PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 1, 1973

# ON ELEMENTARY IDEALS OF $\theta$-CURVES IN THE 3-SPHERE AND 2-LINKS IN THE 4-SPHERE

SHIN'ICHI KINOSHITA

**Let $L$ be a polyhedron in an $n$-sphere $S^n (n \geqq 3)$ that does not separate $S^n$. A topological invariant of the position of $L$ in $S^n$ can be introduced as follows: Let $l$ be an integral $(n-2)$-cycle on $L$. For each nonnegative integer $d$, the $d$th elementary ideal $E_d(l)$ is associated to $l$ on $L$ in $S^n$. If $l$ and $l'$ are homologous on $L$, then $E_d(l)$ is equal to $E_d(l')$. Now the collection of $E_d(l)$ for all possible $l$ is a topological invariant of $L$ in $S^n$.**

**In this paper the following two cases of $E_d(l)$ are considered: (1) $l$ is a 1-cycle on a $\theta$-curve $L$ in $S^3$, and (2) $l$ is a 2-cycle on a 2-link $L$ in $S^4$, i.e., the union of two disjoint 2-spheres in $S^4$, where each of two 2-spheres is trivially imbedded in $S^4$.**

The $d$th elementary ideal $E_d(l)$ of $l$ on $L$ is defined as follows (more precisely see [3]): Let $G$ be the fundamental group $\pi(S^n - L)$ and $H$ the multiplicative infinite cyclic group generated by $t$. Let $\psi$ be a homomorphism of $G$ into $H$ defined by

$$g^{\psi} = t^{\operatorname{link}(g,l)} ,$$

where link $(g, l)$ is the linking number between $g$ and $l$. Using Fox's free differential calculus, we associate to $\psi$ the $d$th elementary ideal $E_d$ of the group $G$, evaluated in the group ring $JH$ of $H$ over integers. This $d$th elementary ideal $E_d$ depends only on $G$ and $\psi$, and hence it depends only on the position of $l$ on $L$ in $S^n$. We shall denote it by $E_d(l)$.

In this paper we shall prove the following two theorems.

THEOREM 1. *Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then there exists a $\theta$-curve $L_f$ in $S^3$, and an integral 1-cycle $l$ on $L_f$ such that*

$$\begin{cases} E_0(l) = E_1(l) = (0) , \\ E_2(l) = (f(t)) \quad \textit{and} \\ E_d(l) = (1) , \quad \textit{if} \quad d > 2 . \end{cases}$$

THEOREM 2. *Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then there exists a 2-link $L_f$ in $S^4$, and an integral 2-cycle $l$ on $L_f$ such that*

(1) *each component of $L_f$ is a trivially imbedded 2-sphere in $S^4$, and that*

(2) *we have*

$$\begin{cases} E_0(l) = E_1(l) = (0)\,, \\ E_2(l) = (f(t)) \quad \textit{and} \\ E_d(l) = (1)\,, \quad \textit{if} \quad d > 2\,. \end{cases}$$

COROLLARY. *Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then there exists an oriented 2-link $L_f$ in $S^4$ such that*

(1) *each component of $L_f$ is a trivial 2-sphere in $S^4$, and that*

(2) *the* dth *elementary ideal of $L_f$, in the usual sense and in the reduced form, is as follows:*

$$\begin{cases} E_0(L_f) = E_1(L_f) = (0)\,, \\ E_2(L_f) = (f(t)) \quad \textit{and} \\ E_d(L_f) = (1)\,, \quad \textit{if} \quad d > 2\,. \end{cases}$$

REMARK. This kind of example was first considered in [1].

The construction of these two examples are closely related. They are also closely related to the construction of 2-spheres in $S^4$ in [2].

1. Let $P$ be the family of all integral polynomials $f(t)$ which can be expressed in the following form:

$$(1) \qquad \begin{aligned} &t^{-(\varepsilon_1+\cdots+\varepsilon_n)}(1 - t^{\delta_1}) + t^{-(\varepsilon_2+\cdots+\varepsilon_n)}(1 - t^{\delta_2}) \\ &\quad + \cdots + t^{-\varepsilon_n}(1 - t^{\delta_n}) + 1\,, \end{aligned}$$

where $\varepsilon_i = \pm 1$ and $\delta_i = \varepsilon_i$ or $\delta_i = 0$ for $i = 1, 2, \cdots, n$. We assume that $1 \in P$.

