PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 1, 1973

SUBMANIFOLDS OF ACYCLIC 3-MANIFOLDS

JOŽE VRABEC

It is proved that, from the viewpoint of "geometric" homology theory, an arbitrary embedding of a closed surface S in any 3-manifold with trivial first homology group looks exactly like the standard embedding of S in the euclidean 3-space. A consequence: every compact subset of a 3-manifold with trivial first homology group can be embedded in a homology 3-sphere. Necessary and sufficient (homological) conditions are given for a compact 3-manifold to be embeddable in some acyclic 3-manifold (or in some homology 3-sphere).

1. Definitions and preliminaries.

Manifolds. We work in the PL category. Each manifold is supposed to have a fixed PL structure. If M is a manifold, then by a submanifold of M or by a surface, simple closed curve, arc, etc., in M we always mean a respective object contained in M as a subpolyhedron (in the chosen PL structure of M). All maps are assumed to be PL. Our manifolds are never automatically assumed to be without boundary, compact, connected, or orientable. However, by a *surface* we mean a compact, connected, orientable 2-manifold. A *cube with n handles* is a 3-manifold homeomorphic to a regular neighborhood of a connected finite linear graph of Euler characteristic $1-n$ in E^3.

We denote the interior of a manifold M by $\operatorname{int} M$ and the boundary by $\operatorname{Bd} M$. However, if M is oriented, then by ∂M we denote the manifold $\operatorname{Bd} M$ oriented coherently with M. The symbol ∂ also denotes the boundary in the homological sense. Let M be an oriented manifold and P a codimension 0 submanifold of M. Whenever we talk of P as an oriented manifold, we assume that P has the orientation inherited from M, unless explicitly stated otherwise. If M is an oriented manifold, then M with the opposite orientation is sometimes denoted by $-M$.

Homology. All homology and cohomology groups, cycles, chains, etc., have integer coefficients. If z_1, z_2 are n-cycles in a space X, then $z_1 \sim z_2$ means "z_1 is homologous to z_2". A compact oriented n-submanifold N of an m-manifold M generates a uniquely determined PL n-chain in M. This chain is a cycle if and only if N is a closed manifold. We shall make no distinction in notation between N and the n-chain it represents. If M is a manifold of dimension at least 2, then every element of $H_1(M)$ can be represented by an oriented closed 1-manifold in $\operatorname{int} M$. If M is a 3-manifold, if $J \subset M$ is a closed oriented 1-manifold, and if $J \sim 0$ in M, then there exists a compact oriented 2-manifold F in M such that $J = \partial F$.

If X, Y are spaces and $f: X \to Y$ a map, then by f_* we denote the homomorphism $H_1(X) \to H_1(Y)$ induced by f.

Let S be an oriented 2-manifold and x, y either two 1-cycles in S or two elements of $H_1(S)$. By $\operatorname{sc}(x, y)$ we denote the (integral) intersection number of x and y. The following is well-known.

LEMMA 1.1. *Let M be an oriented 3-manifold and let J, K be closed oriented 1-manifolds in ∂M. If $J \sim K \sim 0$ in M, then* $\operatorname{sc}(J, K) = 0$.

A polyhedron X is *acyclic* if it is connected and has $H_n(X) = 0$ for $n > 0$. We will call X 1-*acyclic* if it is connected and has $H_1(X) = 0$. Note that any 1-acyclic manifold W is orientable. The reason is that $\pi_1(W)$ contains no subgroups of index 2; a subgroup of index 2 would contain the commutator subgroup of $\pi_1(W)$, but the commutator subgroup is the whole $\pi_1(W)$ since $H_1(W) = 0$. A *homology n-sphere* is an n-manifold whose homology is isomorphic to the homology of the n-sphere. An n-manifold will be called *sub-acyclic* if it can be embedded in an acyclic n-manifold.

2-*Manifolds*. We give the definition of oriented piping (in dimension 2). Let S be a 2-manifold and $J, K \subset S$ two disjoint oriented simple closed curves. Let $A \subset S$ be an arc from a point $x \in J$ to a point $y \in K$; let $\operatorname{int} A \subset \operatorname{int} S - (J \cup K)$. Take a regular neighborhood N of A in S. The intersection $N \cap J$ is a small arc $J_0 \subset J$ containing x in its interior. Similarly, $N \cap K$ is an arc $K_0 \subset K$ with $y \in \operatorname{int} K_0$. Let D be the closure of the component of $N - (J \cup K)$ which contains $\operatorname{int} A$. Then D is a disk and $\operatorname{Bd} D$ consists of J_0, K_0, and two "long" arcs in $\operatorname{Bd} N$. Suppose that D can be oriented coherently with both J_0 and K_0. Then the simple closed curve $L = (J \cup K \cup \operatorname{Bd} D) - \operatorname{int}(J_0 \cup K_0)$ can be oriented so that it induces in $J - \operatorname{int} J_0$ the same orientation as J and in $K - \operatorname{int} K_0$ the same orientation as K. If this is the case, we say that the oriented simple closed curve L is obtained from $J \cup K$ by *piping along A* (or that L is obtained by *piping J to K* or by *piping together J and K*). If we think of J, K, L as 1-cycles and of D as a 2-chain, then $J + K - L = \partial D$. Hence $L \sim J + K$. The following lemma is obvious.

LEMMA 1.2. *If S is an oriented surface, then any two components J and K of ∂S can be piped together along any properly embedded arc $A \subset S$ which joins J and K.*

For a compact 2-manifold S we define the *genus* of S to be the sum of genera of the components of S.

Groups. If G and H are groups, then $G \approx H$ means "G is isomorphic to H". Since we deal only with abelian groups we use the term "free group" in the meaning "free abelian group". Let G be a free (abelian) group. We will call $x \in G$ a *basic* element of G if x is a member of some basis of G. Using standard facts we can prove that x is basic if and only if the subgroup of G generated by x is a nonzero direct summand of G, or, if and only if x is not equal to ny for any integer $n > 1$ and any $y \in G$.

Matrices. If $\boldsymbol{A}$ is any matrix, let $\boldsymbol{A}'$ denote the transposed of $\boldsymbol{A}$. For any positive integer n we denote by $\boldsymbol{I}_n$ and $\boldsymbol{O}_n$ the identity and zero $n \times n$ matrices, respectively. If $n = 2m$, let $\boldsymbol{J}_n$ be the matrix

$$\boldsymbol{J}_n = \begin{bmatrix} \boldsymbol{O}_m & \boldsymbol{I}_m \\ -\boldsymbol{I}_m & \boldsymbol{O}_m \end{bmatrix}.$$

For any two integers i, j let δ_{ij} denote the Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

2. Surfaces in 1-acyclic 3-manifolds. The main result of this section is the following theorem, which is (together with 2.13 and 2.14) an extension of Theorem 32.3 in [2]. Note that if S is a closed 2-manifold in the interior of a 1-acyclic 3-manifold W, then S separates W. The reason is that every simple closed curve in W bounds modulo 2 in W and has therefore zero intersection number modulo 2 with S. Also, since W is orientable and S separates W, S is necessarily orientable.

THEOREM 2.1. *Let W be a 1-acyclic 3-manifold and S a closed surface of genus g in* int W. *Denote by U and V the closures of the two components of $W - S$. Then there exist oriented simple closed curves $J_1, \cdots, J_g, K_1, \cdots, K_g$ in S such that*

(1) *J_i and K_i intersect transversely at a single point, for each i, and $J_i \cap J_j = J_i \cap K_j = K_i \cap K_j = \emptyset$ if $i \neq j$;*

(2) *$J_i \sim 0$ in U and $K_i \sim 0$ in V $(i = 1, \cdots, g)$;*

(3) *the homology classes of $J_1, \cdots, J_g$ form a free basis of $H_1(V)$ and the homology classes of $K_1, \cdots, K_g$ form a free basis of $H_1(U)$.*

The situation described by this theorem reminds us of the standard embedding of S in E^3; in fact, the only difference is that in the latter case we can choose $J_1, \cdots, J_g, K_1, \cdots, K_g$ so that each J_i bounds a disk (not only an orientable surface) in U and each K_i bounds a disk in V.

We postpone the proof of 2.1, which will occupy most of this section, and first prove two consequences of 2.1.

