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SUBNORMAL OPERATORS IN STRICTLY CYCLIC OPERATOR ALGEBRAS

RICHARD BOLSTEIN AND WARREN WOGEN

It is shown that a subnormal operator cannot belong to a strictly cyclic and separated operator algebra unless it is normal and has finite spectrum. Further, a subnormal operator not of this type cannot have a strictly cyclic commutant.

1. Let $\mathscr{H}$ be a complex Hilbert space, and let $\mathscr{A}$ be a subset of the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$. A vector $x \in \mathscr{H}$ with the property that $\mathscr{A}x = \{Ax: A \in \mathscr{A}\}$ is the full Hilbert space is said to be a *strictly cyclic vector* for $\mathscr{A}$, and $\mathscr{A}$ is said to be *strictly cyclic* if such a vector exists. A vector x is called a *separating vector* for $\mathscr{A}$ if no two distinct operators in $\mathscr{A}$ agree at x. The set $\mathscr{A}$ is said to be *strictly cyclic and separated* if there is a vector x which is both strictly cyclic and separating for $\mathscr{A}$.

Strictly cyclic operator algebras have recently been investigated by Mary Embry [2] and Alan Lambert [3]. Let $\mathscr{A}'$ denote the *commutant* of the set $\mathscr{A}$, that is, $\mathscr{A}'$ is the set of all bounded linear operators which commute with every operator in $\mathscr{A}$. Note that if x is a *cyclic vector* for $\mathscr{A}$ (meaning $\mathscr{A}x$ is dense in $\mathscr{H}$), then x is separating for $\mathscr{A}'$.

LEMMA 1. *Let $\mathscr{A}$ be a strictly cyclic subset of $\mathscr{B}(\mathscr{H})$. If $\mathscr{A}$ is abelian, then it is maximal abelian, $\mathscr{A} = \mathscr{A}'$. Thus, a strictly cyclic abelian subset is automatically a weakly closed algebra.*

This lemma, which indicates the severity of the condition of strict cyclicity, is a sharper form of a result of Lambert [3].

Proof. Let x be strictly cyclic for $\mathscr{A}$, and let $B \in \mathscr{A}'$. Then there exists $A \in \mathscr{A}$ such that $Ax = Bx$. But $\mathscr{A} \subset \mathscr{A}'$ by hypothesis, so $A \in \mathscr{A}'$. Since x is separating for $\mathscr{A}'$, we have $B = A \in \mathscr{A}$, and the proof is complete.

If $\mathscr{A}$ is strictly cyclic and abelian, then it is strictly cyclic and separated by Lemma 1. Mary Embry [2] showed that the converse holds if $\mathscr{A}$ is the commutant of a single operator. Thus, if A is normal and $\{A\}'$ is strictly cyclic and separated, then $\{A\}'$ consists of normal operators by Fuglede's theorem. In a private communication to the authors, Mary Embry asked if "normal" could be replaced by "subnormal" in this statement. An operator is called *subnormal* if it is the restriction of a normal operator to an invariant subspace. To this end, we show that if A is subnormal then strict cyclicity of $\{A\}'$ already forces A to be normal, and, moreover, its spectrum is a finite set. Thus, the commutant of a subnormal operator cannot be strictly cyclic and separated unless the underlying Hilbert space is finite-dimensional (since the commutant is then abelian and hence the operator, which is normal, must have simple spectrum). More generally, it is shown that a uniformly closed subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ which has a separating vector x with the property that $\mathscr{A}x$ is a closed subspace of $\mathscr{H}$ (this is the case if x is also strictly cyclic) contains no subnormal operators except possibly for normal operators with finite spectrum.

2. Let μ be a finite positive Borel measure in the plane with compact support X, let $H^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$, and put $H^\infty(\mu) = H^2(\mu) \cap L^\infty(\mu)$. The next theorem, which is used to derive the main result, may be of independent interest.

THEOREM 1. *$H^\infty(\mu) = H^2(\mu)$ if, and only if, X is finite.*

Proof. The sufficiency is trivial. Assume now that X is infinite. Note that the inclusion map of $H^\infty(\mu)$ into $H^2(\mu)$ is continuous. We will show that the inverse map is not continuous, and hence, by the Open Mapping Theorem, that $H^\infty(\mu) \neq H^2(\mu)$.

