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APOSYNDETIC PROPERTIES OF HYPERSPACES

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Let X be a compact connected metric space and $2^X(C(X))$ denote the hyperspace of closed subsets (subcontinua) of X. In this paper the hyperspaces are investigated with respect to the property of aposyndesis. The main result states that each of 2^X and $C(X)$ is aposyndetic. If X is semi-aposyndetic, then each of 2^X and $C(X)$ is mutually aposyndetic. An example is given of a non-semi-aposyndetic continuum for which $C(X)$ is not mutually aposyndetic. In an extension of the main result for $C(X)$ it is shown that $C(X)$ is countable closed set aposyndetic. The techniques utilize the partially ordered structure of 2^X and $C(X)$.

A *continuum* will be a compact connected metric space and X will denote a continuum throughout. Each of 2^X and $C(X)$ is endowed with the finite (Vietoris) topology and since X is a continuum each of 2^X and $C(X)$ is also a continuum (see [5]). If $A_1, \cdots, A_n$ are subsets of X, then $N(A_1, \cdots, A_n) = \{B \in 2^X \mid \text{ for each } i = 1, \cdots, n, B \cap A_i \neq \emptyset, \text{ and } B \subseteqq \bigcup_{i=1}^n A_i\}$. If n is a positive integer, $F_n(X) = \{B \in 2^X \mid B \text{ has at most } n \text{ elements}\}$ and $F(X) = \bigcup_{n=1}^\infty F_n(X)$.

For notational purposes, small letters will denote elements of X, capital letters will denote subsets of X and elements of 2^X, and script letters will denote subsets of 2^X. If $A \subseteqq X$, then A^* (int A) (bd A) will denote the closure (interior) (boundary) of A in X.

The concept of aposyndesis was introduced by F. Burton Jones [3] and several extensions of this concept have been studied. Let $p, q \in X, p \neq q$. X is *aposyndetic at p with respect to q* provided there exists a continuum M such that $p \in \text{int } M$ and $q \in X - M$. If for each $q \in X - p$, X is aposyndetic at p with respect to q, then X is *aposyndetic at p*. If X is aposyndetic at each of its points then X is *aposyndetic*. X is *semi-aposyndetic* if for each pair (p, q) of distinct elements of X, X is aposyndetic at p with respect to q or X is aposyndetic at q with respect to p (L. E. Rogers [9]). X is *mutually aposyndetic* if for each pair (p, q) of distinct elements of X there exist disjoint continua M and N such that $p \in \text{int } M$ and $q \in \text{int } N$ (Hagopian [2]). Let $\mathscr{F}$ be a collection of closed subsets of X. Then X is *$\mathscr{F}$-aposyndetic* if for each $x \in X$ and each $F \in \mathscr{F}$ such that $x \notin F$ there exists a continuum M such that $x \in \text{int } M$ and $M \cap F = \emptyset$ (Bennett [1]). If $\mathscr{F}$ is the collection of finite (countable closed) subsets of X and X is $\mathscr{F}$-aposyndetic then X is said to be *finitely* (*countable closed set*) *aposyndetic*.

Let $(X, \leqq)$ be a partially ordered space. If $x \in X$, then $S(x) =$

$\{y \in X \mid y \leqq x\}$ and $T(x) = \{y \in X \mid x \leqq y\}$ are called the *lower and upper sets of* x respectively. Similarly, if $A \subseteqq X$, then $S(A) = \cup \{S(a) \mid a \in A\}$ and $T(A) = \cup \{T(a) \mid a \in A\}$ are called the *lower and upper sets of* A respectively. If X is compact and the partial order is closed ($\{(x, y) \mid x \leqq y\}$ is closed in $X \times X$), then $S(A)$ and $T(A)$ are closed whenever A is closed (Proposition 4, p. 44 of [6]). A is said to be *decreasing* if $A = S(A)$ and *increasing* if $A = T(A)$.

The following definition is due to L. Nachbin [6]. A partially ordered space is *normally ordered* if for each pair A, B of disjoint closed subsets of X such that $A = S(A)$ and $B = T(B)$, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$ and such that $U = S(U)$ and $V = T(V)$. It is known that any compact Hausdorff space with closed partial order is normally ordered (Theorem 4, p. 48 of [6]).