THEOREM 2.2. *Let W be a* 1-*acyclic* 3-*manifold and S a closed surface of genus g in* int W. *Denote by U and V the closures of the two components of $W - S$. Let V' be a cube with g handles. Then there exists a homeomorphism h*: Bd $V' \to S$ *such that*

(1) *the* 3-*manifold $W' = V' \cup_h U$ is* 1-*acyclic*;

(2) *if J is a closed oriented* 1-*manifold in S, then $J \sim 0$ in V if and only if $h^{-1}(J) \sim 0$ in V'.*

Proof. Assume 2.1. Think of V' as embedded in E^3; let $S' =$ Bd V' and $U' = E^3 - \text{int}\, V'$. Let J_i, K_i $(i = 1, \cdots, g)$ be oriented simple closed curves in S satisfying the conclusions of 2.1. Let $J_i', K_i' \subset S'$ have analogous meaning (with respect to U' and V'). Then there exists a homeomorphism $h: S' \to S$ which maps each J_i' onto J_i and each K_i' onto K_i (not necessarily in an orientation preserving way). Let $W' = V' \cup_h U$.

As is well-known, (1) of 2.1 implies that the homology classes of $J_1, \cdots, J_g, K_1, \cdots, K_g$ form a basis of $H_1(S)$. The homology classes of $K_1, \cdots, K_g$ belong to the kernel of $H_1(S) \to H_1(V)$; it follows from 2.1 (3) that no nontrivial linear combination of the J_i is homologous to 0 in V. Therefore, a 1-cycle in S bounds in V if and only if it is homologous in S to a linear combination of $K_1, \cdots, K_g$. Similarly, a 1-cycle in S' bounds in V' if and only if it is homologous in S' to a linear combination of $K_1', \cdots, K_g'$. Therefore, (2) of 2.2 follows directly from the choice of h.

To prove (1) of 2.2 choose an arbitrary $x \in H_1(W')$. We have to show that $x = 0$. Since S is connected and separates W', x can be represented by a sum $z_1 + z_2$ where z_1 is a 1-cycle in U and z_2 is a 1-cycle in V'. By 2.1 (3), z_1 is homologous in U to a linear combination of $K_1, \cdots, K_g$ and z_2 is homologous in V' to a linear combination of $J_1', \cdots, J_g'$. Since the sewing map h was chosen so that each $J_i' \sim 0$ in U and each $K_i \sim 0$ in V', $z_1 + z_2 \sim 0$ in W'.

THEOREM 2.3. *If C is a compact subset of a* 1-*acyclic* 3-*manifold W, then C can be embedded in a homology* 3-*sphere* (*and thus also in an acyclic* 3-*manifold unless C is itself a homology* 3-*sphere*).

Proof. We may assume that $C \subset \text{int}\, W$. Cover C by a compact connected 3-submanifold M of int W. Take a boundary component S

of M. Denote by U and V the closures of the components of $W - S$; let U be the one which contains M. By 2.2 we can replace V by a cube with handles, V', in such a way that $W' = U \cup V'$ is still 1-acyclic. If we perform a similar surgery along each boundary component of M, we end up with M embedded in a closed 1-acyclic 3-manifold, Σ say. It follows from Poincaré duality that Σ is a homology 3-sphere. If C is not a closed 3-manifold, then there is a point $p \in \Sigma - C$ and hence C lies in the acyclic 3-manifold $\Sigma - p$.

Before we start proving Theorem 2.1 we establish some homological properties of surfaces. Let S be a closed oriented surface and let $a_1, \cdots, a_n \in H_1(S)$. The *intersection number matrix* or the *sc-matrix* of the ordered n-tuple $(a_1, \cdots, a_n)$ is the $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$, where $a_{ij} = \mathrm{sc}\,(a_i, a_j)$. Obviously $\boldsymbol{A}$ is skew-symmetric. The following lemma is proved by a straightforward computation.

LEMMA 2.4. *Let S be a closed oriented surface and $a_1, \cdots, a_m, b_1, \cdots, b_n \in H_1(S)$. Let $\boldsymbol{A}$ be the sc-matrix of $(a_1, \cdots, a_m)$ and $\boldsymbol{B}$ the sc-matrix of $(b_1, \cdots, b_n)$. Suppose that there exists an $m \times n$ matrix $\boldsymbol{T}$ with integer entries such that the column vector $(a_1, \cdots, a_n)'$ is the product of $\boldsymbol{T}$ with the column vector $(b_1, \cdots, b_n)'$. Then $\boldsymbol{A} = \boldsymbol{TBT}'$.*

LEMMA 2.5. *Let S be a closed oriented surface of genus g. Let $a_1, \cdots, a_{2g} \in H_1(S)$ and let $\boldsymbol{A}$ be the* sc-*matrix of $(a_1, \cdots, a_{2g})$. Then $\{a_1, \cdots, a_{2g}\}$ is a basis of $H_1(S)$ if and only if* $\det \boldsymbol{A} = 1$.

Proof. It is well-known that $H_1(S)$ is free of rank $2g$ and that it has a basis $\{b_1, \cdots, b_{2g}\}$ whose sc-matrix is $\boldsymbol{J}_{2g}$. There exists a $2g \times 2g$ matrix $\boldsymbol{T}$ with integer entries such that $(a_1, \cdots, a_{2g})'$ is the product of $\boldsymbol{T}$ with $(b_1, \cdots, b_{2g})'$. From 2.4 we obtain $\det \boldsymbol{A} = (\det \boldsymbol{T})^2$. Obviously $\{a_1, \cdots, a_{2g}\}$ is a basis of $H_1(S)$ if and only if $\boldsymbol{T}$ has an inverse with integer entries, and this is true if and only if $\det \boldsymbol{T} = \pm 1$. The lemma follows.

COROLLARY 2.6. *Let S be a closed surface of genus g. Let A be a subgroup of $H_1(S)$ such that* $\mathrm{sc}\,(x, y) = 0$ *for any $x, y \in A$. Then the rank of A is at most g.*

Proof. Let r be the rank of A. There exists a basis $\{a_1, \cdots, a_r\}$ of A, a basis $\{b_1, \cdots, b_{2g}\}$ of $H_1(S)$, and positive integers $k_1, \cdots, k_r$ such that $a_i = k_i b_i$ $(i = 1, \cdots, r)$. Obviously $\mathrm{sc}\,(b_i, b_j) = 0$ if $i, j \leqq r$. Therefore $\boldsymbol{B}$, the sc-matrix of $(b_1, \cdots, b_{2g})$, contains a zero $r \times r$ block. If $r > g$, then $\det \boldsymbol{B} = 0$; but this is impossible by 2.5.

The next proposition is an algebraic version of Theorem 2.1.

PROPOSITION 2.7. *Let W be a 1-acyclic 3-manifold and S a closed surface of genus g in* $\operatorname{int} W$. *Denote by U and V the closures of the components of $W - S$ and by $i: S \to U$, $j: S \to V$ the inclusions. Let $A = \operatorname{Ker} i_*$, $B = \operatorname{Ker} j_*$. Then*

(1) *$H_1(S) = A \oplus B$ and either of A, B has rank g;*

(2) *$i_* \mid B: B \to H_1(U)$ and $j_* \mid A: A \to H_1(V)$ are isomorphisms;*

(3) *if $x, y \in H_1(S)$ are either both in A or both in B, then* $\operatorname{sc}(x, y) = 0$.

Proof. Consider the Mayer-Vietoris sequence of $(W; U, V)$:

$$\cdots \longrightarrow H_1(S) \xrightarrow{\alpha} H_1(U) \oplus H_1(V) \longrightarrow H_1(W) \longrightarrow \cdots .$$

Since $H_1(W) = 0$, α is an epimorphism. We will show that it is also one-to-one. Recall that α is defined by $\alpha(x) = (i_*(x), -j_*(x))$. Take an $x \in \operatorname{Ker} \alpha = A \cap B$. Represent x by a closed oriented 1-manifold $J \subset S$. Then J bounds compact, oriented, properly embedded 2-manifolds $G' \subset U$ and $G'' \subset V$. $G = G' \cup G''$ is a closed orientable 2-manifold in $\operatorname{int} W$. Let $G_1, \cdots, G_n$ be the components of G and let $G'_r = G' \cap G_r$, $G''_r = G'' \cap G_r$, $J_r = J \cap G_r = \partial G'_r = \partial G''_r$ $(r = 1, \cdots, n)$. To prove that $x = 0$ it suffices to show that each J_r bounds a compact oriented 2-submanifold of S.

