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ON SATURATED FORMATIONS OF SOLVABLE LIE ALGEBRAS

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The concepts of formations, $\mathscr{F}$-projectors and $\mathscr{F}$-normalizers have all been developed for solvable Lie algebras. In this note, for each saturated formation $\mathscr{F}$ of solvable Lie algebras, the class $\mathscr{T}(\mathscr{F})$ of solvable Lie algebras L in which each $\mathscr{F}$-normalizer of L is an $\mathscr{F}$-projector is considered. This is the natural generalization of the Lie algebra analogue to SC groups which were first investigated by R. Carter. It is shown that $\mathscr{T}(\mathscr{F})$ is a formation. Then some properties of $\mathscr{F}$-normalizers of $L \in \mathscr{T}(\mathscr{F})$ are considered.

All Lie algebras considered here are solvable and finite dimensional over a field F. $\mathscr{F}$ will always denote a saturated formation of solvable Lie algebras and L will be a solvable Lie algebra. $N(L)$ is the nil-radical of L and $\Phi(L)$ is the Frattini subalgebra of L. For definitions and properties of all these concepts see [3], [4], and [9]. For SC groups see [6].

We begin with a general lemma.

LEMMA 1. *Let N be an ideal of L and D/N be an $\mathscr{F}$-normalizer of L/N. Then there exists an $\mathscr{F}$-normalizer E of L such that $E + N = D$.*

Proof. Let L be a minimal counterexample and we may assume that N is a minimal ideal of L. If $D/N = L/N$, then any $\mathscr{F}$-normalizer of L has the desired property, hence we may suppose that $D/N \subset L/N$. Suppose first that N is $\mathscr{F}$-central in L. Let $N^*/N = N(L/N)$ and $C = C_L(N)$. Then $N(L) = N^* \cap C$. Let M/N be a maximal $\mathscr{F}$-critical subalgebra of L/N such that D/N is an $\mathscr{F}$-normalizer of M/N. Now either M is $\mathscr{F}$-critical in L or M complements a chief factor of L between N^* and $N(L)$. In the first case, by induction, there exists an $\mathscr{F}$-normalizer E of M such that $E + N = D$ and E is also an $\mathscr{F}$-normalizer in L. In the second case, $L/C \in \mathscr{F}$ and $C + N^*/C$ is operator isomorphic to $N^*/N^* \cap C = N^*/N(L)$. Hence each chief factor of L between N^* and $N(L)$ is $\mathscr{F}$-central which contradicts M being $\mathscr{F}$-abnormal.

Now suppose that N is $\mathscr{F}$-eccentric and assume $N \subseteqq \Phi(L)$. Let M/N be as in the above paragraph. Again, by induction, there exists an $\mathscr{F}$-normalizer E of M such that $E + N = D$. But $N \subseteqq \Phi(L)$ yields that M is $\mathscr{F}$-critical in L using Theorem 2.5 of [4]. Hence E is an

$\mathscr{F}$-normalizer of L and this case is completed.

Finally suppose that N is $\mathscr{F}$-eccentric and assume $N \nsubseteq \Phi(L)$. Then N is complemented by a maximal subalgebra M which must be $\mathscr{F}$-critical in L. Now there must exist an $\mathscr{F}$-normalizer E of M such that $E + N = D$. Again E must be an $\mathscr{F}$-normalizer of L and the result is shown.

COROLLARY. *$\mathscr{T}(\mathscr{F})$ is closed under homomorphisms.*

Proof. Let N be a minimal ideal of L, $L \in \mathscr{T}(\mathscr{F})$. Let D/N be an $\mathscr{F}$-normalizer of L/N. Then $D = E + N$ for some $\mathscr{F}$-normalizer of L. Now E is an $\mathscr{F}$-projector of L and $E + N/N = D/N$ is an $\mathscr{F}$-projector of L/N.

