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PRODUCT INTEGRALS FOR AN ORDINARY DIFFERENTIAL EQUATION IN A BANACH SPACE

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Let Y be a Banach space with norm $|\ |$, and let R^+ be the interval $[0, \infty)$. Let A be a function on R^+ having the properties that if t is in R^+ then $A(t)$ is a function from Y to Y and that the function from $R^+ \times Y$ to Y described by $(t, x) \rightarrow A(t)[x]$ is continuous. Suppose there is a continuous real-valued function α on R^+ such that if t is in R^+ then $A(t) - \alpha(t)I$ is dissipative. Now it is known that if z is in Y, the differential equation $u'(t) = A(t)[u(t)]$; $u(0) = z$ has exactly one solution on R^+. It is shown in this paper that if t is in R^+ then $u(t) = {}_0\prod^t \exp[(ds)A(s)][z] = {}_0\prod^t [I - (ds)A(s)]^{-1}[z]$, where the exponentials are defined by the solutions of the associated family of autonomous equations.

The dissipitavity condition on A is simply that if (t, x, y) is in $R^+ \times Y \times Y$ and c is a positive number then

$$|[I - cA(t)][x] - [I - cA(t)][y]| \geqq [1 - c\alpha(t)]|x - y| . \tag{1}$$

The author and R. H. Martin, Jr. [5] have shown that if (1) holds, and z is in Y, then there is exactly one continuously differentiable function u from R^+ to Y such that

$$u(0) = z \tag{2}$$

and

$$u'(t) = A(t)[u(t)] \tag{3}$$

whenever t is in $(0, \infty)$. In the present article we shall show that u can be expressed as a product integral in each of two forms:

$$u(t) = \prod_0^t \exp[(ds)A(s)][z] \tag{4}$$

and

$$u(t) = \prod_0^t [I - (ds)A(s)]^{-1}[z] . \tag{5}$$

Our work is related to results of J. V. Herod [2, §6] and G. F. Webb [7], [8]. Herod showed that representation (5) is valid if the mapping $(t, x) \rightarrow A(t)[x]$ is bounded on bounded subsets of $R^+ \times Y$. Webb obtained in [7] a representation similar to (4) under a set of hypotheses different from, and independent of, those used here. In

[8], Webb showed that (5) is valid if A is independent of t. (Actually Webb in [8] restricted his attention to the case $\alpha = 0$, but his proofs adapt easily to the general time-independent case.)

II. **Product integrals.** We shall assume throughout that A and α are as in our introduction, and that (1) is true whenever (t, x, y) is in $R^+ \times Y \times Y$ and c is a positive number. Now it follows from either of [5] and [6] that if (t, x) is in $R^+ \times Y$ then there is exactly one solution v of the problem

$$v'(s) = A(t)[v(s)];\ v(0) = x\,. \tag{6}$$

Furthermore, this problem generates an operator semigroup, which we shall denote $\{\exp[sA(t)]: s \text{ is in } R^+\}$, i.e., if s is in R^+ then $\exp[sA(t)]$ is a function from Y to Y such that if x is in Y then $\exp[sA(t)][x] = v(s)$, where v solves (6).

It is clear from (1) that there is no loss in assuming α to be R^+-valued, and we shall. It follows from [6] that if (c, t) is in $R^+ \times R^+$ and $c\alpha(t) < 1$ then $I - cA(t)$ is a bijection on Y, and

$$|[I - cA(t)]^{-1}[x] - [I - cA(t)]^{-1}[y]| \leqq [1 - c\alpha(t)]^{-1}|x - y|$$

whenever (x, y) is in $Y \times Y$. If $\{B_1, \cdots, B_n\}$ is a set of functions from Y to Y, and x is in Y, then $\prod_{j=1}^{0} B_j[x] = x$ and $\prod_{j=1}^{k} B_j[x] = B_k[\prod_{j=1}^{k-1} B_j[x]]$ whenever k is an integer in $[1, n]$. If (t, x, y) is in $R^+ \times Y \times Y$ then the statement

$$y = \prod_0^t [I - (ds)A(s)]^{-1}[x]$$

means that if ε is a positive number then there is a chain $\{r_j\}_{j=0}^{m}$ from 0 to t such that if $\{s_k\}_{k=0}^{n}$ is a refinement of $\{r_j\}_{j=0}^{m}$, and $\{\tilde{s}_k\}_{k=1}^{n}$ is a $[0, t]$-valued sequence such that if k is an integer in $[1, n]$ then $\tilde{s}_k$ is in $[s_{k-1}, s_k]$, then

$$\left| y - \prod_{k=1}^{n} [I - (s_k - s_{k-1})A(\tilde{s}_k)]^{-1}[x] \right| < \varepsilon\,.$$

The statement

$$y = \prod_0^t \exp[(ds)A(s)][x]$$

is defined analogously.