G_r separates W. Let M be the union of G_r and a component of $\operatorname{int} W - G_r$. Orient M so that $\partial M = G'_r \cup (-G''_r)$. Let $M' = M \cap U$, $F = M \cap S$. Then $\operatorname{Bd} M' = F \cup G'_r$. If we orient F so that $\partial M' = (-F) \cup G'_r$, then $\partial F = \partial G'_r = J_r$. We have thus shown that $\operatorname{Ker} \alpha = 0$ and therefore α is an isomorphism. This proves (2) and the first part of (1) of our proposition. Obviously 1.1 implies (3), and (3) together with 2.6 imply the second part of (1).

Now we start proving Theorem 2.1. In the first step we will choose the homology classes for the simple closed curves which we want to construct: a_i will be the homology class of J_i, b_i of K_i. We work with a surface in limbo.

LEMMA 2.8. *Let S be a closed oriented surface of genus g. Suppose that the group $H_1(S)$ is represented as a direct sum $A \oplus B$ so that* $\operatorname{sc}(x, y) = 0$ *for any two elements $x, y \in H_1(S)$ which lie either both in A or both in B. Then there exist bases $\{a_1, \cdots, a_g\}$ of A and $\{b_1, \cdots, b_g\}$ of B such that* $\operatorname{sc}(a_i, b_j) = \delta_{ij}$ *for each i and j.*

ADDENDUM 2.9. *Let $0 \leqq r \leqq g$ and $0 \leqq s \leqq g$. Suppose that $\{a_1, \cdots, a_r\}$ is a basis of a direct summand of A and $\{b_1, \cdots, b_s\}$ is a basis of a direct summand of B such that* $\operatorname{sc}(a_i, b_j) = \delta_{ij}$ $(i = 1, \cdots, r;$

$j = 1, \cdots, s$). *Then we can find* $a_{r+1}, \cdots, a_g, b_{s+1}, \cdots, b_g$ *such that* $a_1, \cdots, a_g, b_1, \cdots, b_g$ *satisfy the conclusion of* 2.8.

Proofs of 2.8 *and* 2.9. First note that 2.6 implies that A and B have rank g. Assume that $r \geqq s$. If $r < g$, choose any elements $a'_{r+1}, \cdots, a'_g$ such that $\{a_1, \cdots, a_r, a'_{r+1}, \cdots, a'_g\}$ is a basis of A. Then set

$$a_i = a'_i - \sum_{k=1}^{s} \operatorname{sc}(a'_i, b_k)a_k , \quad i = r + 1, \cdots, g$$

(if $s = 0$, let $a_i = a'_i$). Obviously $a_1, \cdots, a_g$ again form a basis of A and $\operatorname{sc}(a_i, b_j) = \delta_{ij}$ for $1 \leqq i \leqq g$, $1 \leqq j \leqq s$.

Let us first consider the case $s = 0$. Choose an arbitrary basis $\{b'_1, \cdots, b'_g\}$ of B. Let $\boldsymbol{C}$ be the sc-matrix of $(a_1, \cdots, a_g, b'_1, \cdots, b'_g)$. Then

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{O}_g & \boldsymbol{D} \\ -\boldsymbol{D}' & \boldsymbol{O}_g \end{bmatrix}$$

where $\boldsymbol{D}$ is the $g \times g$ matrix whose (i, j)-entry is $\operatorname{sc}(a_i, b'_j)$. By 2.5, $\det \boldsymbol{C} = (\det \boldsymbol{D})^2 = 1$, therefore, $\boldsymbol{D}'$ has an inverse $\boldsymbol{U} = (u_{ij})$ with integer entries. Put

2.10 $$b_i = \sum_{j=1}^{g} u_{ij}b'_j \quad (i = 1, \cdots, g) .$$

Let

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I}_g & \boldsymbol{O}_g \\ \boldsymbol{O}_g & \boldsymbol{U} \end{bmatrix} .$$

Then, by 2.4, the sc-matrix of $(a_1, \cdots, a_g, b_1, \cdots, b_g)$ is $\boldsymbol{TCT}' = \boldsymbol{J}_{2g}$. This is what we wished to have.

If $s > 0$ we work in the same way except that we do not start with an arbitrary basis $\{b'_1, \cdots, b'_g\}$ of B. Choose $b''_{s+1}, \cdots, b''_g \in B$ such that $b_1, \cdots, b_s, b''_{s+1}, \cdots, b''_g$ form a basis of B. Then set $b'_i = b_i$ for $i = 1, \cdots, s$ and

$$b'_i = b''_i - \sum_{k=1}^{s} \operatorname{sc}(a_k, b''_i)b_k \quad \text{for} \quad i = s + 1, \cdots, g .$$

Then $\{b'_1, \cdots, b'_g\}$ is a basis of B and $\operatorname{sc}(a_i, b'_j) = \delta_{ij}$ unless $i, j > s$. This means that the matrices $\boldsymbol{D}$ and $\boldsymbol{U}$, defined as above, have the form

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{I}_s & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{E} \end{bmatrix} , \qquad \boldsymbol{U} = \begin{bmatrix} \boldsymbol{I}_s & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{V} \end{bmatrix} ,$$

where $\boldsymbol{E}$ is a $(g - s) \times (g - s)$ matrix and $\boldsymbol{V} = (\boldsymbol{E}')^{-1}$. Therefore, the defining formula 2.10 yields the a priori given b_i for $i = 1, \cdots, s$.

Lemma 2.8 and its Addendum are proved.

We have chosen the homology classes of our future simple closed curves J_i and K_i. Now we will show that the chosen homology classes can really be represented by simple closed curves.

PROPOSITION 2.11. *Let S be a closed surface and $x \in H_1(S)$. Then there exist an oriented simple closed curve $J \subset S$ and a positive integer n such that x is the homology class of the 1-cycle nJ.*

This proposition obviously follows from.

LEMMA 2.12. *Let S be a closed surface and $K \subset S$ a closed oriented 1-manifold. Then there exists a sequence $K^{(1)}, K^{(2)}, \cdots, K^{(m)}$ of closed oriented 1-submanifolds of S such that*

(1) $K^{(1)} = K$;

(2) *$K^{(i+1)}$ is obtained from $K^{(i)}$ either by omitting a component of $K^{(i)}$ which separates S or by piping together two components of $K^{(i)}$ $(i = 1, \cdots, m-1)$;*

(3) *any two components of $K^{(m)}$ are homologous in S.*

Proof. We use induction on the number of components of K. If K is connected, then there is nothing to prove. Suppose that 2.12 is true if K has less than n components $(n \geqq 2)$. Choose a closed oriented 1-manifold $K \subset S$ which has n components, say $K_1, \cdots, K_n$. Denote by T the 2-manifold obtained by cutting S along K, and let $p: T \to S$ be the corresponding identification map. Let L_{r1}, L_{r2} be the two components of $p^{-1}(K_r)$ $(r = 1, \cdots, n)$; orient them so that p maps each of them onto K_r in an orientation preserving way.

Case 1. Suppose that T has a component T_0 with connected boundary; let for instance $\mathrm{Bd}\, T_0 = L_{ri}$. Then K_r separates S. By induction hypothesis, 2.12 holds for $K' = K - K_r$. It obviously follows that 2.12 holds for K.

Case 2. Suppose that T has a component T_0 which has more than two boundary components. Then T_0 can be oriented so that two of its boundary components, say L_{ri} and L_{sj}, are oriented coherently with T_0. By 1.2, L_{ri} and L_{sj} can be piped together along any properly embedded arc $A \subset T_0$. We claim that $r \neq s$. Indeed, S is obtained from T by sewing each L_{k1} to L_{k2} by an orientation preserving homeomorphism and hence, if L_{k1} and L_{k2} lie in the same component of T and if T is given any orientation, one of L_{k1}, L_{k2} is oriented coherently and the other incoherently with T. It follows that we can pipe K_r to K_s along the arc $p(A)$. Denote by K_r' the oriented simple closed curve obtained by this piping. By induction hypothesis, 2.12 holds for $K' = (K - K_r - K_s) \cup K'_r$, hence it holds for K.

Case 3. Finally we consider the case when each component of T is bounded by exactly two simple closed curves. If some component of T can be oriented coherently with both its boundary components, then we prove as in Case 2 that 2.12 holds for K. Suppose that no component of T can be oriented coherently with its boundary. Obviously T has n components, say $T_1, \cdots, T_n$. Let $S_i = p(T_i)$ $(i = 1, \cdots, n)$. Since $n > 1$, no S_i is a closed surface and therefore each S_i is bounded by two components of K. We may assume that the numbering has been chosen so that $\operatorname{Bd} S_i = K_i \cup K_{i+1}$ $(i = 1, \cdots, n-1)$ and $\operatorname{Bd} S_n = K_n \cup K_1$. Since no T_i can be oriented coherently with its boundary, the same holds for S_i. This means that each S_i can be oriented so that $\partial S_i = (-K_i) \cup K_{i+1}$ $(i = 1, \cdots, n-1)$. It follows that $K_1 \sim K_2 \sim \cdots \sim K_n$ in S. This concludes the proof of 2.12.