LEMMA 2. *If $L \in \mathscr{T}(\mathscr{F})$ and C is an $\mathscr{F}$-projector of L, then C is an $\mathscr{F}$-normalizer of L.*

Proof. Let N be a minimal ideal of L. $L/N \in \mathscr{T}(\mathscr{F})$ hence $C + N/N$ is an $\mathscr{F}$-normalizer of L/N by induction. Hence $C + N = D + N$ for some $\mathscr{F}$-normalizer D of L. Now D is also an $\mathscr{F}$-projector of L and both C and D are $\mathscr{F}$-projectors of $C + N$. Then C and D are conjugate in $C + N$ by an inner automorphism of $C + N$ induced by an element of N by Lemma 1.11 of [3]. Hence D and C are conjugate in L and the result holds.

Note that $\mathscr{T}(\mathscr{F})$ contains a large class of Lie algebras. In fact by Theorem 3 of [9] we have

LEMMA 3. $\mathscr{N}\mathscr{F} \subseteqq \mathscr{T}(\mathscr{F})$.

In order to obtain that $\mathscr{T}(\mathscr{F})$ is a formation, we record a characterization of $\mathscr{F}$-projectors which is completely analogous to a result in group theory due to Bauman [5]. Since the proofs carry over virtually unchanged, we omit them.

DEFINITION. If M is a subalgebra of L, then a series $0 = L_0 \subset \cdots \subset L_n = L$ is called an M-series if L_i is an ideal in L_{i+1}, if $M \subseteqq N_L(L_i)$ and if each L_{i+1}/L_i is a nontrivial, irreducible M-factor of L.

THEOREM 1. *If C is an $\mathscr{F}$-projector of L and $\{L_i\}$, $0 \leqq i \leqq n$, is any C-series of L, then C covers L_i/L_{i-1} if and only if $C + L_i/L_{i-1} \in \mathscr{F}$.*

Proof. See proof of Theorem 1 of [5].

THEOREM 2. *If* $\{L_i\}$ *is a C-series of* L *such that* C *covers* L_i/L_{i-1} *if and only if* $C + L_i/L_{i-1} \in \mathscr{F}$, *then* C *is an* $\mathscr{F}$*-projector of* L.

Proof. See proof of Theorem 2 of [5].

We intend to use these results in a slightly different form by means of

LEMMA 4. *Let* M *be a subalgebra of* $L, M \in \mathscr{F}$ *and* H/K *be a nontrivial, irreducible M-factor of* L. *Then* $M + H/K \in \mathscr{F}$ *if and only if the split extension of* H/K *by* $M/C_M(H/K)$ *is in* $\mathscr{F}$.

Proof. Since $M + H/H \in \mathscr{F}$, $M + H/K$ will be in $\mathscr{F}$ if and only if the minimal ideal H/K of $M + H/K$ is $\mathscr{F}$-central in $M + H/K$; that is, if and only if the split extension of H/K by $M + H/C_{M+H}(H/K)$ is in $\mathscr{F}$. But

$$\begin{aligned} M/C_M(H/K) &= M/M \cap C_{M+H}(H/K) \cong M + C_{M+H}(H/K)/C_{M+H}(H/K) \\ &= M + H/C_{M+H}(H/K) . \end{aligned}$$

Now the corresponding split extensions of H/K by $M + H/C_{M+H}(H/K)$ and H/K by $M/C_M(H/K)$ are isomorphic and the result holds.