In a partially ordered space, a *chain* is a totally ordered subset. An arc which is also a chain is called an *order arc.*

It is easy to establish that the natural partial order on $2^X(C(X))$ induced by inclusion is a closed partial order. If $A, B \in 2^X(C(X))$, then there exists an order arc in $2^X(C(X))$ from A to B if and only if $A \leqq B$ and each component of B intersects A (Lemma 2.3 and Lemma 2.6 of [4]).

LEMMA 1. *Let* $A \in 2^X$ *and* $M \in C(X)$. *Then* $T(A)$, $T(M) \cap C(X)$, *and* $S(M) \cap C(X)$ *are continua. Consequently, if* $\mathscr{A}$ *is a closed set in* $2^X(C(X))$, *then* $T(\mathscr{A})$ *is a continuum in* $2^X(C(X))$.

Proof. $T(A)$ is a continuum in 2^X because $T(A) = \{B \in 2^X \mid A \subseteqq B\}$ is the continuous image of 2^X under the function $C \rightarrow A \cup C$. If $N \in T(M) \cap C(X)$, there exists an order arc $\mathscr{L}$ in $C(X)$ from N to X, and $\mathscr{L} \subset T(M) \cap C(X)$. It follows that $T(M) \cap C(X)$ is connected, and since the partial order is closed, $T(M) \cap C(X)$ is a continuum. $S(M) \cap C(X) = C(M)$, so $S(M) \cap C(X)$ is a continuum. For each $A \in \mathscr{A}$, $T(A)$ is a continuum in $2^X(C(X))$ and $X \in T(A)$. It follows that $T(\mathscr{A}) = \cup \{T(A) \mid A \in \mathscr{A}\}$ is a continuum in $2^X(C(X))$.

LEMMA 2. $2^X(C(X))$ *is locally connected at* X.

Proof. Let $\mathscr{U}$ be an open set containing X and $N(U_1, \cdots, U_n)$ be a basic open set such that $X \in N(U_1, \cdots, U_n) \subset \mathscr{U}$. If $A \in N(U_1, \cdots, U_n)$, then there exists an order arc $\mathscr{L}$ in $2^X\,(C(X))$ from A to X. Since $A, X \in N(U_1, \cdots, U_n)$, each element of $\mathscr{L}$ is in $N(U_1, \cdots, U_n)$. It follows that $N(U_1, \cdots, U_n)$ is connected. Hence $2^X(CX))$ is locally connected at X.

The following theorem of Nachbin [6] will yield a useful method for constructing continua in the hyperspaces.

THEOREM A. *Let X be a normally ordered space with closed partial order and A be a compact subset of X. Then every continuous, order-preserving, real-valued function on A can be extended to X in such a way as to remain continuous and order-preserving.*

We now have the necessary equipment to prove the main result.

THEOREM 1. *Each of 2^X and $C(X)$ is aposyndetic.*

Proof. Let $A \in 2^X(C(X))$. We will show that $2^X(C(X))$ is aposyndetic at A with respect to each of the other points of $2^X(C(X))$. If $A = X$, then by Lemma 2, $2^X(C(X))$ is locally connected at A and hence aposyndetic at A. So we will assume that A is a proper closed subset (subcontinuum) of X.

Let $B \in 2^X(C(X))$, $B \neq A$. If $B \subset A$ or if B and A are not related under inclusion, then there exists $x \in A$ such that $x \notin B$. Let U be an open set containing x such that $U^* \cap B = \emptyset$. Let V be an open set containing $A - U$ such that $x \notin V^*$. Then $A \in N(U, V)$ and $B \notin N(U^*, V^*) = N(U, V)^*$ (Lemma 2.3.2 of [5]). By Lemma 1, $T(N(U, V)^*)$ is a continuum, and $A \in \operatorname{int} T(N(U, V)^*)$. Furthermore, $B \notin T(N(U, V)^*)$. Hence $2^X(C(X))$ is aposyndetic at A with respect to B.

Now suppose $A, B \in C(X)$ and $A \subset B$. Let $\mathscr{C} = \{A\} \cup \{B\} \cup F_1(X)$. Then $\mathscr{C}$ is a compact subset of $C(X)$. Define $f\colon \mathscr{C} \to [0, 1]$ by

$$f(C) = \begin{cases} 0 & \text{if} \quad C = A \quad \text{or} \quad C \in F_1(X) \\ 1 & \text{if} \quad C = B\,. \end{cases}$$

Since A is a proper subset of B, f is continuous and order-preserving, so by Theorem A, f has a continuous extension $\hat{f}\colon C(X) \to R$ (reals) which is also order-preserving. Since $C(X)$ is a continuum, $\hat{f}(C(X))$ is a closed interval, and since $\hat{f}$ is order-preserving, for some $b \geqq 1$, $\hat{f}(C(X)) = [0, b]$. Let $t \in (0, 1)$ and consider $\hat{f}^{-1}([0, t])$. If $L \in \hat{f}^{-1}([0, t])$, then $S(L) \subset \hat{f}^{-1}([0, t])$. By Lemma 1, $S(L)$ is a continuum. Moreover, $S(L) \cap F_1(X) \neq \emptyset$. Since $F_1(X) \subset \hat{f}^{-1}([0, t])$ and $\hat{f}^{-1}([0, t]) = \bigcup \{S(L) \mid L \in \hat{f}^{-1}([0, t])\}$, it follows that $\hat{f}^{-1}([0, t])$ is connected, and since $\hat{f}$ is continuous, it follows that $\hat{f}^{-1}([0, t])$ is a continuum. Furthermore, $A \in \operatorname{int} \hat{f}^{-1}([0, t])$ and $B \notin \hat{f}^{-1}([0, t])$. Hence $C(X)$ is aposyndetic at A with respect to B. This concludes the proof that $C(X)$ is aposyndetic.

Finally, suppose $A, B \in 2^X$ and $A \subset B$. Let U be an open set such that $A \subset U$ and $B - U^* \neq \emptyset$. Let $\mathscr{K} = N(U)^* \cup N(X - U)$. Observe that $A \in \operatorname{int} \mathscr{K}$ and $B \notin \mathscr{K}$. Now $F(X) \cap \mathscr{K}$ is dense in $\mathscr{K}$.

We will show that $F(X) \cap \mathscr{K}$ is connected.

Let $\{x_1, \cdots, x_n\} \in N(X-U)$. Then for each $i = 1, \cdots, n$, $x_i \in X - U^*$ or $x_i \in \text{bd}\, U$. Let

$$C_i = \begin{cases} \text{the component of } X - U^* \text{ containing } x_i & \text{if } x_i \in X - U^* \\ x_i & \text{if } x_i \in \text{bd}\, U\,. \end{cases}$$

Then $C_i^* \cap \text{bd}\, U \neq \varnothing$, because C_i is a component of the open set $X - U^*$ and hence meets $\text{bd}\,(X - U^*) \subseteqq \text{bd}\, U$. Let $x_i' \in C_i^* \cap \text{bd}\, U$. Let $\mathscr{D}_i = \{\{x_1, \cdots, x_{i-1}, y, x_{i+1}', \cdots, x_n'\} \mid y \in C_i^*\}$. Now $\mathscr{D}_i$ is the continuous image of the continuum C_i^*, so $\mathscr{D}_i$ is a continuum in $N(X - U)$ containing $\{x_1, \cdots, x_{i-1}, x_i, x_{i+1}', \cdots, x_n'\}$ and $\{x_1, \cdots, x_{i-1}, x_i', x_{i+1}', \cdots, x_n'\}$. Then for each $i = 2, \cdots, n$, $\mathscr{D}_{i-1} \cap \mathscr{D}_i \neq \varnothing$. So $\bigcup_{i=1}^{n} \mathscr{D}_i$ is a continuum in $N(X - U)$ containing $\{x_1, \cdots, x_n\}$ and $\{x_1', \cdots, x_n'\}$. For each $i = 1, \cdots, n - 1$, let $\mathscr{L}_i = \{\{x_1', \cdots, x_i', y\} \mid y \in X\}$. Then $\mathscr{L}_i$ is a continuum in $\mathscr{K}$ containing $\{x_1', \cdots, x_i', x_{i+1}'\}$ and $\{x_1', \cdots, x_i'\}$, and for each $i = 2, \cdots, n - 1$, $\mathscr{L}_{i-1} \cap \mathscr{L}_i \neq \varnothing$. Hence $\bigcup_{i=1}^{n-1} \mathscr{L}_i$ is a continuum in $\mathscr{K}$ containing $\{x_1', \cdots, x_n'\}$ and $\{x_1'\}$. Let $\mathscr{M} = \{\{y\} \mid y \in C_1^*\}$. Then $\mathscr{M}$ is a continuum in $N(X - U)$ containing $\{x_1'\}$ and $\{x_1\}$. So

$$\left(\bigcup_{i=1}^{n} \mathscr{D}_i\right) \cup \left(\bigcup_{i=1}^{n-1} \mathscr{L}_i\right) \cup \mathscr{M}$$

is a continuum in $\mathscr{K}$ containing $\{x_1, \cdots, x_n\}$ and $\{x_1\}$.

If $\{x_1, \cdots, x_n\} \in N(U)^*$, we can use a similar construction to show that $\mathscr{K}$ contains a continuum containing $\{x_1, \cdots, x_n\}$ and $\{x_1\}$. So if $C \in F(X) \cap \mathscr{K}$ and $c \in C$, there exists a continuum in $F(X) \cap \mathscr{K}$ containing C and $\{c\}$. Hence $F(X) \cap \mathscr{K}$ can be written as a union of continua, each of which meets $F_1(X)$. Since $U^* \cup (X - U) = X$, it follows that $F_1(X) \subset F(X) \cap \mathscr{K}$, and since $F_1(X)$ is connected, it follows that $F(X) \cap \mathscr{K}$ is connected. Furthermore, $F(X) \cap \mathscr{K}$ is dense in $\mathscr{K}$, so $\mathscr{K}$ is connected. Hence $\mathscr{K}$ is a continuum containing A in its interior which misses B. This concludes the proof that 2^X is aposyndetic.

A continuum X is said to be *unicoherent* provided that whenever A and B are proper subcontinua such that $X = A \cup B$, then $A \cap B$ is connected. S. B. Nadler, Jr. [7] has proved that each of 2^X and $C(X)$ is unicoherent. D. E. Bennett [1] has shown that a unicoherent aposyndetic continuum is finitely aposyndetic. These results and Theorem 1 imply the following corollary.

COROLLARY 1. *Each of 2^X and $C(X)$ is finitely aposyndetic.*

THEOREM 2. *Let X be a semi-aposyndetic continuum. Then each of 2^X and $C(X)$ is mutually aposyndetic.*

Proof. Let $M, N \in C(X)$, $M \neq N$. Suppose $M \subset N$ or that M and N are not related under inclusion and that $N \notin F_1(X)$. Let $\mathscr{A} = \{M\} \cup \{N\} \cup F_1(X)$ and define $f: \mathscr{A} \to [0, 1]$ by

$$f(A) = \begin{cases} 0 & \text{if} \quad A = M \quad \text{or} \quad A \in F_1(X) \\ 1 & \text{if} \quad A = N . \end{cases}$$

Then f is continuous and order-preserving, so by Theorem A f has a continuous extension $\hat{f}: C(X) \to R$ which is also order-preserving. Let $\hat{f}(C(X)) = [0, b] (b \geqq 1)$ and $t_1, t_2 \in (0, 1)$, $t_1 < t_2$. As in the proof of Theorem 1, $\hat{f}^{-1}([0, t_1])$ is a continuum containing M in its interior which misses N. Now suppose $L \in \hat{f}^{-1}([t_2, b])$. Observe that $T(L) \subset \hat{f}^{-1}([t_2, b])$ and $X \in T(L)$. By Lemma 1, $T(L)$ is a continuum. Since $\hat{f}^{-1}([t_2, b]) = \cup \{T(L) \mid L \in \hat{f}^{-1}([t_2, b])\}$, $\hat{f}^{-1}([t_2, b])$ is a union of continua, each of which contains X. Hence $\hat{f}^{-1}([t_2, b])$ is a continuum containing N in its interior which misses M. Since $\hat{f}^{-1}([0, t_1]) \cap \hat{f}^{-1}([t_2, b]) = \varnothing$, it follows that $C(X)$ is mutually aposyndetic at (M, N).