Theorem 2.1 follows from 2.7 and the following proposition.

PROPOSITION 2.13. *Let S be a closed oriented surface of genus g. Suppose that $H_1(S)$ is represented as a direct sum $A \oplus B$ so that* $\operatorname{sc}(x, y) = 0$ *for any $x, y \in H_1(S)$ that lie either both in A or both in B. Then there exist oriented simple closed curves J_i, K_i in S $(i = 1, \cdots, g)$ such that*

(1) *for each i, $J_i \cap K_i$ is a single point and* $\operatorname{sc}(J_i, K_i) = 1$; *if $i \neq j$, then $J_i \cap J_j = J_i \cap K_j = K_i \cap K_j = \emptyset$;*

(2) *the homology classes of $J_1, \cdots, J_g$ form a basis of A and the homology classes of $K_1, \cdots, K_g$ form a basis of B.*

ADDENDUM 2.14. *Suppose that we are given elements $a_1, \cdots, a_{r'} \in A$, $b_1, \cdots, b_{s'} \in B$ $(0 \leqq r' \leqq g,\ 0 \leqq s' \leqq g)$ and oriented simple closed curves $J_1, \cdots, J_r$, $K_1, \cdots, K_s \subset S$ $(0 \leqq r \leqq r',\ 0 \leqq s \leqq s')$ such that the following conditions are satisfied:*

(i) *if $i \leqq \min(r, s)$, $J_i \cap K_i$ is a single point; if $i \neq j$, then $J_i \cap J_j = \emptyset$, $J_i \cap K_j = \emptyset$, $K_i \cap K_j = \emptyset$ (each of these equalities is satisfied for all pairs i, j for which it makes sense);*

(ii) *a_i is the homology class of J_i and b_j is the homology class of K_j $(i = 1, \cdots, r;\ j = 1, \cdots, s)$;*

(iii) *$\{a_1, \cdots, a_{r'}\}$ is a basis of a direct summand of A and $\{b_1, \cdots, b_{s'}\}$ is a basis of a direct summand of B;*

(iv) $\operatorname{sc}(a_i, b_j) = \delta_{ij}$ $(i = 1, \cdots, r';\ j = 1, \cdots, s')$. *Then there exist oriented simple closed curves $J_{r+1}, \cdots, J_g$, $K_{s+1}, \cdots, K_g$ such that J_i represents a_i $(i = r+1, \cdots, r')$, K_j represents b_j $(j = s+1, \cdots, s')$, and $J_1, \cdots, J_g$, $K_1, \cdots, K_g$ satisfy the conclusions of* 2.13.

REMARK. It is not difficult to show that if $J_1, \cdots, J_r$ are disjoint oriented simple closed curves in S such that $S - (J_1 \cup \cdots \cup J_r)$ is connected, then the homology classes of these curves freely generate a direct summand of $H_1(S)$. Therefore, if $r' = r$ and $s' = s$, we can replace the condition (iii) above by: $a_i \in A$, $b_i \in B$, and $S - (J_1 \cup \cdots \cup J_r)$, $S - (K_1 \cup \cdots \cup K_s)$ are connected.

In the proof of 2.13 and 2.14 we shall need the following three lemmas. The proofs of 2.15 and 2.17 are easy and we omit them.

LEMMA 2.15. *Let S be a surface, $J \subset S$ an oriented simple closed curve, and $L \subset S$ an oriented closed* 1*-manifold. Orient S so that $n = \operatorname{sc}(J, L) \geqq 0$. Then L is homologous to an oriented closed* 1*-manifold $K \subset S$ such that $J \cap K$ contains exactly n points.*

The K of 2.15 may have to have more components than L. On the other hand, the following lemma is valid.

LEMMA 2.16. *Let S be an oriented surface, $J \subset \operatorname{int} S$ an oriented simple closed curve, and $L \subset \operatorname{int} S$ an oriented closed* 1*-manifold such that $\operatorname{sc}(J, L) = 1$. Then L is homologous to an oriented simple closed curve $K \subset S$ such that $J \cap K$ contains exactly* 1 *point.*

Proof. By 2.15 we may assume that $J \cap L$ has only one point. We will prove the lemma by induction on the number of components of L. If this number is 1, we can take $K = L$. Suppose that 2.16 is true if L has at most n components ($n \geqq 1$). Take an L with $n + 1$ components, say $L_0, L_1, \cdots, L_n$; let L_0 be the component which intersects J.

Denote by T the 2-manifold obtained by cutting S along all components of L and let $p: T \to S$ be the corresponding identification map. Let L_i', L_i'' be the two boundary components of T composing $p^{-1}(L_i)$ ($i = 0, 1, \cdots, n$). Orient L_i' and L_i'' so that p maps each of them onto L_i in an orientation preserving way. Obviously $p^{-1}(J)$ is an arc connecting L_0' and L_0''. Hence L_0'' and L_0' lie in the same component, T_0 say, of T. Clearly, $\operatorname{Bd} T_0$ intersects $p^{-1}(L - L_0)$. Changing the notation, if necessary, we can assume that $L_1' \subset \operatorname{Bd} T_0$. Orient T_0 coherently with L_1'. Then one of L_0', L_0'' is oriented coherently with T_0 and the other incoherently. Assume that L_0' is oriented coherently with T_0. Let $A \subset T_0$ be a properly embedded arc which misses $p^{-1}(J)$ and joins L_0' to L_1'. By 1.2 we can pipe L_0' to L_1' along A. It follows that in S we can pipe L_0 to L_1 along the arc $p(A)$, whose interior misses $J \cup L$. This piping changes L to a closed oriented 1-manifold, homologous to L, which still intersects J at a single point and has only n components. Therefore, the induction hypothesis implies that 2.16 holds for L.

LEMMA 2.17. *Let S be a closed surface of genus $g > 0$ and let $J, K \subset S$ be two simple closed curves crossing each other at a single point. Denote by T the surface obtained by cutting S along J and K and let $p: T \to S$ be the corresponding identification map. Let S' be the closed surface obtained by attaching a disk to T along the boundary curve $p^{-1}(J \cup K)$ of T; let $k: T \to S'$ be the inclusion. Then k_* is an isomorphism and $p_* k_*^{-1}: H_1(S') \to H_1(S)$ maps $H_1(S')$ isomorphically onto the direct summand of $H_1(S)$ which consists of homology classes of 1-cycles that have zero intersection numbers with both J and K. Moreover, if we orient S and S' so that p preserves orientation, then $p_* k_*^{-1}$ preserves intersection numbers.*

Proofs of 2.13 *and* 2.14. By 2.8 and 2.9 we can assume that $r' = s' = g$. We also assume that $r \geqq s$.

The proof is by induction on the genus of S. If this genus is 0, there is nothing to prove. Suppose that 2.13 and 2.14 are true if the genus of S is less than $g (g > 0)$ and consider a situation with the genus of S equal to g.

If $r > 0$, then we already have J_1. If $r = 0$, choose any oriented simple closed curve representing a_1 (it follows from 2.11 that one such exists) and call it J_1. If $s > 0$ (and hence $r > 0$ by our assumption), then we already have K_1. Suppose that $s = 0$. Represent b_1 by a closed oriented 1-manifold L. By 2.15 we can assume that $L \cap (J_2 \cup \cdots \cup J_r) = \emptyset$. Applying 2.16 to $S - (J_2 \cup \cdots \cup J_r)$, J_1, and L we can find an oriented simple closed curve $K_1 \sim L$ such that $J_1 \cap K_1$ is a single point and $K_1 \cap (J_2 \cup \cdots \cup J_r) = \emptyset$.

