PACIFIC JOURNAL OF MATHEMATICS
Vol, 47, No. 1, 1973

A VERY WEAK TOPOLOGY FOR THE MIKUSINSKI FIELD OF OPERATORS

HARRIS S. SHULTZ

Using a generalized Laplace transformation the Mikusinski field is given a topology T such that sequences which converge in the sense of Mikusinski converge with respect to T, such that the mapping $q \rightarrow q^{-1}$ is continuous and such that the series $\sum (-\lambda)^n s^n / n!$ converges to the translation operator $e^{-\lambda s}$.

In [3] it is shown that the notion of convergence defined in [8] for the Mikusinski field of operators is not topological. Topologies for the Mikusinski field are given in [1], [3], and [9]. In the present paper we endow this field with a topology T such that sequences which converge in the sense of Mikusinski converge with respect to T, such that the identity

$$e^{-\lambda s} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} s^k \qquad (\lambda > 0)$$

holds and such that the mapping $q \rightarrow q^{-1}$ is continuous. The author wishes to acknowledge that this paper constitutes proofs of assertions proposed by Gregers Krabbe [7].

Let L denote the family of complex-valued functions which are locally integrable on $[0, \infty)$. Under addition and convolution L is an integral domain. If Q denotes the quotient field of L then Q is the Mikusinski field of operators. Elements of Q will be denoted $\{f(t)\}$: $\{g(t)\}$ and the injection of L into Q will be denoted $f \rightarrow \{f(t)\}$. We define S to be the set of all f in L for which the integral

$$\int_0^\infty e^{-zt} f(t) dt$$

converges for some z. For f in S let

$$\bar{f}(z) = \int_0^\infty e^{-zt} f(t) dt$$

and $\bar{S} = \{\bar{f} \colon f \in S\}$. Each element of $\bar{S}$ is holomorphic in some right half-plane. Let B denote the set of all sequences $(\bar{f}_n)$ of nonzero elements of $\bar{S}$ for which there exists f in L such that

$$(1) \qquad f_n = f \text{ on } (0, n) \qquad \text{for all } n .$$

For a given f the set of all elements of B satisfying (1) will be denoted $\hat{f}$. Let B^* denote the set of all elements $(\bar{g}_n)$ of B such that $(\bar{g}_n) \in \hat{g}$ where g is a nonzero element of L. Finally, let X denote the set of all sequences $(\bar{f}_n/\bar{g}_n)$ where $(\bar{f}_n) \in B$ and $(\bar{g}_n) \in B^*$. Then X consists of sequences of functions which are meromorphic in some right half-plane.

LEMMA. *Let $(\bar{f}_n) \in \hat{f}$, $(\bar{g}_n) \in \hat{g}$, $(\bar{F}_n) \in \hat{F}$ and $(\bar{G}_n) \in \hat{G}$ and suppose that g and G are nonzero elements of L. Then $f_n^* G_n = F_n^* g_n$ on $(0, n)$ for all n if and only if $\{f\{t\}\}$: $\{g(t)\} = \{F(t)\}$: $\{G(t)\}$.*

Proof. Since $f_n^* G_n = f^* G$ on $(0, n)$ and $F_n^* g_n = F^* g$ on $(0, n)$, the statements $f_n^* G_n = F_n^* g_n$ on $(0, n)$ for all n and $f^* G = F^* g$ are equivalent.

THEOREM 1. *There exists a mapping Φ of X onto Q such that if q belongs to Q, say $q = \{f(t)\}$: $\{g(t)\}$, and if $(\bar{f}_n)$ and $(\bar{g}_n)$ belong, respectively, to $\hat{f}$ and $\hat{g}$, then $\Phi((\bar{f}_n/\bar{g}_n)) = q$.*

Proof. Let $(\bar{f}_n/\bar{g}_n) \in X$. If $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$ define

$$\Phi((\bar{f}_n/\bar{g}_n)) = \{f(t)\}: \{g(t)\} .$$

If $(\bar{f}_n/\bar{g}_n) = (\bar{F}_n/\bar{G}_n)$ then $\bar{f}_n \bar{G}_n = \bar{F}_n \bar{g}_n$ (all n), that is, $\overline{f_n^* G_n} = \overline{F_n^* g_n}$ (all n). Therefore, $f_n^* G_n = F_n^* g_n$ (all n) and hence, by the lemma,

$$\Phi((\bar{F}_n/\bar{G}_n)) = \Phi((\bar{f}_n/\bar{g}_n)) .$$

Thus, Φ is well-defined. Now, for any $q \in Q$ there exist f and g in L such that $q = \{f(t)\}$: $\{g(t)\}$. Let $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$. Then $(\bar{f}_n/\bar{g}_n) \in X$ and $\Phi((\bar{f}_n/\bar{g}_n)) = q$. Therefore, Φ is "onto."

