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ON A GENERALIZATION OF MARTINGALES DUE TO BLAKE

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It is shown that any uniformly integrable fairer with time game (stochastic process) converges in L_1.

1. Introduction. Let $(\Omega, \mathscr{U}, P)$ be a probability space and $\{\mathscr{U}_n\}_{n\geqq 1}$ an increasing family of sub σ-algebras of $\mathscr{U}$. Let $\{X_n\}_{n\geqq 1}$ be a stochastic process adapted to $\{\mathscr{U}_n\}_{n\geqq 1}$ (see, [2, p. 65]). Following Blake [1] we refer to $\{X_n\}_{n\geqq 1}$ as a game and define

DEFINITION. The game $\{X_n\}_{n\geqq 1}$ will be said to become *fairer with time* if for every $\varepsilon > 0$

$$P[| E(X_n/\mathscr{U}_m) - X_m | > \varepsilon] \rightarrow 0$$

as $n, m \rightarrow \infty$ with $n \geqq m$. Any martingale is, trivially, a fairer with time game and thus this concept generalizes that of martingales. Blake, in [1], gave a set of sufficient conditions under which any uniformly integrable fairer with time game $\{X_n\}_{n\geqq 1}$ is convergent in L_1. We show that these sufficient conditions are not needed; in fact, we show that any uniformly integrable, fairer with time game converges in L_1.

2. THEOREM 2.1. *Any uniformly integrable fairer with time game $\{X_n\}_{n\geqq 1}$ converges in L_1.*

Proof. To facilitate understanding, we break up the proof into a few important steps numbered (S1) through (S5). For every m and $n \geqq m$ define $Y_{m,n} = E(X_n/\mathscr{U}_m)$. Let Γ stand for the family $\{Y_{m,n},$ for all m and $n \geqq m\}$.

(S1) Γ is uniformly integrable.

Since $\{X_n\}_{n\geqq 1}$ is uniformly integrable there exists a function f defined on the nonnegative real axis which is positive, increasing and convex, such that

$$\lim_{t\to\infty} \frac{f(t)}{t} = +\infty$$

and $\sup_n E[f \circ |X_n|] < \infty$. (See [2, II T 22].) Now,

$$\begin{aligned} E[f \circ | Y_{m,n} |] &= E[f \circ | E(X_n/\mathscr{U}_m) |] \\ &\leqq E[f \circ E(| X_n |/\mathscr{U}_m)] \text{ (since } f \text{ is nondecreasing)} \\ &\leqq E[E(f \circ | X_n |/\mathscr{U}_m)] \\ &= E[f \circ | X_n |] . \end{aligned}$$

Therefore,

$$\sup_{Y_{m,n} \in \Gamma} E[f \circ | Y_{m,n} |] \leqq \sup_n E[f \circ | X_n |] < \infty .$$

Another application of II T 22 of [2] ensures that Γ is uniformly integrable. Hence (S1).

(S2) Given $\varepsilon > 0$, there exists M such that for all $m \geqq M$, one has

$$E(| X_m - Y_{m,n} |) \leqq 2\varepsilon \quad \text{for all} \quad n \geqq m .$$

Since Γ is uniformly integrable given $\varepsilon > 0$ there exists $\delta > 0$ such that $P(A) < \delta$ implies $\int_A | Y_{m,n} | dP \leqq \varepsilon/2$, for all $Y_{m,n} \in \Gamma$. Choose M so large that $m \geqq M$ and $n \geqq m$ implies $P[| X_m - E(X_n/U_m) | > \varepsilon] < \delta$. Then, it is not difficult to see that

$$E[| X_m - Y_{m,n} |] \leqq 2\varepsilon \quad \text{for all } m \geqq M \quad \text{and} \quad n \geqq m .$$

(S3) For every fixed m, the sequence $\{Y_{m,n}\}$ converges in L_1 to an $\mathscr{U}_m$ measurable random variable Z_m.

Let $m \leqq n < n'$.

$$\begin{aligned} E[| Y_{m,n} - Y_{m,n'} |] &= E[| E(X_n/\mathscr{U}_m) - E(X_{n'}/\mathscr{U}_m) |] \\ &= E[| E(X_n - X_{n'}/\mathscr{U}_m) |] \\ &= E[| E(\{E(X_n - X_{n'}/\mathscr{U}_n)\}/\mathscr{U}_m) |] \\ &\leqq E[E(\{| E(X_n - X_{n'}/\mathscr{U}_n) |\}/\mathscr{U}_m)] \\ &= E[| E(X_n - X_{n'}/\mathscr{U}_n) |] \\ &= E[| X_n - Y_{n,n'} |] . \end{aligned}$$

Now from (S2) it follows that given $\varepsilon > 0$ for all sufficiently large n and n'

$$E[| Y_{m,n} - Y_{m,n'} |] \leqq E[| (X_n - Y_{n,n'}) |] \leqq 2\varepsilon .$$

Hence, for m fixed, the sequence $\{Y_{m,n}\}$ is Cauchy in the L_1-norm. So, there exists, an integrable random variable Z_m, such that, $Y_{m,n} \xrightarrow[n\to\infty]{L_1} Z_m$. Without loss of generality we can take Z_m to be $\mathscr{U}_m$ measurable. (Note that each $Y_{m,n}$ is $\mathscr{U}_m$ measurable and there is a subsequence $\{Y_{m,n'}\}$ converging almost surely to Z_m.)