We can therefore assume that we already have a "good" pair J_1, K_1, either preassigned or constructed as described above. Define T, p, S', and k as in the statement of 2.17, with J_1 and K_1 taking the roles of J and K, respectively. It follows from 2.17 that S', $g' = g - 1$, $A' = k_* p_*^{-1}(A)$, $B' = k_* p_*^{-1}(B)$, $a'_i = k_* p_*^{-1}(a_i)$ and $b'_i = k_* p_*^{-1}(b_i)$ $(i = 2, \cdots, g)$, $J'_i = kp^{-1}(J_i)$ and $K'_j = kp^{-1}(K_j)$ $(i = 2, \cdots, r;$ $j = 2, \cdots, s)$ satisfy the hypotheses of 2.13 and 2.14. By induction hypothesis we can represent each a'_i by an oriented simple closed curve $J'_i \subset k(T) \subset S'$ and each b'_j by an oriented simple closed curve $K'_j \subset k(T) \subset S'$ $(i = r + 1, \cdots, g;$ $j = s + 1, \cdots, g)$ such that $J'_2, \cdots, J'_g, K'_2, \cdots, K'_g$ satisfy (1) of 2.13. Let $J_i = pk^{-1}(J'_i)$, $K_j = pk^{-1}(K'_j)$ $(i = r + 1, \cdots, g;$ $j = s + 1, \cdots, g)$. Then $J_1, \cdots, J_g$, $K_1, \cdots, K_g$ satisfy the conclusions of 2.13 and 2.14.

We conclude this section with a proof of the following theorem.

THEOREM 2.18. *Let U be a cube with g handles. Denote* Bd U *by S and let $i: S \to U$ be the inclusion. Let $\{a_1, \cdots, a_g\}$ be any basis of* Ker i_*. *Then we can represent each a_r by an oriented simple* *closed curve* $J_r \subset S$ $(r = 1, \cdots, g)$ *such that* $J_1, \cdots, J_g$ *bound disjoint disks in* U.

This result is implicitly contained in pp. 296–299 of [2]. But perhaps it is worth while stating and proving it explicitly. Let us first consider the following weaker lemma.

LEMMA 2.19. *Let* U, S, *and* i *be as in* 2.18. *Then every* $x \in \operatorname{Ker} i_*$ *can be represented by a* 1-*cycle* nJ *where* n *is a positive integer and* $J \subset S$ *is an oriented simple closed curve which bounds a disk in* U.

Proof. We can assume that $x \neq 0$. Let $\{K_1, \cdots, K_g\}$ be a collection of oriented simple closed curves in S which bound disjoint disks in U and whose union does not separate S. Then the homology classes of $K_1, \cdots, K_g$ form a basis of $\operatorname{Ker} i_*$. Therefore, there exist integers $n_1, \cdots, n_g$ such that the 1-cycle $n_1K_1 + \cdots + n_gK_g$ represents x. This obviously implies that x can be represented by an oriented closed 1-manifold K such that the components of K bound disjoint disks in U. Therefore, 2.19 easily follows from 2.12 and the following obvious lemma.

LEMMA 2.20. *Let* U *be a* 3-*manifold and let* L_1, L_2 *be oriented simple closed curves in* $\operatorname{Bd} U$ *bounding disjoint properly embedded disks* E_1 *and* E_2, *respectively, in* U. *Suppose that* L_1 *can be piped to* L_2 *along an arc* $A \subset \operatorname{Bd} U$ *and let* L *be the simple closed curve obtained by this piping. Let* N *be a neighborhood of* A *in* U *containing the "pipe"* $L - (L_1 \cup L_2)$. *Then* L *bounds a properly embedded disk* $E \subset U$ *which is contained in* $E_1 \cup E_2 \cup N$.

Proof of 2.18. Suppose that 2.18 is false and take the smallest g for which 2.18 fails. By 2.19, $g > 1$.

Embed U into E^3. Let $V = E^3 - \operatorname{int} U$ and let $j: S \to V$ be the inclusion. Choose an orientation for S. It follows from 2.7, 2.8, and 2.9 that there exists a basis $\{b_1, \cdots, b_g\}$ of $\operatorname{Ker} j_*$ such that $\operatorname{sc}(a_r, b_s) = \delta_{rs}$ $(r, s = 1, \cdots, g)$. By 2.19 we can represent a_1 by an oriented simple closed curve $J_1 \subset S$ which bounds a properly embedded disk $D_1 \subset U$. Obviously $U - D_1$ is connected.

Represent a_r by an oriented closed 1-manifold $K_r \subset S$ and b_r by an oriented closed 1-manifold $L_r \subset S$ $(r = 2, \cdots, g)$; by 2.15 we may assume that $J_1 \cap K_r = J_1 \cap L_r = \varnothing$. Choose compact, oriented, properly embedded 2-manifolds $F_r \subset U$, $G_r \subset V$ such that $\partial F_r = K_r$, $\partial G_r = L_r$. If F_r intersects D_1, we can put F_r in general position with D_1, remove the part of F_r which lies in a regular neighborhood of D_1, and then patch the resultant holes in F_r by disjoint disks "parallel" to D_1. In this manner we can replace F_r by another compact, oriented, properly embedded 2-manifold in U such that it is bounded by K_r and misses D_1. Therefore, we will assume that the originally chosen $F_2, \cdots, F_g$ were already disjoint from D_1.

Choose a regular neighborhood N of D_1 in E^3 and let $U' = U - \operatorname{int} N$, $V' = V \cup N$, $S' = \operatorname{Bd} U' = \operatorname{Bd} V'$. Let i' and j' be the inclusions of S' in U' and V', respectively. U' is again a cube with handles ([2], 6.2). Let $T = S \cap S'$. Then $S' - \operatorname{int} T$ consists of two disjoint disks, which we denote by D_1' and D_1''. Let $J_1' = \operatorname{Bd} D_1'$, $J_1'' = \operatorname{Bd} D_1''$. Orient J_1' and J_1'' so that $J_1' \sim J_1 \sim J_1''$ in S. Let $a_r', b_r' \in H_1(S')$ be the homology classes of K_r, L_r, respectively ($r = 2, \cdots, g$). Since K_r bounds F_r in U' and L_r bounds G_r in V' we have $a_r' \in \operatorname{Ker} i_*'$, $b_r' \in \operatorname{Ker} j_*'$. If we give S' the orientation which on T agrees with the chosen orientation of S, then $\operatorname{sc}(a_r', b_s') = \operatorname{sc}(K_r, L_s) = \delta_{rs}$. Thus it follows from 2.5 that the a_r' and b_r' form a basis of $H_1(S')$ and therefore, $\{a_2', \cdots, a_g'\}$ is a basis of $\operatorname{Ker} i_*'$.

By supposition, 2.18 is true for cubes with $g - 1$ handles. Thus there exist oriented simple closed curves $J_2', \cdots, J_g' \subset S'$ and disjoint properly embedded disks $D_2', \cdots, D_g'$ in U' such that for each r the following are true:

(a) J_r' is in the homology class a_r' and hence $J_r' \sim K_r$ in S';

(b) $J_r' = \operatorname{Bd} D_r'$.

Without loss of generality we can assume

(c) $J_r' \subset \operatorname{int} T$.

Note that $T - (J_2' \cup \cdots \cup J_g')$ is connected.

It is easy to see that (a) above implies that $K_r \sim J_r' + n_r' J_1' + n_r'' J_1''$ in T for some integers n_r', n_r''. Hence $K_r \sim J_r' + n_r J_1$ in S, where $n_r = n_r' + n_r''$. We will therefore try to replace each $J_r' + n_r J_1$ by a homologous oriented simple closed curve bounding a disk in U.

Suppose that $n_2 \neq 0$. Let for instance $n_2 > 0$. We can show, by the same argument as twice before, that it is possible to pipe J_2' to J_1' along an arc whose interior misses $J_1' \cup J_1'' \cup J_2' \cup \cdots \cup J_g'$. (If $n_2 < 0$, we pipe J_2' to $-J_1'$.) By this piping we obtain an oriented simple closed curve J_2''; 2.20 implies that J_2'' bounds a properly embedded disk $D_2'' \subset U$ which is disjoint from $D_1, D_3', \cdots, D_g'$. We replace J_2' by J_2'' and D_2' by D_2''. Now we have $K_2 \sim J_2'' + m_2 J_1$, where $|m_2| = |n_2| - 1$. It should now be clear how to finish the proof of 2.18 by induction on the number $|n_2| + \cdots + |n_g|$.