Let $A, B \in 2^X$, $A \neq B$. Suppose that $A \subset B$ or that A and B are not related under inclusion and $B \notin F_1(X)$. Let $x \in B - A$ and $y \in B$, $x \neq y$. Let U be an open set containing A and y such that $x \notin U^*$. Let V_y be an open set containing y such that $V_y^* \subset U$. Let V_x be an open set containing x such that $V_x^* \subset \operatorname{int}(X - U)$. Let W be an open set containing $B - (V_x \cup V_y)$ such that $x, y \notin W^*$. (If $B - (V_x \cup V_y) = \varnothing$, replace $N(V_x, V_y, W)$ by $N(V_x, V_y)$ in the remainder of the argument.) Then $B \in N(V_x, V_y, W)$ and $N(V_x, V_y, W)$ is disjoint from the sets $N(U)^*$ and $N(X - U)$. As in the proof of Theorem 1, $N(U)^* \cup N(X - U)$ is a continuum containing A in its interior which misses B. By Lemma 1, $T(N(V_x, V_y, W)^*)$ is a continuum, and $B \in \operatorname{int} T(N(V_x, V_y, W)^*)$. If $C \in T(N(V_x, V_y, W)^*)$, then C meets V_x^* and V_y^*, so C meets U and $\operatorname{int}(X - U)$. Therefore $C \notin N(U)^* \cup N(X - U)$. So $T(N(V_x, V_y, W)^*) \cap (N(U)^* \cup N(X - U)) = \varnothing$. Hence 2^X is mutually aposyndetic at (A, B).

Finally, suppose $A, B \in F_1(X)$, $A \neq B$. We will write $A = \{p\}$, $B = \{q\}$. Since X is semi-aposyndetic, we assume that X is aposyndetic at p with respect to q. Then there exists a subcontinuum M of X such that $p \in \operatorname{int} M$ and $q \in X - M$. Let V be an open set containing q such that $V^* \cap M = \varnothing$. By Lemma 1, $T(N(V)^*)$ is a continuum in $2^X(C(X))$ and $\{q\} \in \operatorname{int} T(N(V)^*)$. Now $2^M(C(M))$ is a subcontinuum of $2^X(C(X))$ and $\{p\} \in \operatorname{int} 2^M(C(M))$. Moreover, $2^M(C(M))$ and $T(N(V)^*)$ are disjoint, since M and V^* are disjoint. Hence $2^X(C(X))$ is mutually aposyndetic at $(\{p\}, \{q\})$. This concludes the proof.

In the preceding theorem we have shown that if X is any continuum and at least one of A and B is not an element of $F_1(X)$, then each of 2^X and $C(X)$ is mutually aposyndetic at the pair (A, B). We now give an example of a non-semi-aposyndetic continuum for which $C(X)$ fails to be mutually aposyndetic at certain pairs of elements belonging to $F_1(X)$.

EXAMPLE 1. Let X be the planar continuum which is the union of $S^1 = \{(r, \theta) \mid r = 1\}$ and $S = \{(r, \theta) \mid \theta \geqq 0 \text{ and } r = 1 + 1/(1 + \theta)\}$. J. T. Rogers, Jr. [8] has shown that for this X, $C(X)$ is homeomorphic to the cone over X. Moreover, the homeomorphism carries $F_1(X)$ onto the base of the cone.

Observe that if $p, q \in S^1$, then X is not semi-aposyndetic at (p, q). To show that $C(X)$ is not mutually aposyndetic at $(\{p\}, \{q\})$ it will suffice to show that $X \times I(I = [0, 1])$ is not mutually aposyndetic at (p', q') where $p' = (p, 0)$ and $q' = (q, 0)$.

Suppose $M_{p'}$ and $M_{q'}$ are disjoint continua containing p' and q' respectively in their interiors. Let $N_{p'}$ be the component of $(S^1 \times I) \cap M_{p'}$ which contains p'. Let $U_1, \cdots, U_n$ be a finite cover of $N_{p'}$ by spherical open sets such that $(\bigcup_{i=1}^n U_i) \cap M_{q'} = \emptyset$. Using the fact that each component of $(S^1 \times I) - (\bigcup_{i=1}^n U_i)$ is arcwise connected, it can be established that no component of $(S^1 \times I) - (\bigcup_{i=1}^n U_i)$ meets both $S^1 \times \{0\}$ and $S^1 \times \{1\}$. It follows that $\bigcup_{i=1}^n U_i$ contains a simple closed curve C which separates $S^1 \times I$ between $S^1 \times \{0\}$ and $S^1 \times \{1\}$. Furthermore, for some $c \in C$, $\bigcup_{i=1}^n U_i$ contains an arc $[p', c]$ such that $[p', c] \cap C = \{c\}$.