THEOREM. *Let z be in Y, and let u solve* (2) *and* (3). *Then each of* (4) *and* (5) *is true whenever t is in R^+.*

Let m_- be that function from $Y \times Y$ to the real numbers given by

$$m_-[x, y] = \lim_{\delta \to 0-} (1/\delta)(|x + \delta y| - |x|) .$$

Now (1) is equivalent to requiring that

$$m_-[x - y, A(t)[x] - A(t)[y]] \leqq \alpha(t) |x - y|$$

whenever (t, x, y) is in $R^+ \times Y \times Y$ (compare [1, p. 3]). Also, if f is a function from a subset of R^+ to Y, if c is in the domain of f, if $f'_-(c)$ (the left derivative of f at c) exists, and if P is given on the domain of f by $P(t) = |f(t)|$, then $P'_-(c)$ exists and $P'_-(c) = m_-[f(c), f'_-(c)]$ (compare [1, p. 3]). If (x, y, z) is in $Y \times Y \times Y$ then $m_-[x, y + z] \leqq m_-[x, y] + |z|$ (see [4, Lemma 6]). We are now prepared to prove our theorem.

Proof of the theorem. Let b be a positive number, and let β be a positive upper bound for the set $\{\alpha(t): t \text{ is in } [0, b]\}$. Let ε be a positive number, and let δ be a positive number such that $(\delta/\beta)(e^{\beta b} - 1) < \varepsilon$. Now $\{u(t): t \text{ is in } [0, b]\}$ is a compact subset of Y, so the function described by $(t, x) \to A(t)[x]$ is uniformly continuous on $[0, b] \times \{u(t): t \text{ is in } [0, b]\}$. In particular, there is a positive number η such that if (r, s, t) is in $[0, b] \times [0, b] \times [0, b]$ and $|r - s| < \eta$ then $|A(r)[u(t)] - A(s)[u(t)]| < \delta$. Let $\{t_k\}_{k=0}^{n}$ be a chain from 0 to b such that $t_k - t_{k-1} < \eta$ whenever k is an integer in $[1, n]$, and let $\{\tilde{t}_k\}_{k=1}^{n}$ be a $[0, b]$-valued sequence such that if k is an integer in $[1, n]$ then $\tilde{t}_k$ is in $[t_{k-1}, t_k]$. Let v be that function from $[0, b]$ to Y having the property that if k is an integer in $[1, n]$ and t is in $[t_{k-1}, t_k]$ then

$$v(t) = \exp [(t - t_{k-1})A(\tilde{t}_{k-1})] \prod_{j=1}^{k-1} \exp [(t_j - t_{j-1})A(\tilde{t}_j)][z] .$$

Clearly now v is continuous. Also, v is left differentiable on $(0, b]$: if k is an integer in $[1, n]$ and t is in $(t_{t-1}, t_k]$ then

$$v'_-(t) = A(\tilde{t}_{k-1})[v(t)] .$$

Let P be given on $[0, b]$ by $P(t) = |v(t) - u(t)|$. Now $P(0) = 0$. Suppose that t is in $(0, b]$ and k is an integer in $[1, n]$ and t is in $(t_{k-1}, t_k]$. Now

$$\begin{aligned} P'_-(t) &= m_-[v(t) - u(t), v'_-(t) - u'(t)] \\ &= m_-[v(t) - u(t), A(\tilde{t}_{k-1})[v(t)] - A(t)[u(t)]] \\ &= m_-[v(t) - u(t), A(\tilde{t}_{k-1})[v(t)] - A(\tilde{t}_{k-1})[u(t)] \\ &\quad + A(\tilde{t}_{k-1})[u(t)] - A(t)[u(t)]] \end{aligned}$$

$$\begin{aligned}&\leqq m_-[v(t)-u(t), A(\tilde{t}_{k-1})[v(t)]-A(\tilde{t}_{k-1})[u(t)]]\\&\quad+|A(\tilde{t}_{k-1})[u(t)]-A(t)[u(t)]|\\&\leqq \beta P(t)+\delta\,.\end{aligned}$$

Hence [3, Theorem 1.4.1, p. 15],

$$P(t) \leqq \int_0^t \delta e^{\beta(t-s)}ds = (\delta/\beta)(e^{\beta t}-1)$$

whenever t is in $[0, b]$. In particular,

$$\begin{aligned}&\left|u(b)-\prod_{k=1}^{n}\exp[(t_k-t_{k-1})A(\tilde{t}_k)][z]\right|\\&\quad=|u(b)-v(b)|\\&\quad=P(b)\\&\quad\leqq(\delta/\beta)(e^{\beta b}-1)<\varepsilon\,.\end{aligned}$$

Thus we have proved that representation (4) is valid.