For each nonempty open subset Ω of the complex plane let $M(\Omega)$ denote the set of all functions which are meromorphic in Ω. We equip $M(\Omega)$ with the topology of uniform convergence on compact subsets of Ω with respect to the chordal metric. Thus $\varphi_\mu \to \varphi$ in $M(\Omega)$ if and only if

$$\lim_{\mu} \left[\sup_{z \in K} \frac{|\varphi_\mu(z) - \varphi(z)|}{\sqrt{1 + |\varphi_\mu(z)|^2} \sqrt{1 + |\varphi(z)|^2}} \right] = 0$$

for all compact subsets K of Ω. Let $M = \bigcup M(\Omega)$ where Ω varies over the nonempty open subsets of the complex plane and equip M with the finest topology for which all of the injections $M(\Omega) \to M$ are continuous. Let Y denote the set of all sequences in M and equip Y with the product topology. We may then endow its subset X with the relative topology. Finally, Q is given the quotient topology (relative to Φ and the topology of X). Let T denote this topology. Thus, T is the finest topology on Q for which the function $\Phi\colon X \to Q$ is continuous.

THEOREM 2. *If q_k converges to q in the sense of Mikusinski then q_k converges to q with respect to the topology T.*

Proof. Suppose q_k converges to q in the sense of Mikusinski. Then there exists g, f and f_k $(k = 1, 2, \cdots)$ in L such that $\{g(t)\}q = \{f_k(t)\}$ and $\{g(t)\}q = \{f(t)\}$ and such that f_k converges to f uniformly on compact subsets of $[0, \infty)$. Define

$$\bar{f}_{k,n}(z) = \int_0^n e^{-zt} f_k(t) dt$$

and

$$\bar{f}_n(z) = \int_0^n e^{-zt} f(t) dt \,.$$

Then $(\bar{f}_{k,n}) \in \hat{f}_k$ and $(\bar{f}_n) \in \hat{f}$. Moreover, $\bar{f}_{k,n}$ and $\bar{f}_n$ are entire functions and

$$\lim_{k \to \infty} [\sup_{z \in K} |\bar{f}_{k,n}(z) - \bar{f}_n(z)|] = 0 \qquad (n = 1, 2, \cdots)$$

for any compact set K. Let $(\bar{g}_n) \in \hat{g}$ and, for each n, choose a nonempty open set Ω_n such that $\bar{g}_n$ is holomorphic and nonvanishing in Ω_n. Then $\bar{f}_{k,n}/\bar{g}_n$ is holomorphic in Ω_n and

$$\lim_{k \to \infty} \bar{f}_{k,n}/\bar{g}_n = \bar{f}_n/\bar{g}_n$$

in $M(\Omega_n)$ and therefore in M. Thus,

$$\lim_{k \to \infty} (\bar{f}_{k,n}/\bar{g}_n) = (\bar{f}_n/\bar{g}_n) \quad \text{in } X \,.$$

But $\Phi((\bar{f}_{k,n}/\bar{g}_n)) = q_k$ and $\Phi((\bar{f}_n/\bar{g}_n)) = q$ by Theorem 1. Therefore, since Φ is continuous, it follows that

$$\lim_{k \to \infty} q_k = q \,.$$

Let us define

$$h_\beta(t) = \frac{t^{\beta-1}}{(\beta-1)!} \qquad (\beta = 1, 2, \cdots)$$

$$s^0 = \text{the identity element of } Q$$

$$s^\beta = \{h_\beta(t)\}^{-1} \qquad (\beta = 1, 2, \cdots) \,.$$

We also define $e^{-\lambda s} = s\{f(t)\}$, where

$$f(t) = \begin{cases} 0 & 0 \leqq t < \lambda \\ 1 & 0 < \lambda \leqq t\,. \end{cases}$$

Then s is the *differential* operator and $e^{-\lambda s}$ is the *translation* operator.

THEOREM 3. $e^{-\lambda s} = \sum_{k=0}^{\infty} (-\lambda)^k/k!\, s^k$.

Proof. If f and h_β are defined as above then $\bar{f}(z) = e^{-\lambda z}/z$ and $\bar{h}_\beta(z) = z^{-\beta}$ $(\beta = 1, 2, \cdots)$. Let

$$\varphi_k(z) = \frac{(-\lambda)^k}{k!} z^k \qquad (k = 0, 1, 2, \cdots)\,.$$

Then

$$\frac{\bar{f}(z)}{\bar{h}_1(z)} = e^{-\lambda z} = \sum_{k=0}^{\infty} \varphi_k(z)$$

where the convergence is uniform on compact sets. Therefore,

$$\bar{f}/\bar{h}_1 = \sum_{k=0}^{\infty} \varphi_k \qquad \text{(convergence in } M)\,.$$

That is,

$$\bar{f}/\bar{h}_1 = \lim_{N\to\infty} \sum_{k=0}^{N} \varphi_k \qquad \text{(convergence in } M)\,.$$