(S4) $\{Z_m, \mathscr{U}_m\}_{m \geqq 1}$ is a uniformly integrable martingale.

The fact that $\{Z_m\}_{m \geqq 1}$ is uniformly integrable follows trivially because the closure in L_1 of a uniformly integrable collection is uniformly integrable. (See, [2, II T20].) To show $\{Z_m, \mathscr{U}_m\}$ is a martingale it is enough to show that for every m, $E(Z_{m+1}/\mathscr{U}_m) = Z_m$ a.s. Since

$$\begin{aligned} E[| E(Y_{m+1,n}/\mathscr{U}_m) &- E(Z_{m+1}/\mathscr{U}_m) |] \\ &= E[| E\{(Y_{m+1,n} - Z_{m+1})/\mathscr{U}_m\} |] \\ &\leqq E[E\{| (Y_{m+1,n} - Z_{m+1}) |/\mathscr{U}_m\}] \\ &= E[| Y_{m+1,n} - Z_{m+1} |] \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty , \end{aligned}$$

there exists a subsequence n' of $\{n: n \geqq m\}$ such that

$$E(Y_{m+1,n'}/\mathscr{U}_m) \xrightarrow{\text{a.s.}} E(Z_{m+1}/\mathscr{U}_m) .$$

We can assume (— if necessary, by choosing a further subsequence, —) that $Y_{m,n'} \xrightarrow{\text{a.s.}} Z_m$. Now,

$$\begin{aligned} E(Z_{m+1}/\mathscr{U}_m) &= \lim_{n'\to\infty} E(Y_{m+1,n'}/\mathscr{U}_m) \quad \text{a.s.} \\ &= \lim_{n'\to\infty} E(\{E(X_{n'}/\mathscr{U}_{m+1})\}/\mathscr{U}_m) \quad \text{a.s.} \\ &= \lim_{n'\to\infty} E(X_{n'}/\mathscr{U}_m) \quad \text{a.s.} \\ &= \lim_{n'\to\infty} Y_{m,n'} \quad \text{a.s.} \\ &= Z_m \qquad \text{a.s.} \end{aligned}$$

Hence (S4). (S5) $\{X_n\}_{n\geqq 1}$ converges in L_1.

Since $\{Z_n, \mathscr{U}_n\}_{n\geqq 1}$ is an uniformly integrable martingale, there exists an integrable random variable Z_∞ such that $Z_n \xrightarrow[n\to\infty]{L_1} Z_\infty$. We shall show that $X_n \xrightarrow[n\to\infty]{L_1} Z_\infty$. From (S3) and (S2) it is easy to check that given $\varepsilon > 0$ there exists M such that for all $m \geqq M$

$$\int | X_m - Z_m | \, dP \leqq 2\varepsilon .$$

Therefore, for sufficiently large m,

$$\int | X_m - Z_\infty | \, dP \leqq \int | X_m - Z_m | \, dP + \int | Z_m - Z_\infty | \, dP \leqq 3\varepsilon ,$$

say. Hence (S5) and the theorem.

Since any game (stochastic process) $\{X_n\}_{n\geqq 1}$ converging in L_1 can be taken to be a game fairer with time, by setting $\mathscr{U}_n \equiv \mathscr{U}$ in n, we get the following corollary.

COROLLARY 2.1. *Let $\{X_n\}_{n\geqq 1}$ be a game. It converges in L_1 if and only if it is uniformly integrable and fairer with time with respect to some increasing family of sub σ-algebras $\{\mathscr{U}_n\}_{n\geqq 1}$ to which it is adapted.*

Let $p > 1$.

THEOREM 2.2. *Let $\{X_n\}_{n\geqq 1}$ be a fairer with time game with $\{|X_n|^p\}_{n\geqq 1}$ uniformly integrable. Then $\{X_n\}_{n\geqq 1}$ converges in Lp.*

Proof. Noting that the function f defined on the nonnegative real axis by $f(t) = t^p$ is positive, increasing and convex and $\lim_{t\to\infty}(f(t)/t) = +\infty$, in view of II T 22 of [2], it is clear that $\{X_n\}_{n\geqq 1}$ is uniformly integrable. Hence by Theorem 2.1 it converges in L_1; in particular, $\{X_n\}_{n\geqq 1}$ converges in probability. Therefore, $\{X_n\}_{n\geqq 1}$ converges in L_p. (See Proposition II 6.1 of [3].)

COROLLARY 2.2. *The game $\{X_n\}_{n\geqq 1}$ converges in L_p if and only if $\{|X_n|^p\}_{n\geqq 1}$ is uniformly integrable and $\{X_n\}_{n\geqq 1}$ is fairer with time with respect to some increasing family of sub σ-algebras $\{\mathscr{U}_n\}_{n\geqq 1}$ to which it is adapted.*

REMARK. In view of our Theorem 2.1, the second convergence theorem of Blake in [1] becomes redundant.

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