THEOREM 3. $\mathscr{T}(\mathscr{F})$ *is a formation.*

Proof. $\mathscr{T}(\mathscr{F})$ is closed under homomorphisms has been noted already. Hence let N_1 and N_2 be ideals of L such that $L/N_1, L/N_2 \in \mathscr{T}(\mathscr{F})$. We may assume $N_1 \cap N_2 = 0$ and show that $L \in \mathscr{T}(\mathscr{F})$. Let D be an $\mathscr{F}$-normalizer of L. Then $D + N_1/N_1$ is an $\mathscr{F}$-normalizer of L/N_1, hence is an $\mathscr{F}$-projector of L/N_1 and the corresponding statement holds for $D + N_2/N_2$. Consider a D-series of L which passes through N_1 and $N_1 + N_2$. There is a D-series of L which passes through N_2 and $N_1 + N_2$ which is the same as the original D-series above $N_1 + N_2$ and corresponds to the original D-series below $N_1 + N_2$ in the natural way. In particular, a factor H/K in the new D-series which is between N_2 and $N_1 + N_2$ corresponds to $H \cap N_1/K \cap N_1$ in the original D-series and we claim that D covers (avoids) H/K if and only if D covers (avoids) $H \cap N_1/K \cap N_1$. For if D avoids H/K, then $D \cap H \subseteqq K$, hence $D \cap H \cap N_1 \subseteqq K \cap N_1$ and D avoids $H \cap N_1/K \cap N_1$. Suppose that D covers H/K. Then $H \subseteqq K + D$. In order to show that D covers $H \cap N_1/K \cap N_1$ it is sufficient to show that $D + (K \cap N_1) \supseteqq H \cap N_1$. Since $H \subseteqq K + D, D \subseteqq N_L(K)$ and $H \subseteqq N_1 + N_2$, it follows that $H \subseteqq K + (D \cap (N_1 + N_2))$. Using the corollary on p. 241 of [9], $H \subseteqq K + ((D \cap N_1) + (D \cap N_2)) = K + (D \cap N_1)$. Then, since $D \cap N_1 \subset N_L(K)$ it follows that $H \cap N_1 \subseteqq (K + (D \cap N_1)) \cap N_1 \subseteqq (K \cap N_1) + (D \cap N_1) \subseteqq (K \cap N_1) + D$, hence D covers $H \cap N_1/K \cap N_1$.

By Theorem 1 and Lemma 4, a factor H/K above N_1 in the original D-series is covered by $D + N_1/N_1$ (hence D) if and only if the split extension of H/K by $D + N_1/C_{D+N_1}(H/K)$ is in $\mathscr{F}$. That is, H/K is covered by D if and only if the split extension of H/K by $D/C_D(H/K)$ is in $\mathscr{F}$. A similar statement holds above N_2. Every D-factor in the original series is operator isomorphic to a D-factor above N_1 or above N_2 and, using the result of the above paragraph, in the original D-series a factor H/K is covered by D if and only if the split extension of H/K by $D/C_D(H/K)$ is in $\mathscr{F}$. Now by Lemma 4 and Theorem 2, D is an $\mathscr{F}$-projector of L and $\mathscr{T}(\mathscr{F})$ is a formation.

The following example shows that $\mathscr{N}\mathscr{N} \subset \mathscr{T}(\mathscr{N})$ and that $\mathscr{T}(\mathscr{N})$ is not closed under taking ideals. It is a variant of an example on p. 52 of [7].

EXAMPLE. Let F be a field of characteristic $p \geqq 2$ and let A be a vector space over F with basis $e_0, \cdots, e_{p-1}$. Define linear transformations x, y, z on A by

$$x(e_i) = ie_i$$
$$y(e_i) = e_{i+1}$$

and

$$z(e_i) = e_i$$

(subscripts mod p). Then $[x, y] = xy - yx = y$ and $[x, z] = [y, z] = 0$. Let B be the three dimensional Lie algebra generated by x, y, z. Let L be the semi-direct sum of A and B with the natural product. As on p. 53 of [7], B acts irreducibly on A so that A is a minimal ideal of L. Evidently A is self-centralizing in L, hence A is the unique minimal ideal of L and $N(L) = A$. Hence each $\mathscr{N}$-critical maximal subalgebra of L complements A. Furthermore, L is clearly of nilpotent length three.