3. Compact 3-submanifolds of acyclic 3-manifolds. In this section we will prove the following two theorems.

THEOREM 3.1. *A compact connected* 3-*manifold M whose boundary* *has m components* ($m > 0$) *is subacyclic if and only if it satisfies the following conditions* (1), (2), *and either* (3′) *or* (3″):

(1) *M is orientable;*

(2) *$H_1(M)$ is free;*

(3′) *$H_2(M)$ is free of rank $m-1$;*

(3″) *$H_1(\mathrm{Bd}\, M) \to H_1(M)$ is onto.*

THEOREM 3.2. *Let M be a compact, connected, subacyclic 3-manifold and J a closed oriented 1-manifold lying in a boundary component S of M. Let F be an oriented surface and $h: \partial F \to J$ an orientation preserving homeomorphism. Then the polyhedron $P = F \cup_h M$ can be embedded in an acyclic 3-manifold if and only if J satisfies one of the following two conditions.*

(1) *The homology class of J in M is a basic element of $H_1(M)$.*

(2) *There exist compact 2-submanifolds $G, H \subset S$ such that $G \cup H = S$, $G \cap H = J$, and there exists an orientation of S such that $\partial G = -\partial H = J$.*

The proof of 3.1 in one direction is quite easy. Suppose that M lies in an open acyclic 3-manifold W. Then M is orientable. Let V be the closure of $W - M$. The Mayer-Vietoris sequence of $(W; M, V)$ contains the following subsequence

$$0 \longrightarrow H_1(\mathrm{Bd}\, M) \longrightarrow H_1(M) \oplus H_1(V) \longrightarrow 0 .$$

It follows that $H_1(M)$ is free and that $H_1(\mathrm{Bd}\, M) \to H_1(M)$ is onto.

The other direction of 3.1 will be proved by induction on m. First we show that the conditions (3′) and (3″) of 3.1 are equivalent.

LEMMA 3.3. *Let M be a compact connected 3-manifold with m boundary components and suppose that* (1) *and* (2) *of* 3.1 *are satisfied. Then* (3′) *and* (3″) *of* 3.1 *are equivalent and they imply that $H_2(M, \mathrm{Bd}\, M) \approx H_1(M)$ and that the following sequence is split exact:*

$$0 \longrightarrow H_2(M, \mathrm{Bd}\, M) \xrightarrow{\partial_*} H_1(\mathrm{Bd}\, M) \xrightarrow{i_*} H_1(M) \longrightarrow 0$$

(*here ∂_* and i_* are the homomorphisms from the homology sequence of the pair* $(M, \mathrm{Bd}\, M)$).

Proof. Since $H_1(M)$ and $H_2(M)$ are free, duality and the Universal Coefficient Theorem yield the following two relations

$$H_1(M, \mathrm{Bd}\, M) \approx H^2(M) \approx H_2(M),\ H_2(M, \mathrm{Bd}\, M) \approx H^1(M) \approx H_1(M) .$$

Consider the exact sequence for the reduced homology of the pair $(M, \mathrm{Bd}\, M)$:

$$\cdots \longrightarrow H_2(\mathrm{Bd}\, M) \longrightarrow H_2(M) \longrightarrow H_2(M, \mathrm{Bd}\, M) \longrightarrow$$
$$\longrightarrow H_1(\mathrm{Bd}\, M) \longrightarrow H_1(M) \longrightarrow H_1(M, \mathrm{Bd}\, M) \longrightarrow \widetilde{H}_0(\mathrm{Bd}\, M) \longrightarrow 0\,.$$

Suppose that $H_1(\mathrm{Bd}\, M) \to H_1(M)$ is onto. Then $\widetilde{H}_0(\mathrm{Bd}\, M) \approx H_1(M, \mathrm{Bd}\, M) \approx H_2(M)$ and hence $H_2(M)$ is free of rank $m - 1$. Now suppose that $H_2(M)$ is free of rank $m-1$. Then $H_1(M, \mathrm{Bd}\, M) \to \widetilde{H}_0(\mathrm{Bd}\, M)$ is an epimorphism of free groups of the same rank and thus it is actually an isomorphism. It follows that $H_1(\mathrm{Bd}\, M) \to H_1(M)$ is onto.

We conclude the proof of 3.3 by showing that (3′) of 3.1 implies that $\partial_*\colon H_2(M, \mathrm{Bd}\, M) \to H_1(\mathrm{Bd}\, M)$ is one-to-one. It suffices to show that $H_2(\mathrm{Bd}\, M) \to H_2(M)$ is onto, and this follows from the fact that the image of $H_2(\mathrm{Bd}\, M) \to H_2(M)$ is free of rank $m-1$ (this is true for any 3-manifold M which has exactly m compact orientable boundary components) and that $H_2(M, \mathrm{Bd}\, M)$ is torsion free.

Now we start proving the remaining direction of 3.1.

LEMMA 3.4. *Let M be a compact connected 3-manifold having precisely m boundary components and satisfying* (1), (2), *and* (3′) *of* 3.1. *Suppose that there exists an oriented simple closed curve $K \subset \mathrm{Bd}\, M$ such that the homology class of K in M is a basic element of $H_1(M)$. Then M can be embedded in a compact connected 3-manifold M' which has again m boundary components, again satisfies* (1), (2), *and* (3′) *of* 3.1, *and whose boundary has smaller genus than* Bd M.

Proof. Denote by S the component of Bd M which contains K and let A be a regular neighborhood of K in S. Let M' be the 3-manifold obtained by attaching a 2-handle H to M along A. Since K does not separate S, M' has exactly m boundary components and Bd M' has smaller genus than Bd M. Obviously M' is compact, connected, and orientable. By considering the Mayer-Vietoris sequence of $(M'; M, H)$ for reduced homology we can prove that $H_1(M')$ is free and $H_2(M') \approx H_2(M)$.

LEMMA 3.5. *Theorem* 3.1 *is valid for* $m = 1$.

Proof. Suppose that this is false. Among all 3-manifolds M which are counterexamples to 3.1 for $m = 1$ choose one whose boundary has the smallest genus. Because of 2.3, $H_1(M)$ is nontrivial. Choose a basic element $x \in H_1(M)$. It follows from 3.1 (3″) and 2.11 that x can be represented by a simple closed curve $K \subset \mathrm{Bd}\, M$. But then 3.4 yields a 3-manifold M' which is a counterexample to 3.1 for $m = 1$ and whose boundary has smaller genus than Bd M. This contradicts our choice of M.

LEMMA 3.6. *Theorem* 3.1 *is valid for* $m = 2$.

Proof. Suppose that the lemma is false. Choose a 3-manifold M which is a counterexample to 3.1 for $m = 2$ and whose boundary has the smallest possible genus. Our plan is to find a simple closed curve $K \subset \mathrm{Bd}\, M$ representing a basic element of $H_1(M)$; as in 3.5 this will lead to a contradiction.

Let S' and S'' be the two components of $\mathrm{Bd}\, M$, let g' be the genus of S' and g'' the genus of S'', and let $i'\colon S' \to M$, $i''\colon S'' \to M$, $i\colon \mathrm{Bd}\, M \to M$ be inclusions. By 3.3, $H_1(M)$ has rank $g' + g''$ and therefore $g' + g'' > 0$.

SUBLEMMA 1. $\mathrm{Ker}\, i'_* = \mathrm{Ker}\, i''_* = 0$.

Proof. Suppose that e.g. $\mathrm{Ker}\, i'_* \neq 0$. Since $H_1(M)$ is free, $\mathrm{Ker}\, i'_*$ is a direct summand of $H_1(S')$. Therefore, it follows from 2.11 that there exists a nonseparating oriented simple closed curve $J \subset S'$ such that $J \sim 0$ in M. Let $K \subset S'$ be a simple closed curve intersecting J transversely at exactly one point. Choose an orientation for M, orient $\mathrm{Bd}\, M$ coherently with M, and then orient K so that $\mathrm{sc}\,(J, K) = 1$.

We claim that K represents a basic element of $H_1(M)$. Suppose that for some oriented closed 1-manifold $L \subset M$ and for some positive integer n, K is homologous to nL in M. Since M satisfies (3″) of 3.1 we can assume that $L \subset \mathrm{Bd}\, M$. Then $K - nL$ is a 1-cycle in $\mathrm{Bd}\, M$, homologous to 0 in M. By 1.1, $1 - n\, \mathrm{sc}\,(J, L) = \mathrm{sc}\,(J, K - nL) = 0$. Hence $n = 1$ and consequently K represents a basic element of $H_1(M)$. As we know, this leads to a contradiction and hence our supposition above must be wrong. Sublemma 1 is proved.

Identify $H_1(\mathrm{Bd}\, M)$ with $H_1(S') \oplus H_1(S'')$ and let $p'\colon H_1(\mathrm{Bd}\, M) \to H_1(S')$, $p''\colon H_1(\mathrm{Bd}\, M) \to H_1(S'')$ be natural projections.

