k}{4\,|x_k|}\right) \end{cases}$$

showing that $(1/j)\sum_{n=1}^{j} x^{(n)} \to x$. Thus, the dentability condition fails, proving the proposition.

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