Consider first any $\mathscr{N}$-normalizer E of L which is also an $\mathscr{N}$-normalizer of B. Such $\mathscr{N}$-normalizer exists since B is a maximal $\mathscr{N}$-critical subalgebra of L. By the covering-avoidance property of $\mathscr{N}$-normalizers of $B, E = ((z, x + \alpha y))$ where $\alpha \in F$. Now B is of nilpotent length 2, hence E is a Cartan subalgebra of B. Now since $z \in E$, it is easily verified that E is a Cartan subalgebra of L.

Now in general, each $\mathscr{N}$-normalizer of L is an $\mathscr{N}$-normalizer of some $\mathscr{N}$-critical maximal subalgebra M of L and M must complement A. But L is of nilpotent length 3 and L/A is of nilpotent length 2, hence M must be conjugate to B by Theorem 8 of [8].

Consequently, any $\mathscr{N}$-normalizer of L is a Cartan subalgebra of L and $L \in \mathscr{T}(\mathscr{N})$.

Now the ideal $P = A + ((x, y))$ of L is not in $\mathscr{T}(\mathscr{N})$. For $((x)) \subset ((x, y)) \subset P$ is a maximal $\mathscr{N}$-critical chain of P, hence $((x))$ is an $\mathscr{N}$-normalizer of P. However, the normalizer of $((x))$ in P is $((x, e_0))$. Hence $L \notin \mathscr{T}(\mathscr{N})$.

We recall that each $\mathscr{F}$-normalizer is contained in an $\mathscr{F}$-projector (Theorem 6 of [9]). However, the usual converse result, namely each $\mathscr{F}$-projector contains an $\mathscr{F}$-normalizer has not been obtained, even for $\mathscr{N}\mathscr{N}\mathscr{F}$-Lie algebras. We now show that this result holds if $L \in \mathscr{N}\mathscr{T}(\mathscr{F})$. First we record the following result which is needed.

THEOREM 4. *Let $L \in \mathscr{N}\mathscr{T}(\mathscr{F})$. Then each $\mathscr{F}$-normalizer of L is contained in a unique $\mathscr{F}$-projector of L.*

Proof. Same as the proof of Theorem 9 of [9].

THEOREM 5. *Let $L \in \mathscr{N}\mathscr{T}(\mathscr{F})$. Then each $\mathscr{F}$-projector of L contains an $\mathscr{F}$-normalizer of L.*

Proof. Let N be a minimal ideal of L and let C be an $\mathscr{F}$-projector of L. Then $C + N/N$ is an $\mathscr{F}$-projector of L/N and $C + N/N$ contains an $\mathscr{F}$-normalizer D/N of L/N by induction. Let $T = C + N$ and let F be an $\mathscr{F}$-normalizer of L such that $F + N = D \subseteqq T$. Then F is contained in an $\mathscr{F}$-projector G of L and $D/N \subseteqq G + N/N$. Hence $G + N = C + N$ by Theorem 4 and G and C are $\mathscr{F}$-projectors of T. By Lemma 1.11 of [3], G and C are conjugate in T by an inner automorphism of T induced by an element of N. Hence G and C are conjugate in L and the result holds.

$\mathscr{F}$-normalizers have the covering-avoidance property but the converse is not true in general. However, if $L \in \mathscr{T}(\mathscr{F})$, then the converse is true.

THEOREM 6. *Let $L \in \mathscr{T}(\mathscr{F})$. If D is a subalgebra of L which covers the $\mathscr{F}$-central chief factors of L and avoids the $\mathscr{F}$-eccentric chief factors of L, then D is an $\mathscr{F}$-normalizer of L.*