SUBLEMMA 2. The compositions

$$p'\partial_*\colon H_2(M, \mathrm{Bd}\, M) \longrightarrow H_1(S')\,, \qquad p''\partial_*\colon H_2(M, \mathrm{Bd}\, M) \longrightarrow H_1(S'')$$

are monomorphisms and hence $g' = g''$.

Proof. Let $x \in H_2(M, \mathrm{Bd}\, M)$ be such that $p'\partial_*(x) = 0$. Then $i''_* p''\partial_*(x) = i'_* p'\partial_*(x) + i''_* p''\partial_*(x) = i_*\partial_*(x) = 0$. This equality and Sublemma 1 imply that $p''\partial_*(x) = 0$. Therefore, $\partial_*(x) = 0$ and hence, by 3.3, $x = 0$. Similarly we show that $p''\partial_*$ is one-to-one.

By 3.3, $H_2(M, \mathrm{Bd}\, M)$ has rank $g' + g''$. Since $p'\partial_*$ is one-to-one, $g' + g'' \leqq 2g'$; similarly, $g' + g'' \leqq 2g''$. Hence $g' = g''$.

Denote $g' = g''$ by g.

SUBLEMMA 3. There exist oriented simple closed curves $J, K \subset \mathrm{Bd}\, M$, one lying in S' and the other in S'' and neither homologous to 0 in $\mathrm{Bd}\, M$, and there exists a positive integer r such that $J + rK \sim 0$ in M.

Proof. Choose a basis $\{a'_1, \cdots, a'_{2g}\}$ of $H_1(S')$, a basis $\{b'_1, \cdots, b'_{2g}\}$ of $E' = p'\partial_* H_2(M, \mathrm{Bd}\, M)$, and positive integers $n'_1, \cdots, n'_{2g}$ such that $b'_j = n'_j a'_j$ $(j = 1, \cdots, 2g)$. Let n' be the greatest common divisor of $n'_1, \cdots, n'_{2g}$.

Suppose that $n' = 1$. Then E' contains a basic element of $H_1(S')$. Indeed; suppose that no element of E' is basic for $H_1(S')$. Then there exists a prime q such that $E' \subset qH_1(S')$ (see e.g. [1], 5.1.1). Since $n' = 1$ there is an s $(1 \leqq s \leqq 2g)$ such that n'_s is not divisible by q. Then, as a'_s is a basic element of $H_1(S')$, $b'_s = n'_s a'_s$ is not equal to qx for any $x \in H_1(S')$ and this contradicts our previous conclusion. Thus there really exists a $b' \in E'$ which is a basic element of $H_1(S')$. Let $b'' = p''\partial_*(p'\partial_*)^{-1}(b') \in H_1(S'')$. By Sublemma 2, $b'' \neq 0$. It follows from 2.11 that there exist oriented simple closed curves $J \subset S'$, $K \subset S''$ and a positive integer r such that J represents b' and rK represents b''. Since $(b', b'') = \partial_*(p'\partial_*)^{-1}(b') \in \mathrm{Ker}\, i_*$, $J + rK \sim 0$ in M. Thus Sublemma 3 is true in this case.

Now suppose that $n' > 1$. For each j let $b''_j = p''\partial_*(p'\partial_*)^{-1}(b'_j) \in H_1(S'')$; let a''_j be the basic element of $H_1(S'')$ and n''_j the positive integer such that $b''_j = n''_j a''_j$. Let n'' be the greatest common divisor of $n''_1, \cdots, n''_{2g}$. If $n'' = 1$ we show as above that Sublemma 3 is valid. Suppose that $n'' > 1$. We will show that this leads to a contradiction. Choose a prime divisor q of n''. By 2.5 the determinant of the intersection number matrix of $(a'_1, \cdots, a'_{2g})$ is equal to 1. Therefore, there exists an entry of this matrix, say $\mathrm{sc}\,(a'_s, a'_t)$, which is not divisible by q. Note that each pair (b'_j, b''_j) lies in $\mathrm{Ker}\, i_*$. Therefore, 1.1 implies that $\mathrm{sc}\,((b'_s, b''_s), (b'_t, b''_t)) = 0$ and hence $\mathrm{sc}\,(b'_s, b'_t) = -\mathrm{sc}\,(b''_s, b''_t)$. Since the number $n'_s n'_t\, \mathrm{sc}\,(a'_s, a'_t) = -n''_s n''_t\, \mathrm{sc}\,(a''_s, a''_t)$ is divisible by q and $\mathrm{sc}\,(a'_s, a'_t)$ is not, one of n'_s, n'_t, say n'_s, is divisible by q. Let for instance $n'_s = qk'$, $n''_s = qk''$. Put $a' = k'a'_s$, $a'' = k''a''_s$. Then the basic element $(p'\partial_*)^{-1}(b'_s)$ of $H_2(M, \mathrm{Bd}\, M)$ is mapped by ∂_* to $(b'_s, b''_s) = q(a', a'')$. This contradicts the fact that ∂_* embeds $H_2(M, \mathrm{Bd}\, M)$ as a direct summand into $H_1(\mathrm{Bd}\, M)$. Sublemma 3 is proved.

We conclude the proof of 3.6 with

SUBLEMMA 4. K represents a basic element of $H_1(M)$.

Proof. Assume that $J \subset S'$, $K \subset S''$. Let $u \in H_1(S')$, $v \in H_1(S'')$ be the homology classes of J, K, respectively; let $x = \partial_*^{-1}(u, rv) \in$

$H_2(M, \operatorname{Bd} M)$. Since u is a basic element of $H_1(S')$, x is basic for $H_2(M, \operatorname{Bd} M)$. Let $H_2(M, \operatorname{Bd} M) = A \oplus B$ where A is the subgroup generated by x. Let A' be the subgroup of $H_1(S')$ generated by u, A'' the subgroup of $H_1(S'')$ generated by v, B' the smallest direct summand of $H_1(S')$ containing $p'\partial_*(B)$, and B'' the smallest direct summand of $H_1(S'')$ containing $p''\partial_*(B)$. Then $H_1(S') = A' \oplus B'$ and $H_1(S'') = A'' \oplus B''$.

We have to show that $i''_*(v) = i_*(0, v)$ is a basic element of $H_1(M)$. It follows from Sublemma 1 that $i''_*(v) \neq 0$. Suppose that there exists an integer $n > 1$ and an element $z \in H_1(M)$ such that $i''_*(v) = nz$. It follows from 3.3 that there exist elements $a \in H_1(S')$, $b \in H_1(S'')$, $y \in H_2(M, \operatorname{Bd} M)$ such that $z = i_*(a, b)$ and $(na, nb - v) = \partial_*(y)$. Let $a = \alpha u + a_0$, $b = \beta v + b_0$, $y = \lambda x + y_0$, where α, β, λ are integers and $a_0 \in B'$, $b_0 \in B''$, $y_0 \in B$. Then we have $\partial_*(y) = \lambda\partial_*(x) + \partial_*(y_0)$ or $(na, nb - v) = \lambda(u, rv) + \partial_*(y_0)$. Applying on both sides of this equation the natural projection $H_1(\operatorname{Bd} M) \to A'$ we obtain $n\alpha u = \lambda u$; projecting to A'' yields $(n\beta - 1)v = \lambda rv$. The former equation implies that λ is divisible by n, which contradicts the latter equation. Sublemma 4 and Lemma 3.6 are proved.

The following lemma is a special case of Theorem 3.2.

LEMMA 3.7. *Let M' be a compact, connected, subacyclic 3-manifold, S a boundary component of M', and $A \subset S$ a separating annulus. Let M be the 3-manifold obtained by attaching a 2-handle to M' along A. Then M is subacyclic.*

Proof. Embed M' in a homology 3-sphere Σ. Let U be the closure of the component of $\Sigma - S$ which intersects M'. Denote by V the 3-manifold obtained by attaching a 2-handle H to U along A. Then there exists a natural embedding of M into V and therefore in order to prove our lemma it suffices to show that V is subacyclic. Obviously V is orientable. Since $H_2(U) = 0$ (by the already proved part of 3.1), the following is a section of the Mayer-Vietoris sequence of $(V; U, H)$ for reduced homology:

$$0 \longrightarrow H_2(V) \longrightarrow H_1(A) \longrightarrow H_1(U) \longrightarrow H_1(V) \longrightarrow 0\ .$$

As A separates S the homomorphism $H_1(A) \to H_1(U)$ is trivial. Hence $H_2(V) \approx H_1(A)$ and $H_1(V) \approx H_1(U)$. Since $\operatorname{Bd} V$ has two components it follows from 3.6 that V is subacyclic.

We conclude the proof of 3.1 by proving

LEMMA 3.8. *Suppose that* 3.1 *holds for* $m < k$ $(k > 2)$. *Then* 3.1 *is true for* $m = k$.

Proof. Let M be a compact, connected 3-manifold whose boundary has k components and which satisfies conditions (1), (2), and (3′) of 3.1 (with $m = k$). Let J be a properly embedded arc in M whose endpoints lie in different components of Bd M. Let N be a regular neighborhood of J in M and let M' be the closure of $M - N$. Then M' is a compact, connected, orientable 3-manifold with $k - 1$ boundary components. Let S be the component of Bd M' which intersects N. We can think of N as a 2-handle attached to M' along the annulus $A = M' \cap N$, which separates S. By considering the Mayer-Vietoris sequence of $(M; M', N)$ for reduced homology we can prove that M' satisfies (2) and (3′) of 3.1 (with $m = k - 1$). By the hypothesis of the lemma this implies that M' is subacyclic. Hence, by 3.7, M is subacyclic.

Proof of 3.2. We consider all possible situations with respect to the homological properties of J in M. We divide these situations in two larger groups. First we consider

Case 1. Suppose that $J \sim 0$ in S. In this case there exists a unique pair of compact 2-submanifolds of S, say G, H, such that $G \cup H = S$, $G \cap H = J$. Indeed; choose a point $x_0 \in S - J$ and let G be the closure of the set of all points in $S - J$ that can be reached from x_0 by some arc in S which misses J or crosses J an even number of times; let $H = S - \operatorname{int} G$. If at least one of G, H is such that it cannot be oriented coherently with J, then the polyhedron P cannot be embedded in any acyclic 3-manifold. Suppose that e.g. G cannot be oriented coherently with J. Then $F \cup G \subset P$ is a non-orientable closed 2-manifold and therefore, as observed in the beginning of § 2, $F \cup G$ cannot be embedded in any acyclic 3-manifold.

Now suppose that both G and H can be oriented coherently with J. Then there exists an orientation for S such that, for the induced orientations in G and H, $J = \partial G = -\partial H$. Give S this orientation. Let $G_1, \cdots, G_n$ be the components of G. For $i = 1, \cdots, n$ do the following. Let $K_i = h^{-1}(\text{Bd}\, G_i) \subset \text{Bd}\, F$. Choose a disk with holes $G'_i \subset G_i$ such that Bd $G'_i = J_i \cup \text{Bd}\, G_i$ where J_i is a simple closed curve in int G_i. Similarly choose a disk with holes $F'_i \subset F$ such that Bd $F'_i = K_i \cup K'_i$ where K'_i is a simple closed curve in int F; let $F'_1, \cdots, F'_n$ be pairwise disjoint. Orient J_i coherently with G'_i and K'_i coherently with F'_i.

Let $C = S \times I$, where $I = [0, 1]$, be an outer collar of M on S and let $M' = M \cup C$, $S' = S \times 1 \subset \text{Bd}\, M'$ ($S \times 0$ is identified with S in the natural way). Since F'_i is homeomorphic to G'_i there exists a proper embedding h'_i of F'_i into $G'_i \times I \subset C$ such that $h'_i \mid K_i = h \mid K_i$ and $h'_i(K'_i) = J'_i = J_i \times 1 \subset S'$. In particular, choose a function $f_i\colon F'_i \to$

I such that $f_i(K_i) = 0$, $f_i(K'_i) = 1$, $f_i(\operatorname{int} F'_i) = \operatorname{int} I$; extend $h \mid K_i$ to a homeomorphism $h_i\colon F'_i \to G'_i$ and then set $h'_i(x) = (h_i(x), f_i(x)) \in G'_i \times I$ $(x \in F'_i)$.

Let $H' = (H \cup \bigcup G'_i) \times 1 \subset S'$, $G' = S' - \operatorname{int} H'$, $J' = \bigcup J'_i$. Orient S' and J' so that the natural homeomorphisms $S' \to S$, $J'_i \to J_i$ preserve orientations. Then $J' = -\partial G' = \partial H'$. Denote by F'' the closure of $F - \bigcup F'_i$ and let $h'\colon \partial F'' \to J'$ be defined by $h' \mid K'_i = h'_i \mid K'_i$ $(i = 1, \cdots, n)$. Then h' is an orientation reversing homeomorphism. There is an obvious embedding of P into $P' = F' \cup_{h'} M'$. Thus, if we can prove that P' can be embedded in some acyclic 3-manifold W, then P can be embedded in W. Note that each component of G' has connected boundary and that this implies that H' is connected. This means that we have reduced our problem to the case when one of G, H, say G, is connected. If we apply the procedure described above to this situation, we reduce the problem to the case when J is a separating simple closed curve in S.

Let us therefore assume that $J \subset S$ is a separating simple closed curve. Let A be a regular neighborhood of J in S. Denote by M' the 3-manifold obtained by attaching a 2-handle to M along A. Obviously P can be embedded in M'. By 3.7, M' is subacyclic and therefore P can be embedded in some acyclic 3-manifold.

Case 2. Suppose that J is not homologous to 0 in S.

If $J \sim 0$ in M, then P certainly cannot be embedded in any acyclic 3-manifold. Suppose that there exists an embedding $i\colon P \to W$ where W is an open acyclic 3-manifold. Let U be the closure of the component of $W - i(S)$ which contains $i(\operatorname{int} M)$ and let V be the closure of the other component of $W - i(S)$. Then $i(J)$ bounds in both U and V. But this contradicts 2.7 (1).

Now suppose that the homology class of J in M is equal to kx for some integer $k > 1$ and some nonzero $x \in H_1(M)$. In this case the image of x under $H_1(M) \to H_1(P)$ is a nonzero element of order k in $H_1(P)$. Since 3.1 implies that a compact subpolyhedron of an acyclic 3-manifold has free first homology group, P cannot be embedded in any acyclic 3-manifold.

Finally suppose that J represents a basic element of $H_1(M)$. Consider the Mayer-Vietoris sequence of $(P; M, F)$ for reduced homology:

$$0 \longrightarrow H_2(M) \longrightarrow H_2(P) \longrightarrow H_1(J) \xrightarrow{\alpha} H_1(M) \oplus H_1(F)$$
$$\xrightarrow{\beta} H_1(P) \longrightarrow \tilde{H}_0(J) \longrightarrow 0 .$$

It is not difficult to prove that α embeds $H_1(J)$ as a direct summand into $H_1(M) \oplus H_1(F)$. Therefore, the homomorphism $H_2(P) \to H_1(J)$ is trivial and, as $H_1(M)$ and $H_1(F)$ are free, the image of β is free. It follows that $H_2(P) \approx H_2(M)$ and that $H_1(P) \approx \operatorname{Im} \beta \oplus \tilde{H}_0(J)$ is free.

Identify F with $F \times 0 \subset F \times I$. Give $\operatorname{Bd}(F \times I)$ the orientation induced by the orientation of F and choose an orientation for S. Extend h to an orientation reversing embedding $g: (\operatorname{Bd} F) \times I \to S$ and construct $V = (F \times I) \cup_g M$. Then V is a compact, connected, orientable 3-manifold containing P, and P is a deformation retract of V. It follows that $H_2(V) \approx H_2(M)$ and that $H_1(V)$ is free. Suppose that $\operatorname{Bd} M$ has m components. Then, by 3.1, $H_2(V) \approx H_2(M)$ is free of rank $m - 1$. This implies that $\operatorname{Bd} V$ has at most m components. On the other hand, $\operatorname{Bd} V$ has at least as many components as $\operatorname{Bd} M$. Thus V has exactly m boundary components. Now it follows from 3.1 that V is subacyclic. This concludes the proof of 3.2.

REFERENCES

1. P. J. Hilton and S. Wylie, *Homology Theory: An Introduction to Algebraic Topology*, Cambridge Univ. Press, Cambridge, 1960.

2. C. D. Papakyriakopoulos, *A reduction of the Poincaré conjecture to group theoretic conjectures*, Ann. of Math., (2) **77** (1963), 250–305.

Received July 17, 1972. Section 2 of this paper contains some of the results of the author's Ph. D. thesis written at the University of Wisconsin under the supervision of Professor J. W. Cannon. The author thanks the referee for his comments and suggestions.

UNIVERSITY OF LJUBLJANA, LJUBLJANA, YUGOSLAVIA