LEMMA. *We have $f(t) \in P$, if and only if $f(1) = 1$.*

*Proof*. If $f(t) \in P$, then clearly we have $f(1) = 1$. Suppose that $f(1) = 1$. Then we have

$$\begin{aligned} f(t) - 1 &= (1 - t)(a_m t^m + \cdots + a_0) \\ &\quad - (1 - t)(b_m t^m + \cdots + b_0) \\ &= (1 - t)(a_m t^m + \cdots + a_0) \\ &\quad + (1 - t^{-1})(b_m t^{m+1} + \cdots\cdot + b_0 t)\,, \end{aligned}$$

where $a_i, b_i \geqq 0$ for $i = 1, 2, \cdots, n$. This means that $f(t)$ with $f(1) = 1$ can be obtained from 1 by applying a finite number of operation:

$$g(t) \longrightarrow g(t) + t^p(1 - t^\delta)\,,$$

where $p \geqq 0$ and $\delta = \pm 1$.

We assume $1 \in P$. Hence we should prove that if $f(t) \in P$, then $f(t) + t^p(1 - t^\delta) \in P$. Suppose that $f(t)$ has form (1). Now let

$$p = -(\varepsilon'_1 + \cdots + \varepsilon'_k + \varepsilon'_{k+1} + \cdots + \varepsilon'_{k+n}),$$

where $\varepsilon'_{k+i} = \varepsilon_i$ for $i = 1, 2, \cdots, n$ and let

$$\delta'_1 = \delta,\ \delta'_2 = \cdots = \delta'_k = 0 \quad \text{and} \quad \delta'_{k+i} = \delta_i$$

for $i = 1, 2, \cdots, n$. Then clearly we have

$$t^{-(\varepsilon'_1+\cdots+\varepsilon'_k+\varepsilon'_{k+1}+\cdots+\varepsilon'_{k+n})}(1 - t^{\delta'_1}) + \cdots + t^{\varepsilon'_{k+n}}(1 - t^{\delta'_{k+n}}) = t^p(1 - t^\delta) + f(t).$$

Hence the proof is complete.

2. Let $f(t)$ be an integral polynomial with $f(1) = 1$. Suppose that $f(t)$ is expressed as (1). Now we construct a 1-dimensional polyhedron $K_f$ in $E^3(\subset S^3)$ as follows: The left-most side of $K_f$ is shown in Fig. 1. Then for each $i$ ($i = 1, \cdots, n$) we add step by step one of the four figures in Fig. 2. This depends on values of $\varepsilon_i$ and

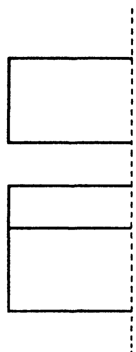

Fig. 1.

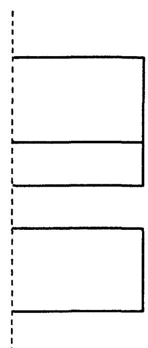

Fig. 3.

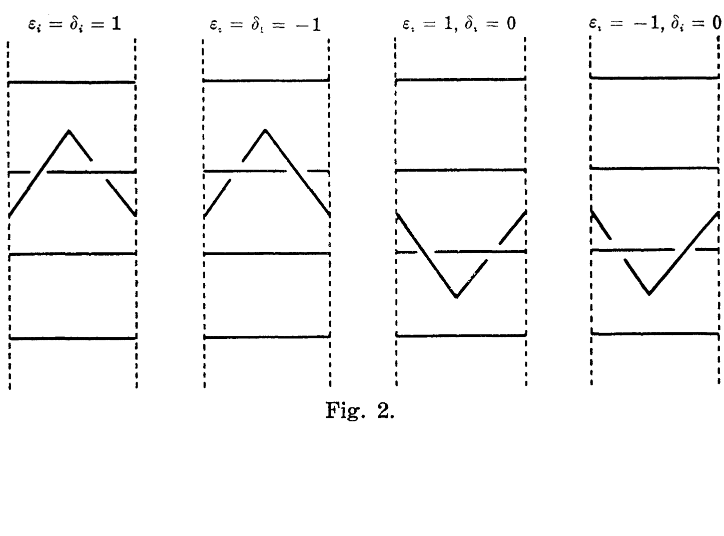


Fig. 2.

$\delta_i$ as in Fig. 2. The right-most side of $K_f$ is shown in Fig. 3.

Now we give a presentation of the fundamental group of $E^3 - K_f$ (and that of $S^3 - K_f$, too). We use the Wirtinger presentation. If $a_0, \cdots, a_n, c_0, \cdots, c_m, d_0, \cdots, d_m, (m + m' = n)$ are paths in Fig. 4, and

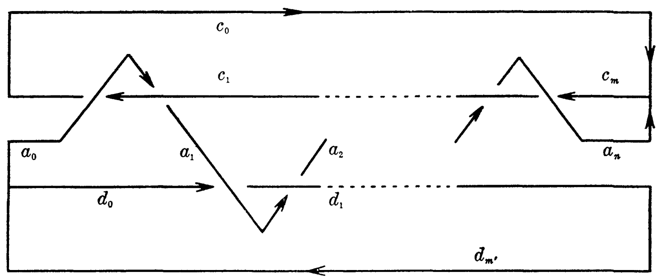


Fig. 4.

also, as usual, the paths which represent elements of the fundamental group in question, then the presentation is given as follows:

Generators: $\begin{cases} a_0, \cdots, a_n, \\ c_0, \cdots, c_m, \\ d_0, \cdots, d_{m'} (m + m' = n) . \end{cases}$

Relations:

(i) If $\varepsilon_i = 1, \delta_i = 1$, then

$$\begin{cases} c_{j-1} = a_{i-1} c_j a_{i-1}^{-1} , \\ a_i = c_j a_{i-1} c_j^{-1} , \end{cases}$$

(ii) If $\varepsilon_i = -1, \delta_i = -1$, then

$$\begin{cases} c_j = a_i c_{j-1} a_i^{-1} , \\ a_{i-1} = c_{j-1} a_i c_{j-1}^{-1} , \end{cases}$$

(iii) If $\varepsilon_i = 1, \delta_i = 0$, then

$$\begin{cases} d_j = a_{i-1} d_{j-1} a_{i-1}^{-1} , \\ a_i = d_j a_{i-1} d_j^{-1} , \end{cases}$$

(iv) If $\varepsilon_i = -1, \delta_i = 0$, then

$$\begin{cases} a_{i-1} = d_{j-1} a_i d_{j-1}^{-1} , \\ d_{j-1} = a_i d_j a_i^{-1} , \end{cases}$$

for each $i = 1, \cdots, n$, and

$$c_0 c_m^{-1} a_n = 1 .$$

3. Let $k_f$ be a 1-cycle on $K_f$ such that

$$\begin{cases} \operatorname{link}(a_i, k_f) = 0\,, & \text{for} \quad i = 0, 1, \cdots, n\,, \\ \operatorname{link}(c_i, k_f) = 1\,, & \text{for} \quad i = 0, 1, \cdots, m\,, \\ \operatorname{link}(d_i, k_f) = 1\,, & \text{for} \quad i = 0, 1, \cdots, m'\,. \end{cases}$$

We consider the elementary ideals of $k_f$ on $K_f$ in $S^3$. For each pair $a_{i-1}$ and $a_i$ the corresponding two rows in the Alexander matrix are elementary equivalent to the following:

(1) If $\varepsilon_i = \delta_i$, then

$$\begin{array}{cccc} a_{i-1} & a_i & c_{j-1} & c_j \end{array}$$
$$\begin{bmatrix} 1 - t^{\varepsilon_i} & 0 & -1 & 1 \\ t^{\varepsilon_i} & -1 & 0 & 0 \end{bmatrix}.$$

(2) If $\delta_i = 0$, then

$$\begin{array}{cccc} a_{i-1} & a_i & d_{j-1} & d_j \end{array}$$
$$\begin{bmatrix} 1 - t^{\varepsilon_i} & 0 & 1 & -1 \\ t^{\varepsilon_i} & -1 & 0 & 0 \end{bmatrix}.$$

From the last relation we have the following entries to the Alexander matrix.