Thus,

$$(\bar{f}/\bar{h}_1, \bar{f}/\bar{h}_1, \cdots) = \lim_{N\to\infty} \left(\sum_{k=0}^{N} \varphi_k, \sum_{k=0}^{N} \varphi_k, \cdots \right)$$

where the convergence is in X. But $\Phi((\bar{f}/\bar{h}_1, \bar{f}/\bar{h}_1, \cdots)) = e^{-\lambda s}$ and

$$\Phi\left(\left(\sum_{k=0}^{N} \varphi_k, \sum_{k=0}^{N} \varphi_k, \cdots \right) \right) = \sum_{k=0}^{N} \frac{(-\lambda)^k}{k!} s^k\,.$$

Since Φ is continuous it follows that

$$e^{-\lambda s} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-\lambda)^k}{k!} s^k\,.$$

Let Q^* denote the set of nonzero elements of Q and define Γ: $Q^* \to Q^*$ by the equation $\Gamma(q) = q^{-1}$ (all q in Q^*).

THEOREM 4. *The function Γ is continuous.*

Proof. Let $X^* = \{x \in X\colon \Phi(x) \in Q^*\}$. Since Q^* has the quotient topology (relative to Φ and the topology of X^*) it suffices to show that the composition $\Gamma \circ \Phi$ is continuous [5, p. 95, Theorem 9]. Suppose x_μ is a net in X^* which converges to x in X^*. Let $x_\mu = (\bar{f}_{\mu,n}/\bar{g}_{\mu,n})$ and $x = (\bar{f}_n/\bar{g}_n)$. If $(\bar{f}_{\mu,n}) \in \hat{f}$ then $f_\mu \neq 0$ (since $x_\mu \in X^*$) and therefore $(\bar{f}_{\mu,n}) \in B^*$. Similarly, $(\bar{f}_n) \in B^*$. Therefore, $(\bar{g}_{\mu,n}/\bar{f}_{\mu,n})$ and $(\bar{g}_n/\bar{f}_n)$ belong to X^*. Since $x_\mu \rightarrow x$ it follows that $\bar{f}_{\mu,n}/\bar{g}_{\mu,n} \rightarrow \bar{f}_n/\bar{g}_n$ in M for each n. Therefore, for each n there exists Ω_n such that

$$\bar{f}_{\mu,n}/\bar{g}_{\mu,n} \longrightarrow \bar{f}_n/\bar{g}_n$$

in $M(\Omega_n)$. Since the reciprocals $\bar{g}_{\mu,n}/\bar{f}_{\mu,n}$ and $\bar{g}_n/\bar{f}_n$ are also meromorphic in Ω_n, the identity

$$\frac{\left|\frac{1}{z} - \frac{1}{w}\right|}{\sqrt{1 + \left|\frac{1}{z}\right|^2}\sqrt{1 + \left|\frac{1}{w}\right|^2}} = \frac{|z - w|}{\sqrt{1 + |z|^2}\,\sqrt{1 + |w|^2}}$$

implies that $\bar{g}_{\mu,n}/\bar{f}_{\mu,n} \rightarrow \bar{g}_n/\bar{f}_n$ in $M(\Omega_n)$ and therefore in M. Since this is true for each n it follows that $(\bar{g}_{\mu,n}/\bar{f}_{\mu,n}) \rightarrow (\bar{g}_n/\bar{f}_n)$ in X^*. Therefore, $\Phi((\bar{g}_{\mu,n}/\bar{f}_{\mu,n})) \rightarrow \Phi((\bar{g}_n/\bar{f}_n))$ in Q^*. But, by Theorem 1, $\Phi((\bar{g}_{\mu,n}/\bar{f}_{\mu,n})) = \Gamma(\Phi(x_\mu))$ and $\Phi((\bar{g}_n/\bar{f}_n)) = \Gamma(\Phi(x))$. Therefore,

$$\Gamma(\Phi(x_\mu)) \longrightarrow \Gamma(\Phi(x)) ,$$

from which we may conclude that the function $\Gamma \circ \Phi$ is continuous.

References

1. J. Anthony, *Topologies for quotient fields of commutative integral domains*, Pacific J. Math., **36** (1971), 585-601.

2. L. Berg, *Asymptotische Auffassung der Operatorenrechung*, Studia Math., **21** (1962), 215-229.

3. T. K. Boehme, *On Mikusinski operators*, Studia Math., **33** (1969), 127-140.

4. W. A. Ditkin, *On the theory of operational calculus*, Dokl. Akad. Nauk SSSR, **123** (1958), 395-396.

5. J. Kelley, *General Topology*, Van Nostrand, Princeton, New Jersey, 1955.

6. G. Krabbe, *Ratios of Laplace transforms, Mikusinski operational calculus*, Math. Annalen, **162** (1966), 237-245.

7. ———, *A very weak topology for the Mikusinski field*, Notices Amer. Math. Soc., **19** (1972), 358.

8. J. Mikusinski, *Operational Calculus*, Pergamon Press, New York, 1959.

9. R. Struble, *A genuine topology for the field of Mikusinski operators*, Canad. Math. Bull., **11** (1968), 297-299.

Received April 26, 1972.

CALIFORNIA STATE UNIVERSITY, FULLERTON