Proof. Let N be a minimal ideal of L. Then $D + N/N$ has the covering-avoidance property in $L/N \in \mathscr{T}(\mathscr{F})$. By induction, $D + N/N$ is an $\mathscr{F}$-normalizer of L/N and $D + N = E + N = T$ for some $\mathscr{F}$-normalizer E of L. Since $L \in \mathscr{T}(\mathscr{F})$, E is an $\mathscr{F}$-projector of L and then also of T. If N is $\mathscr{F}$-central in L, then $N \subseteqq D$ and $N \subseteqq E$, hence $D = E$. Suppose N is $\mathscr{F}$-eccentric. Then $D \cap N = 0 = E \cap N$. Now $T \in \mathscr{N}\mathscr{F}$, hence E is an $\mathscr{F}$-normalizer of T by Theorem 3 of [9]. Furthermore, in a given chief series of T passing through N, E

covers all chief factors above N and avoids all chief factors below N and the same is true for D. Since E is an $\mathscr{F}$-normalizer of T, each chief factor below N must be $\mathscr{F}$-eccentric and each chief factor above N must be $\mathscr{F}$-central. Hence, by Theorem 4 of [9], D must be an $\mathscr{F}$-normalizer of T. By Theorem 3 of [9], D must also be an $\mathscr{F}$-projector of T. Now D and E are conjugate in T (hence in L) by an inner automorphism induced by an element of N. Hence D is an $\mathscr{F}$-normalizer of L.

Henceforth we shall be concerned with the case $\mathscr{F} = \mathscr{N}$. Here we have the following stronger form of Theorem 4.

THEOREM 7. *Let $L \in \mathscr{N}\mathscr{T}(\mathscr{N})$ and D be an $\mathscr{N}$-normalizer of L. Then there exists a Cartan subalgebra C of L which contains every subalgebra H of L in which D is subinvariant. In particular, D is contained in a unique Cartan subalgebra of L. C is the Fitting null component of D acting on L.*

Proof. $D + N(L)/N(L)$ is subinvariant in $H + N(L)/N(L)$ and $D + N(L)/N(L)$ is an $\mathscr{N}$-normalizer of $L/N(L) \in \mathscr{T}(\mathscr{N})$. Hence $D + N(L)/N(L) = H + N(L)/N(L)$ is a Cartan subalgebra of $L/N(L)$. Let $T = D + N(L) = H + N(L)$ and let S be the Fitting null component of D acting on T. Evidently $N_T(S) = S$ and $H \subseteqq S$. Furthermore, $S = S \cap T = S \cap (D + N(L)) = D + (S \cap N(L))$. Each element of D induces a nilpotent derivation on S and $S \cap N(L)$ is a nilpotent ideal of S. Then, using Engel's theorem, S is nilpotent. Hence S is a Cartan subalgebra of T and also of L by Lemma 1.8 of [3]. If K is another Cartan subalgebra of L containing D, then D is subinvariant in K, hence $K = S$. The last past of the theorem follows from the next lemma.

LEMMA 5. *Let L be a solvable Lie algebra and D be a nilpotent subalgebra of L. Let F be the Fitting null component of D acting on L. Then D is subinvariant in F.*

Proof. We may suppose that $F = L$. Let A be a minimal ideal of L. Now in $D + A$, A is an abelian ideal and each element of D induces a nilpotent derivation of $D + A$. Hence, using Engel's theorem, $D + A$ is nilpotent and D is subinvariant in $D + A$. But $D + A/A$ satisfies the conditions in L/A, hence $D + A/A$ is subinvariant in L/A by induction. Therefore, D is subinvariant in L.

For Lie algebras of nilpotent length three, a result somewhat stronger than Theorem 7 holds. The proof is the same as the proof of Theorem 7, using Theorem 1 of [8] instead of the defining property of $\mathscr{T}(\mathscr{N})$, and may be omitted.

THEOREM 8. *Let L be of nilpotent length three (or less) and let D be a nilpotent subalgebra of L which can be joined to L by a maximal chain of subalgebras, each self-normalizing in the next. Then there exists a Cartan subalgebra C of L which contains every subalgebra H of L in which D is subinvariant. In particular, D is contained in a unique Cartan subalgebra C of L and C is the Fitting null component of D acting on L.*

We may use this to find a Lie algebra analogue to Theorem 10 of [2].

THEOREM 9. *Let M be a self-normalizing maximal subalgebra of L. Suppose that L is of nilpotent length three. Then each Cartan subalgebra of M is of the form $M \cap C$ for some Cartan subalgebra C of L.*

Proof. Let D be a Cartan subalgebra of M. Then D is contained in a Cartan subalgebra C of L by Theorem 8 and Lemma 1 of [8]. Now $M \cap C$ is nilpotent and D is a Cartan subalgebra of $M \cap C$. Hence $D = M \cap C$.

The final result is of a slightly different nature. We consider the following: If an $\mathscr{N}$-normalizer D of L is contained in the self-normalizing maximal subalgebra M of L, then is D contained in an $\mathscr{N}$-normalizer of M. The analogous question for finite groups is answered negatively in [1]. The Lie algebra case also has a negative answer as in shown in the following result. The second part of this example is also an analogue to the example of [1].

THEOREM 10. *There exists a solvable Lie algebra $L \in \mathscr{N}\mathscr{N}\mathscr{N}$ which has an $\mathscr{N}$-normalizer D, ideal A and maximal subalgebra M containing D such that*

(1) *D is not contained in an $\mathscr{N}$-normalizer of M*

(2) $N_{L/A}(D + A/A) \supset N_L(D) + A/A$.

Proof. This example is also a variant of an example found on p. 52 of [7]. Let F be a field of characteristic $p > 2$. Let A be the Lie algebra over F with basis $a_0, a_1, \cdots, a_{p-1}, b, c_0, c_1, \cdots, c_{p-1}$ and products $[a_i, b] = c_i$ for $i = 0, \cdots, p-1$ and all other products of basis elements equal to 0. Define linear transformations x, y on A such that

$$\begin{aligned} x(a_i) &= a_{i+1} & y(a_i) &= ia_i \\ x(b) &= 0 & y(b) &= 0 \\ x(c_i) &= c_{i+1} & y(c_i) &= ic_i \end{aligned}$$

(everything mod p). Then x and y are derivations of A and $[y, x] = x$. Let B be the 2-dimensional Lie algebra generated by x and y and let L be the semi-direct sum of A and B with the natural product.

Let $R = ((c_0, \cdots, c_{p-1}))$ and $S = ((c_0, \cdots, c_{p-1}, b))$. The same argument used in [7] shows that R and A/S are $\mathscr{N}$-eccentric chief factors of L and S/R is clearly and $\mathscr{N}$-central chief factor of L. Let $M = ((x, y, b, c_0, \cdots, c_{p-1}))$, $M_1 = ((x, y, b))$ and $M_2 = ((y, b))$. Each of these is a maximal $\mathscr{N}$-critical subalgebra of the preceding and M is maximal, $\mathscr{N}$-critical in L. Now $\exp a_0$ is an automorphism of L since char $F \neq 2$. Then $C = M_2^{\exp a_0} = ((y, b + c_0)) \subsetneqq M$ and D is an $\mathscr{N}$-normalizer of L.

Now the $\mathscr{N}$-normalizers of M have dimension 2 by the covering-avoidance property of $\mathscr{N}$-normalizers, hence, if D is contained in an $\mathscr{N}$-normalizer of M, then it is one of them. If this is the case, then, since $b \in Z(M)$, $b \in D$ and $\dim D > 2$, a contradiction.

For the second part, note that

$$N_{L/R}(M_2 + R/R) = ((y + R, b + R, a_0 + R)) .$$

However, an element of the form $\alpha a_0 + t, \alpha \in F, t \in R$ is not in $N_L(M_2)$ unless $\alpha = 0$, since $[b, \alpha a_0 + t] = -\alpha c_0$. Hence

$$N_L(M_2) + R/R \subset N_{L/R}(M_2 + R/R) .$$

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