Let $N_{q'}$ be the component of $(S^1 \times I) \cap M_{q'}$ which contains q' and let $V_1, \cdots, V_m$ be a finite cover of $N_{q'}$ by spherical open sets disjoint from $\bigcup_{i=1}^n U_i$. In the analogous manner, $\bigcup_{i=1}^m V_i$ contains a simple closed curve D which separates $S^1 \times I$ between $S^1 \times \{0\}$ and $S^1 \times \{1\}$ and for some $d \in D$, $\bigcup_{i=1}^m V_i$ contains an arc $[q', d]$ such that $D \cap [q', d] = \{d\}$. It can now be shown (this involves a consideration of some properties of S^2) that $([p', c] \cup C) \cap ([q', d] \cup D) \neq \emptyset$, a contradiction. Hence $C(X)$ is not mutually aposyndetic at $(\{p\}, \{q\})$.

The final theorem extends the main result and Corollary 1 for $C(X)$. First we need the following lemma.

LEMMA 3. *Let $M \in C(X)$ and $\mathscr{A}$ be a countable closed set in $C(X)$. Then there exists a decreasing open set $\mathscr{U}$ in $C(X)$ such that $M \in \mathscr{U}$ and $(\operatorname{bd} \mathscr{U}) \cap \mathscr{A} = \emptyset$.*

Proof. Let $\varepsilon > 0$ and let d denote the metric on X. For each $x \in M$ let $S_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$. Let $U_\varepsilon = \bigcup_{x \in M} S_\varepsilon(x)$ and $\mathscr{U}_\varepsilon = N(U_\varepsilon)$. Then $\mathscr{U}_\varepsilon$ is a decreasing open set in $C(X)$ which contains M. If $L \in \operatorname{bd} \mathscr{U}_\varepsilon$, then there exists $y \in L$ such that

$$y \in \Big(\bigcup_{x \in M} S_\varepsilon(x)\Big)^* - \Big(\bigcup_{x \in M} S_\varepsilon(x)\Big).$$

It follows that for each $x \in M$, $d(x, y) \geqq \varepsilon$ and for some $x_0 \in M$, $d(x_0, y) = \varepsilon$. Therefore if $\varepsilon_1 \neq \varepsilon_2$ and $L_1 \in \operatorname{bd}(\mathscr{U}_{\varepsilon_1})$ and $L_2 \in \operatorname{bd}(\mathscr{U}_{\varepsilon_2})$, then $L_1 \neq L_2$. Since $\mathscr{A}$ is countable there exists $\varepsilon > 0$ such that $\operatorname{bd}(\mathscr{U}_\varepsilon) \cap \mathscr{A} = \varnothing$.

THEOREM 3. *$C(X)$ is countable closed set aposyndetic.*

Proof. Let $M \in C(X)$ and $\mathscr{A}$ be a countable closed set in $C(X)$ such that $M \notin \mathscr{A}$. If $M = X$, then by Lemma 2, $C(X)$ is locally connected at M and hence $C(X)$ is countable closed set aposyndetic at M.

Suppose M is a nondegenerate proper subcontinuum of X. Let $\mathscr{A}_S = \mathscr{A} \cap S(M)$. By Lemma 3, for each $L \in \mathscr{A}_S$ there exists a decreasing open set $\mathscr{U}_L$ such that $L \in \mathscr{U}_L$ and $\operatorname{bd}(\mathscr{U}_L) \cap (\mathscr{A} \cup \{M\}) = \varnothing$. Since $\mathscr{A}_S$ is compact there exist $\mathscr{U}_{L_1}, \cdots, \mathscr{U}_{L_n}$ such that $\mathscr{A}_S \subset \bigcup_{i=1}^n \mathscr{U}_{L_i}$. Let $\mathscr{A}_F = F_1(X) \cap \mathscr{A}$. For each $x \in \mathscr{A}_F$ let $\mathscr{V}_x$ be a decreasing open set such that $\{x\} \in \mathscr{V}_x$ and $(\operatorname{bd} \mathscr{V}_x) \cap (\mathscr{A} \cup \{M\}) = \varnothing$. Since $\mathscr{A}_F$ is compact there exist $\mathscr{V}_{x_1}, \cdots, \mathscr{V}_{x_m}$ such that $\mathscr{A}_F \subset \bigcup_{i=1}^m \mathscr{V}_{x_i}$.