Since X is compact and infinite, its set X' of accumulation points is compact and nonempty. Choose $\lambda_0 \in X'$ such that $|\lambda_0| = \max\{|\lambda|: \lambda \in X'\}$, and let $D_1 = \{\lambda: |\lambda| \leqq |\lambda_0|\}$. By the choice of λ, $X\backslash D_1$ is a countable set. Therefore, we can choose a closed disk D_2 which contains D_1 and is tangent to D_1 at λ_0, in such a way that the boundary of D_2 intersects X only at λ_0. Now note that we may as well assume that D_2 is the closed unit disc Δ, and that $\lambda_0 = 1$.

Now $X\backslash\Delta$ is a countable set $\{y_1, y_2, \cdots\}$, and if this set infinite, we must have $\lim y_n = 1$. Let $K = \Delta \cup (X\backslash\Delta)$. Then K is a compact set which does not separate the plane. Define a sequence of functions $\{f_n\}$ on K by

$$f_n(z) = \begin{cases} z^n: & z \in \Delta \\ 0: & z = y_i,\ 1 \leqq i \leqq n \\ 1: & z = y_i,\ i > n\ . \end{cases}$$

Then, for each n, f_n is continuous on K and analytic in its interior. By Mergelyan's theorem, each f_n is the uniform limit on K of a sequence of polynomials. Hence each $f_n \in H^\infty(\mu)$.

Let χ denote the function which has the value 1 at the point 1 and the value zero elsewhere. Clearly, $f_n \to \chi$ pointwise, and hence in the metric of $L^2(\mu)$ by dominated convergence. In particular, $\chi \in H^\infty(\mu)$. However, the point 1 is an accumulation point of the support of μ, and hence $\|f_n - \chi\|_\infty = 1$ for every n. Thus, $\{f_n\}$ converges to χ in $H^2(\mu)$ but not in $H^\infty(\mu)$.

THEOREM 2. *Let S be a subnormal operator on the Hilbert space $\mathscr{H}$, let $\mathscr{A}$ be the uniformly closed algebra generated by S. If $\mathscr{A}$ has a separating vector x such that $\mathscr{A}x$ is a closed subspace of $\mathscr{H}$, then the spectrum of S is a finite set, and hence S is normal.*

Proof. Let $\mathscr{B}$ be the uniformly closed algebra generated by S and the identity operator I. Since $\mathscr{B}x$ is the sum of $\mathscr{A}x$ and the one-dimensional space spanned by x, and since we assume that $\mathscr{A}x$ is closed, we also have that $\mathscr{B}x$ is a closed subspace of $\mathscr{H}$.

Now $\mathscr{B}x$ is invariant under S and the restriction operator $S_0 = S \mid \mathscr{B}x$ is subnormal. Since the uniformly closed algebra $\mathscr{B}_0$ generated by S_0 and I contains $\mathscr{B} \mid \mathscr{B}x$, it follows that x is a strictly cyclic vector for $\mathscr{B}_0$, that is, $\mathscr{B}_0 x = \mathscr{B}x$. By the representation theorem for subnormal operators with a cyclic vector, Bram [1], S_0 is unitarily equivalent to the operator of multiplication by the identity function on some $H^2(\mu)$ space. Furthermore, the unitary equivalence can be constructed so that x corresponds to the constant function 1.

Now $\mathscr{B}_0$ corresponds via the unitary equivalence to the algebra of multiplication operators $M_\phi : f \to \phi f$ on $H^2(\mu)$, where ϕ belongs to the $L^\infty(\mu)$-closure of the polynomials. Since any such function ϕ belongs to $H^\infty(\mu)$, it follows that the constant function 1 is a strictly cyclic vector for $\{M_\phi : \phi \in H^\infty(\mu)\}$, and hence that $H^\infty(\mu) = H^2(\mu)$. By Theorem 1, $H^2(\mu)$ is finite-dimensional.