$$\begin{array}{ccc} a_n & c_0 & c_m \\ [1 & 1 & -1] \end{array}$$

Hence we have matrix (*) as an Alexander matrix of $k_f$ on $K_f$ in $S^3$. Matrix (*) is elementary equivalent to (**). Note that we add a suitable number of rows of zeros. Matrix (**) can be reduced to (***) by elementary operations. Now it is easy to see that

$$(*) \quad \left(\begin{array}{c|c|c} \begin{matrix} t^{\varepsilon_1} & -1 & & \\ & \ddots & \ddots & \\ & & t^{\varepsilon_n} & -1 \end{matrix} & 0 & 0 \\ \hline \begin{matrix} & & & 1 \end{matrix} & \begin{matrix} 1 & & -1 \end{matrix} & 0 \\ \hline 1 - t^{\varepsilon_i} & \begin{matrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{matrix} & 0 \\ \hline 1 - t^{\varepsilon_i} & 0 & \begin{matrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{matrix} \end{array}\right). \quad \begin{array}{l} \\ \\ \\ \varepsilon_i = \delta_i \\ \\ \delta_i = 0 \end{array}$$

$$(**)\quad \left(\begin{array}{c|c|c} \begin{matrix} t^{\varepsilon_1} & -1 & & \\ & \ddots & \ddots & \\ & & t^{\varepsilon_n} & -1 \end{matrix} & 0 & 0 \\ \hline 1-t^{\delta_1} \cdots 1-t^{\delta_n} \; 1 & 0 & 0 \\ \hline \begin{matrix} \ddots & & \\ \cdots\cdots 1 - t^{\delta_i} & \cdots\cdots & \\ & \ddots & \end{matrix} & \begin{matrix} \ddots & & \\ \cdots\cdots \begin{Bmatrix} -1 & 1 \\ 0 & 0 \end{Bmatrix} & \cdots & \\ & & \ddots \end{matrix} & 0 \end{array}\right) \begin{Bmatrix} \varepsilon_i = \gamma_i \\ \delta_i = 0 \end{Bmatrix}.$$

$$(***)\quad \left(\begin{array}{c|c|c} \begin{matrix} t^{\varepsilon_1} & -1 & & \\ & \ddots & \ddots & \\ & & t^{\varepsilon_n} & -1 \end{matrix} & 0 & 0 \\ \hline 1-t^{\delta_1} \cdots 1-t^{\delta_n} \; 1 & 0 & 0 \end{array}\right).$$

$$\begin{cases} E_0(k_f) = E_1(k_f) = (0)\,, \\ E_2(k_f) = (t^{-(\varepsilon_1+\cdots+\varepsilon_n)}(1 - t^{\delta}1) + \cdots \\ \qquad\qquad + t^{-\varepsilon_n}(1 - t^{\delta}n) + 1) = (f(t))\,, \\ E_d(k_f) = (1)\,, \quad \text{if} \quad d > 2\,. \end{cases}$$

4. *Proof of Theorem* 1. Let $f(t)$ with $f(1) = 1$ be given. First construct $K_f$ in $S^3$ and $k_f$ on $K_f$ as in 2 and 3. The construction of the corresponding $\theta$-curve $L_f$ is shown in Fig. 5. The 1-cycle $l_f$ on

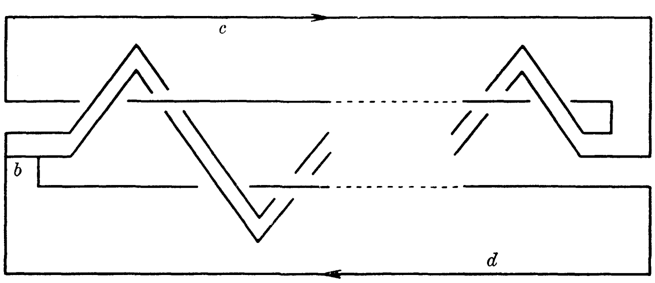


Fig. 5.

$L_f$ has coefficient 1 on the oriented arc $c$ and on the oriented arc $d$, respectively, and coefficient 0 on the arc $b$. It is easy to see that

$\pi(S^3 - L_f)$ is isomorphic to $\pi(S^3 - K_f)$ and $E_d(l_f) = E_d(k_f)$ for every nonnegative integer $d$.

REMARK. It is proved in [3] that if $l$ is a 1-cycle on a $\theta$-curve $L$ in $S^3$, then we have

$$\begin{cases} E_0(l) = E_1(l) = (0)\,, & \text{and} \\ (E_d(l))^\circ = (1)\,, & \text{if} \quad d \geqq 2\,, \end{cases}$$

where $\circ$ is a trivializer (i.e., the operation to let $t = 1$ in $E_d(l)(t)$).