Now let b and β be as before. Let c be a positive number such that $c\beta < 1/2$. Now if t is in $[0, b]$ and r is in $[0, c]$ then

$$\begin{aligned}&|[I-rA(t)]^{-1}[x]-[I-rA(t)]^{-1}[y]|\\&\quad\leqq[1-r\beta]^{-1}|x-y|\\&\quad\leqq(1+2r\beta)|x-y|\\&\quad\leqq e^{2r\beta}|x-y|\end{aligned}$$

whenever (x, y) is in $Y \times Y$.

Now let $K = \{u(t): t \text{ is in } [0, b]\}$, and recall that K is compact. Let ε be a positive number. By the aforementioned uniform continuity, there is a positive number η_1 such that if (s, t, x, y) is in $[0, b] \times [0, b] \times K \times K$ and $|s-t| < \eta_1$ and $|x-y| < \eta_1$ then $|A(s)[x]-A(t)[y]| < (\varepsilon/b)e^{-2\beta b}$. Let η_2 be a positive number such that if (s, t) is in $[0, b] \times [0, b]$ and $|s-t| < \eta_2$ then $|u(s)-u(t)| < \eta_1$. Let $\delta = \min\{\eta_1, \eta_2, c\}$. Suppose that $0 \leqq r \leqq s \leqq t \leqq b$ and $t-r < \delta$. Let $\{\xi_k\}_{k=0}^n$ be a chain from r to t, and let $\{\tilde{\xi}_k\}_{k=1}^n$ be a $[r, t]$-valued sequence such that if k is an integer in $[1, n]$ then $\tilde{\xi}_k$ is in $[\xi_{k-1}, \xi_k]$. Now

$$\begin{aligned}&\left|\sum_{k=1}^{n}(\xi_k-\xi_{k-1})A(\tilde{\xi}_k)[u(\tilde{\xi}_k)]-(t-r)A(s)[u(t)]\right|\\&\quad\leqq\sum_{k=1}^{n}(\xi_k-\xi_{k-1})|A(\tilde{\xi}_k)[u(\tilde{\xi}_k)]-A(s)[u(t)]|\\&\quad\leqq\sum_{k=1}^{n}(\xi_k-\xi_{k-1})(\varepsilon/b)e^{-2\beta b}=(t-r)(\varepsilon/b)e^{-2\beta b}\,.\end{aligned}$$

It is now clear that

$$\left| \int_r^t A(\xi)[u(\xi)]d\xi - (t - r)A(s)[u(t)] \right| \\ \leqq (t - r)(\varepsilon/b)e^{-2\beta b} .$$

Let $\{t_k\}_{k=0}^n$ be a chain from 0 to b, and suppose that $t_k - t_{k-1} < \delta$ whenever k is an integer in $[1, n]$. Let $\{\tilde{t}_k\}_{k=1}^n$ be a $[0, b]$-valued sequence such that if k is an integer in $[1, n]$ then $\tilde{t}_k$ is in $[t_{k-1}, t_k]$. Now

$$\begin{aligned}
&\left| \prod_{k=1}^n [I - (t_k - t_{k-1})A(\tilde{t}_k)]^{-1}[z] - u(b) \right| \\
&\leqq \sum_{k=1}^n \left| \prod_{j=k+1}^n [I - (t_j - t_{j-1})A(\tilde{t}_j)]^{-1}[u(t_k)] \right. \\
&\qquad \left. - \prod_{j=k}^n [I - (t_j - t_{j-1})A(\tilde{t}_j)]^{-1}[u(t_{k-1})] \right| \\
&\leqq \sum_{k=1}^n e^{2\beta(b-t_k)} |u(t_k) - [I - (t_k - t_{k-1})A(\tilde{t}_k)]^{-1}[u(t_{k-1})]| \\
&\leqq e^{2\beta b} \sum_{k=1}^n |[I - (t_k - t_{k-1})A(\tilde{t}_k)][u(t_k)] - u(t_{k-1})| \\
&= e^{2\beta b} \sum_{k=1}^n |u(t_k) - u(t_{k-1}) - (t_k - t_{k-1})A(\tilde{t}_k)[u(t_k)]| \\
&= e^{2\beta b} \sum_{k=1}^n \left| \int_{t_{k-1}}^{t_k} u'(\xi)d\xi - (t_k - t_{k-1})A(\tilde{t}_k)[u(t_k)] \right| \\
&= e^{2\beta b} \sum_{k=1}^n \left| \int_{t_{k-1}}^{t_k} A(\xi)[u(\xi)]d\xi - (t_k - t_{k-1})A(\tilde{t}_k)[u(t_k)] \right| \\
&\leqq e^{2\beta b} \sum_{k=1}^n (t_k - t_{k-1})(\varepsilon/b)e^{-2\beta b} = \varepsilon .
\end{aligned}$$

The proof of the theorem is complete.

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