Let

$$\mathscr{A}_0 = \mathscr{A} \cap \Big[\Big(\bigcup_{i=1}^n \mathscr{U}_{L_i'}\Big) \cup \Big(\bigcup_{i=1}^m \mathscr{V}_{x_i}\Big)\Big] = \mathscr{A} \cap \Big[\Big(\bigcup_{i=1}^n \mathscr{U}_{L_i}^*\Big) \cup \Big(\bigcup_{i=1}^m \mathscr{V}_{x_i}^*\Big)\Big]$$

and let $\mathscr{A}_1 = \mathscr{A} - \mathscr{A}_0$. Then $\mathscr{A}_0$ and $\mathscr{A}_1$ are disjoint closed subsets of $\mathscr{A}$. Define $f: \mathscr{A} \cup \{M\} \cup F_1(X) \to [0, 1]$ by

$$f(A) = \begin{cases} 0 & \text{if } A \in \mathscr{A}_0 \text{ or if } A \in F_1(X) \\ 1/2 & \text{if } A = M \\ 1 & \text{if } A \in \mathscr{A}_1 . \end{cases}$$

Then f is continuous and order-preserving, so by Theorem A, f has a continuous extension $\hat{f}: C(X) \to [0, b]$ $(b \geqq 1)$ which is also order-preserving. $\hat{f}$ has the property that if $t \in [0, b]$, then $\hat{f}^{-1}([0, t])$ and $\hat{f}^{-1}([t, b])$ are subcontinua of $C(X)$. Since $C(X)$ is unicoherent, $\hat{f}^{-1}([0, 3/4]) \cap \hat{f}^{-1}([1/4, b])$ is a continuum containing M in its interior which misses $\mathscr{A}$.

Now suppose that for some $x_0 \in X$, $M = \{x_0\}$. Let $\mathscr{A}_F = \mathscr{A} \cap F_1(X)$. For each $\{x\} \in \mathscr{A}_F$ let $\mathscr{U}_x$ be a decreasing open set such that $\{x\} \in \mathscr{U}_x$ and $(\operatorname{bd} \mathscr{U}_x) \cap (\mathscr{A} \cup \{M\}) = \varnothing$. Since $\mathscr{A}_F$ is compact there exist $x_1, \cdots, x_n$ such that $\mathscr{A}_F \subset \bigcup_{i=1}^n \mathscr{U}_{x_i}$. Let $\mathscr{A}_0 = \mathscr{A} \cap (\bigcup_{i=1}^n \mathscr{U}_{x_i}) = \mathscr{A} \cap (\bigcup_{i=1}^n \mathscr{U}_{x_i}^*)$ and let $\mathscr{A}_1 = \mathscr{A} - \mathscr{A}_0$. Then $\mathscr{A}_0$ and $\mathscr{A}_1$ are disjoint closed subsets of $\mathscr{A}$. Define $f: (\bigcup_{i=1}^n \mathscr{U}_{x_i}^*) \cup \mathscr{A} \cup F_1(X) \to [0, 1]$ by

$$f(A) = \begin{cases} 0 & \text{if } A \in \Big(\bigcup_{i=1}^n \mathscr{U}_{x_i}^*\Big) \cup F_1(X) \\ 1 & \text{if } A \in \mathscr{A}_1 . \end{cases}$$

Then f is continuous and order-preserving, so f has a continuous, order-preserving extension $\hat{f}: C(X) \to [0, b]$ $(b \geqq 1)$. Let $t \in (0, 1)$. Then $\hat{f}^{-1}([0, t])$ is a continuum containing M in its interior which misses $\mathscr{A}_1$.

Let $\mathscr{M} = C(X) - \bigcup_{i=1}^n \mathscr{U}_{x_i}$. $\mathscr{M}$ is a closed set containing M in its interior which misses $\mathscr{A}_0$. Since each of $\mathscr{U}_{x_1}, \cdots, \mathscr{U}_{x_n}$ is decreasing, it follows that $\mathscr{M}$ is increasing. So $\mathscr{M} = T(\mathscr{M})$ and by Lemma 1, $T(\mathscr{M})$ is a continuum. Observe that $C(X) = \hat{f}^{-1}([0, t]) \cup \mathscr{M}$. Since $C(X)$ is unicoherent, $\hat{f}^{-1}([0, t]) \cap \mathscr{M}$ is a continuum containing M in its interior which misses $\mathscr{A}$. Hence $C(X)$ is countable closed set aposyndetic.

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