5. *Proof of Theorem* 2. Let $f(t)$ with $f(1) = 1$ be given. First construct $K_f$ in $S^3$ and $k_f$ on $K_f$ as in 2 and 3. Then construct the corresponding two arcs $C$ and $D$ in $E^3_+$ as in Fig. 6, where

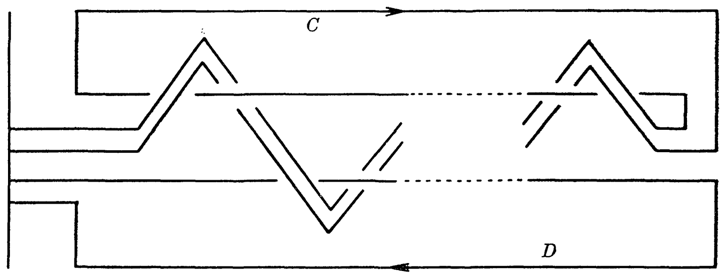


Fig. 6.

$$E^3_+ = \{(x_1, x_2, x_3) \mid x_1 \geqq 0\}\,.$$

Then the usual construction of the spinning of these arcs around the plane

$$\{(x_1, x_2, x_3, x_4) \mid x_1 = 0, x_4 = 0\}$$

gives rise to a 2-link $L_f$ in $S^4$.

Now the arc $C$ represents a trivial knot in $E^3_+$. A part of the step to see this is shown in Fig. 7. From this it follows that the 2-sphere $S^2_C$, which is the result of spinning $C$, is trivial in $S^4$. Clearly the same is true for the 2-sphere $S^2_D$, the result of spinning $D$.

We have

$$\pi(S^3 - K_f) \cong \pi(E^3_+ - C \cup D) \cong \pi(S^4 - L_f)\,,$$

and to find a 2-cycle $l_f$ on $L_f$ that corresponds to $k_f$ on $K_f$ is easy. Then we have

$$E_d(k_f) = E_d(l_f)$$

for every $d \geqq 0$. Hence the proof is complete.

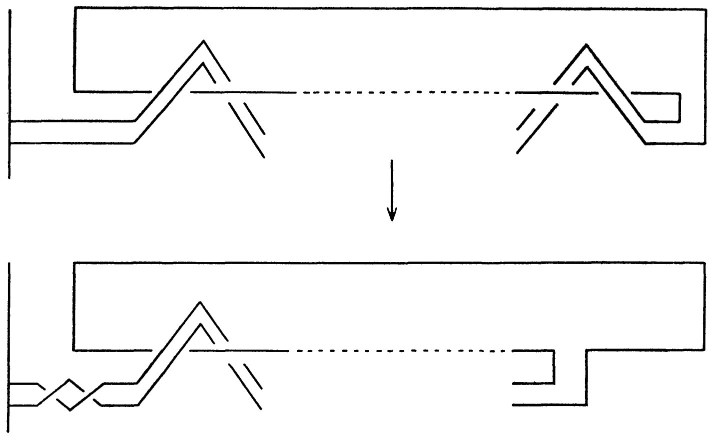

Fig. 7.

*Proof of Corollary.* We have $L_f = S_C^2 \cup S_D^2$ in $S^4$ in the example above. Then $l_f = l_c + l_d$, where $l_c$ and $l_d$ are fundamental cycles of $S_C^2$ and $S_D^2$, respectively. This completes the proof.

REMARK. Let $L$ be a 2-link in $S^4$. Then it is known that for each 2-cycle $l$ on $L$ we always have

$$\begin{cases} E_0(l) = E_1(l) = 0 \ , \\ (E_d(l))^\circ = (1) \ , \quad \text{if} \quad d \geqq 2 \ , \end{cases}$$

where $\circ$ is a trivializer. (See [3] and [4].)

## REFERENCES

1. E. H. Van Kampen, *Zur Isotopie zweidimensionaler Flaechen im* $R^4$, Abh. Math. Sem. Univ. Hamburg, **6** (1927), 216.
2. S. Kinoshita, *On the Alexander polynomials of* 2-*spheres in a* 4-*sphere*, Ann. of Math., **74** (1961), 518-531.
3. ———, *On elementary ideals of polyhedra in the* 3-*sphere*, to appear in the Pacific J. Math., **42** (1972), 89-98.
4. Y. Shinohara and D. W. Sumners, *Homology invariants of cyclic coverings with application to links*, Trans. Amer. Math. Soc., **163** (1972), 101-121.

Received August 3, 1972. The author of this paper is partially supported by NSF Grant GP-19964.

FLORIDA STATE UNIVERSITY