It follows that $\mathscr{B}x$ is finite-dimensional, and, since $\mathscr{A} \subset \mathscr{B}$, so is $\mathscr{A}x$. Since x separates $\mathscr{A}$, it follows that $\mathscr{A}$ is finite-dimensional. So there is a polynomial p such that $p(S) = 0$. Since $p(\sigma(S)) = \sigma(p(S)) = \{0\}$, $\sigma(S)$ in finite and hence S is normal.

COROLLARY 1. *Let $\mathscr{A}$ be a uniformly closed subalgebra of $\mathscr{B}(\mathscr{H})$ which has a separating vector x such that $\mathscr{A}x$ is a closed subspace of $\mathscr{H}$. (This is the case if $\mathscr{A}$ is strictly cyclic and separated.) Then $\mathscr{A}$ contains no subnormal operator with infinite spectrum.*

Proof. Suppose $S \in \mathscr{A}$ is subnormal, and let $\mathscr{A}(S)$ be the uniformly closed algebra generated by S. Since $\mathscr{A}(S) \subset \mathscr{A}$, x separates $\mathscr{A}(S)$. Since the linear transformation $A \to Ax$ of $\mathscr{A}$ onto $\mathscr{A}x$ is continuous and one-to-one, and since $\mathscr{A}x$ is closed by hypothesis, the transformation has a continuous inverse by the Open Mapping Theorem.

Therefore, $\mathscr{A}(S)x$ is closed, and the result follows from Theorem 2.

COROLLARY 2. *The commutant of a subnormal operator S is strictly cyclic if, and only if, S is normal and has finite spectrum.*

Proof. Suppose $\{S\}'$ has a strictly cyclic vector x. Then x separates $\{S\}''$, and it follows from [2, Lemma 2.1 (i)] that $\{S\}''x$ is a closed subspace. Thus, by Corollary 1, S has finite spectrum and hence is normal.

Conversely, if $\sigma(S) = \{\lambda_1, \cdots, \lambda_n\}$, then each λ_j is an eigenvalue and $\mathscr{H}$ is the direct sum of the corresponding eigensubspaces $\mathscr{H}_j$. It follows that $\{S\}' = \mathscr{B}(\mathscr{H}_1) \oplus \cdots \oplus \mathscr{B}(\mathscr{H}_n)$. Hence any vector $x = x_1 + \cdots + x_n$ where $0 \neq x_j \in \mathscr{H}_j$, $j = 1, \cdots, n$, is strictly cyclic for $\{S\}'$.

COROLLARY 3. *Let S be a subnormal operator on a Hilbert space $\mathscr{H}$. If $\{S\}'$ is strictly cyclic and separated, then $\mathscr{H}$ is finite-dimensional.*

Proof. By Corollary 2, S is normal, its spectrum is finite, and $\{S\}' = \mathscr{B}(\mathscr{H}_1) \oplus \cdots \oplus \mathscr{B}(\mathscr{H}_n)$ with notation as in the proof of that corollary. If x is strictly cyclic for $\{S\}'$, then $x = x_1 + \cdots + x_n$ where $0 \neq x_j \in \mathscr{H}_j$, all j. If some $\mathscr{H}_j$ has dimension greater than 1, then there is a nonzero operator B_j on $\mathscr{H}_j$ which annihilates x_j, and hence there is a nonzero $B \in \{S\}'$ such that $Bx = 0$. Therefore, if $\{S\}'$ is strictly cyclic and separated, each $\mathscr{H}_j$ is one-dimensional and hence $\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_n$ is finite-dimensional.

COROLLARY 4. *Let S be a subnormal operator on a Hilbert space $\mathscr{H}$. If $\{S\}''$ is strictly cyclic, then $\mathscr{H}$ is finite-dimensional.*

Proof. If x is strictly cyclic for $\{S\}'' \subset \{S\}'$, then it is strictly cyclic and separating for $\{S\}'$ and the result follows from Corollary 3.

An operator A is said to be *strictly cyclic* if the weakly closed algebra generated by A and I has this property. Since this algebra is contained in the second commutant of A, it follows that the second commutant of a strictly cyclic operator is strictly cyclic. In view of Corollary 4, we have

COROLLARY 5. *There exist no strictly cyclic subnormal operators on an infinite-dimensional Hilbert space.*

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GEORGE MASON UNIVERSITY
AND
UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL