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FREE LATTICE-ORDERED MODULES

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The aim of this paper is to show that the theory of free lattice-ordered groups developed by E. C. Weinberg in the abelian case can be generalized to modules over a totally ordered Ore domain A. The main result is that for every torsion-free ordered A-module M, there exists a free f-module over M. The generalization given will be seen to be, in a certain sense, the best possible.

All rings and modules considered will be assumed to be unital. Let A be a partially ordered ring and A_+ its order. If M is a left A-module, we say that $P \subseteqq M$ is an order on M if:

$P + P \subseteqq P$, $A_+P \subseteqq P$, and $P \cap -P = \{0\}$. If such a P is given, we say that M is a partially ordered (or ordered) module. If P is a total order on M, that is, if $M = P \cup -P$, we say that M is a totally ordered module. Let M and N be partially ordered A-modules and let f be a mapping from M to N. Then f is an o-homomorphism if f is a monotonic homomorphism of A-modules. The o-homomorphism f is an o-isomorphism if f is one-to-one and if f^{-1} is an o-homomorphism.

1. Some properties of f-modules. In this section, A will denote a directed p.o. ring. An A-module M which is lattice-ordered by the order P is called a lattice-ordered module or l-module. Products of lattice-ordered modules are defined in a natural way. If M and N are l-modules, an homomorphism f from M to N is called a l-homomorphism if, for $x, y \in M$:

$$f(x \vee y) = f(x) \vee f(y) \quad \text{and} \quad f(x \wedge y) = f(x) \wedge f(y) .$$

An f-module M is a lattice-ordered module which is a subdirect product of totally ordered modules. This definition was first introduced in [1] and [3].

Recall that a convex l-subgroup S in a commutative l.o. group G is called prime if G/S is totally ordered. The following theorem gives useful characterizations of f-modules.

THEOREM 1. *Let M be a lattice-ordered module over a unital directed ring A. The following are equivalent:*

(1) *M is an f-module.*

(2) *For $x, y \in M$ and $0 \leqq \lambda \in A$, $\lambda(x \vee y) = \lambda x \vee \lambda y$ and $\lambda(x \wedge y) = \lambda x \wedge \lambda y$.*

(3) $x \wedge y = 0$ *implies* $\lambda x \wedge y = 0$ *for all* $0 \leqq \lambda \in A$.
(4) *Every minimal prime subgroup of M is a submodule.*

Proof. (1) implies (2): This is clear since (2) is satisfied in a totally ordered module.

(2) implies (3): If $x \wedge y = 0$, then we have:

$$0 \leqq \lambda x \wedge y \leqq (\lambda \vee I)x \wedge (\lambda \vee I)y = (\lambda \vee I)(x \wedge y) = 0 .$$

(3) implies (4): Let S be a minimal prime subgroup. Then, $x \in S$ if and only if there exists $y \notin S$ with $x \wedge y = 0$. [2]. Thus, if $x \in S$ and $0 \leqq \lambda \in A$, we have $\lambda x \in S$. Since A is directed, S is a submodule.

(4) implies (1): Let $(S_i)_{i \in I}$ be the family of all minimal prime subgroups of M. Then each quotient M/S_i is a totally ordered module and M is a subdirect product of these modules.

If A is not unital, then (1), (3), and (4) are equivalent but condition (2) is weaker (see [3]).

In the sequal, we shall be concerned mainly with torsion-free modules, that is modules in which $\lambda x = 0$ implies $\lambda = 0$ or $x = 0$. The following property is useful:

PROPOSITION 1. *If A is totally ordered, every torsion-free f-module F is a subdirect product of torsion-free totally ordered modules.*

Let S be a minimal prime subgroup of F. Suppose that $\lambda \neq 0$ and $\lambda x \in S$. We may assume $\lambda > 0$, as A is totally ordered. As in the proof of Theorem 1, there exists $y \notin S$ with $\lambda x \wedge y = 0$. This implies $\lambda(x \wedge y) = \lambda x \wedge \lambda y = 0$, and hence $x \wedge y = 0$. As $y \notin S$, we obtain $x \in S$. This proves that M/S is torsion-free and the theorem follows.

As in the theory of ordered groups, P is an isolated order on M if $\lambda > 0$ and $\lambda x \in P$ implies $x \in P$.

PROPOSITION 2. *Every torsion-free f-module is isolated.*

Proof. If $\lambda > 0$ and $\lambda x \geqq 0$, we have $\lambda(-x \vee 0) = -\lambda x \vee 0 = 0$, hence $-x \vee 0 = 0$ and $x \geqq 0$.

Conversely, it is clear that when A is totally ordered, every isolated module is torsion-free.

2. Embedding an order in a total order. In this section, we consider only torsion-free modules over a totally ordered unital ring A. This is not as restrictive as it seems, since the existence of a nontrivial torsion-free module implies that A has no zero divisors, and an f-ring with no zero divisors is totally ordered.

LEMMA 1. *Let M be a torsion-free A-module. For every $x \in M$, A_+x is an order.*

Proof. Suppose that $y \in A_+x \cap -A_+x$, so that $y = \lambda x = -\mu x$. The relation $(\lambda + \mu)x = 0$ implies $\lambda + \mu = 0$ or $x = 0$. In the first case, $\lambda = -\mu \in A_+ \cap -A_+$ so in each case $y = 0$.

LEMMA 2. *Let P and Q be two orders on M. Then $P - Q$ is an order if and only if $P \cap Q = 0$.*

Proof. The condition is necessary, since $P \cap Q \subseteqq (P-Q) \cap (Q-P)$. For the converse, suppose $P \cap Q = 0$ and let $y \in (P - Q) \cap (Q - P)$. Then $y = p - q = q' - p'$, and $p + p' = q + q' \in P \cap Q = 0$. Hence, $p = -p' \in P \cap -P = 0$, $q = -q' \in Q \cap -Q = 0$, and it follows that $y = 0$.

The ring A is said to be a left Ore domain if A admits a left quotient field. Equivalently, A has no zero divisors and satisfies the following condition:

(I) If $\rho \neq 0$ and $\sigma \neq 0$, $A\rho \cap A\sigma \neq 0$.

Clearly, when A is totally ordered, this condition can be replaced by the following:

(II) If $0 < \rho$ and $0 < \sigma$, $A_+\rho \cap A_+\sigma \neq 0$.

THEOREM 2. *Let A be a totally ordered ring with no divisors of zero. The following are equivalent:*

(1) *A is a left Ore domain.*

(2) *In a torsion-free A-module, every order is contained in a total order.*

(3) *In a torsion-free A-module, every order is contained in an isolated order.*

Proof. (1) implies (2): By Zorn's lemma, every order is contained in a maximal order. It remains to show that each maximal order P is total. If not, suppose $b \notin P \cup -P$. As $P \subset P + A_+b$ (strictly), $P + A_+b$ fails to be an order. By Lemma 2, $P \cap -A_+b \neq 0$ and there exists $\rho > 0$ with $\rho b \in -P$. Similarly, $P - A_+b$ is not an order, $P \cap A_+b \neq 0$, and there exists $\sigma > 0$ with $\sigma b \in P$. By condition (II), there exists $\lambda > 0$ and $\mu > 0$ with $\lambda\rho = \mu\sigma > 0$. Hence $\lambda\rho b = \mu\sigma b \in P \cap -P = 0$. This implies $b = 0$, which is a contradiction. Hence P is a total order.

(2) implies (3): This is clear from Proposition 2.

(3) implies (1): Consider A as a left-module on itself. Take $0 < \rho$ and $0 < \sigma$. If $A_+\rho \cap A_+\sigma = 0$, $A_+\rho - A_+\sigma$ is an order by Lemma 2. Hence it is contained in an isolated order P, and thus $\rho 1 \in P$ and $\sigma(-1) \in P$. Then $1 \in P$ and $-1 \in P$, which is a contradiction.

COROLLARY 1. *Let A be a totally ordered left Ore domain. Let f be an o-homomorphism of the torsion-free module M ordered by P into a torsion-free totally ordered module T. There exists a total order P_0 which contains P such that $f(x) > 0$ implies $x \in P_0$.*

To see that $S = \{x \mid f(x) > 0\} \cup \{0\}$ is an order on M, note that $S + S \subseteq S$ and $S \cap -S = \{0\}$. Also for $\lambda > 0$ and $0 \neq x \in S$, $f(x) > 0$ and hence $f(\lambda x) = \lambda f(x) > 0$ since T is torsion-free. As $P \cap -S = 0$, $P + S$ is an order by Lemma 2. The corollary then follows from Theorem 1.

COROLLARY 2. *Let A be a totally ordered left Ore domain and let M be a torsion-free A-module ordered by P. The intersection of all total orders containing P is the set $\bar{P}$ of elements $x \in M$ for which there exists $\lambda > 0$ with $\lambda x \in P$.*

Each total order containing P is isolated and hence contains $\bar{P}$. Suppose $x \notin \bar{P}$, so that $P \cap A_+x = 0$. By Lemma 2, $P - A_+x$ is an order. By Theorem 2, $P - A_+x$ is contained in a total order Q. Since $-x \in Q$ and $x \neq 0$, $x \notin Q$.

THEOREM 3. *Let A be a totally ordered left Ore domain. If M is an A-module ordered by P, these are equivalent:*

(1) *P is isolated.*

(2) *M is torsion-free and P is an intersection of total orders.*

(3) *M can be embedded in a direct product of totally ordered torsion-free modules.*

(4) *M can be embedded in a torsion-free f-module.*

Proof. (1) implies (2): This follows directly from Corollary 2, as $P = \bar{P}$.

(2) implies (3): Let $(P_i)_{i \in I}$ be the set of all total orders containing P. If we denote by M_i the module M ordered by P_i, there is a canonical embedding of M into the direct product of the modules M_i.

(3) implies (4): Clear.

(4) implies (1): This follows from Proposition 1.

3. Free f-modules. Let A be a totally ordered left Ore domain, and let M be a torsion-free A-module ordered by P. A torsion-free f-module L will be called *free over* M if:

(1) There exists an injective o-homomorphism φ from M to L.

(2) For every torsion-free f-module F and every o-homomorphism f from M to F, there exists a unique l-homomorphism $\bar{f}$ from L to F such that $\bar{f}\circ\varphi = f$.

It is not difficult to show that L is determined up to an l-isomorphism. To show that such an L exists, we use the two following lemmas:

LEMMA 3. *If $x_{\alpha\beta}(\alpha \in R, \beta \in S)$ and $x_{\gamma\delta}(\gamma \in U, \delta \in V)$ are two finite families of elements in a lattice-ordered module,*

$$\bigvee_{\alpha\in R}\bigwedge_{\beta\in S} x_{\alpha\beta} - \bigvee_{\gamma\in U}\bigwedge_{\delta\in V} x_{\gamma\delta} = \bigvee_{(\alpha,\sigma)\in R\times(V^{S\times U})}\ \bigwedge_{(\beta,\gamma)\in S\times U} (x_{\alpha\beta} - x_{(\gamma)(\sigma(\beta,\gamma))}) .$$

Proof.

$$\begin{aligned}\bigvee_R\bigwedge_S x_{\alpha\beta} - \bigvee_U\bigwedge_V x_{\gamma\delta} &= \bigvee_R\bigwedge_S\bigwedge_U\bigvee_V (x_{\alpha\beta} - x_{\gamma\delta}) = \bigvee_R\bigwedge_{S\times U}\bigvee_V (x_{\alpha\beta} - x_{\gamma\delta}) \\ &= \bigvee_R\ \bigvee_{\sigma\in(V^{S\times U})}\ \bigwedge_{S\times U} (x_{\alpha\beta} - x_{(\gamma)(\sigma(\beta,\gamma))}) \\ &= \bigvee_{(\alpha,\sigma)\in R\times(V^{S\times U})}\ \bigwedge_{(\beta,\gamma)\in S\times U} (x_{\alpha\beta} - x_{(\gamma)(\sigma(\beta,\gamma))}) .\end{aligned}$$

LEMMA 4. *Let N be a f-module and K a submodule of N. The f-submodule generated by K is the set K' of all elements $\bigvee_{\alpha\in R}\bigwedge_{\beta\in S} x_{\alpha\beta}$ with $x_{\alpha\beta}\in K$.*

Proof. By Lemma 3, K' is an l-subgroup of N. If $\lambda \geqq 0$, it follows from Theorem 1 that: $\lambda\bigvee_R\bigwedge_S x_{\alpha\beta} = \bigvee_R\bigwedge_S \lambda x_{\alpha\beta}$. Since the ring is assumed to be directed, K' is a submodule.

THEOREM 4. *Let $(P_i)_{i\in I}$ be the set of all total orders on M containing P, and denote by M_i the module M ordered by P_i. Let φ be the canonical map of M into $\prod_{i\in I} M_i$. Then the f-submodule L of $\prod_{i\in I} M_i$ generated by $\varphi(M)$ is free over M.*

Proof. Suppose f is an o-homomorphism from M into a torsion-free f-module F. If $x\in L$, then by Lemma 4, $x = \bigvee_R\bigwedge_S \varphi(x_{\alpha\beta})$ where $x_{\alpha\beta}\in M$.

Let $\bar{f}(x) = \bigvee_R\bigwedge_S f(x_{\alpha\beta})$. To show that $\bar{f}$ is a mapping, it is sufficient to show, by Lemma 3, that $\bigvee_R\bigwedge_S\varphi(x_{\alpha\beta}) = 0$ implies $\bigvee_R\bigwedge_S f(x_{\alpha\beta}) = 0$.

By Proposition 1, we may assume that F is totally ordered. By

Corollary 1 of Theorem 2, there exists a total order P_0 containing P such that $f(x) > 0$ implies $x \in P_0$.

If $\bigvee_R \bigwedge_S f(x_{\alpha\beta}) > 0$, there exists $\alpha \in R$ such that for each $\beta \in S$, $f(x_{\alpha\beta}) > 0$, which implies $x_{\alpha\beta} \in P_0$. It follows that $\bigvee_R \bigwedge_S x_{\alpha\beta} > 0$ (modulo P_0) and $\bigvee_R \bigwedge_S \varphi(x_{\alpha\beta}) \neq 0$. Alternatively, if $\bigvee_R \bigwedge_S f(x_{\alpha\beta}) < 0$, there exists for each $\alpha \in R$, a $\beta \in S$ such that $f(x_{\alpha\beta}) < 0$. Thus $x_{\alpha\beta} \in -P_0$ and it follows that $\bigvee_R \bigwedge_S x_{\alpha\beta} < 0$ (with respect to P_0). Hence $\bigvee_R \bigwedge_S \varphi(x_{\alpha\beta}) \neq 0$. Now, it is clear that $\bar{f}$ is a mapping. By Lemma 3, $\bar{f}$ is a group homomorphism. The theorem follows easily.

COROLLARY. *Let A be a totally ordered ring with no divisors of zero. The following are equivalent:*

(1) *A is a left Ore domain.*

(2) *For every torsion-free ordered module M, there exists a free f-module over M.*

Proof. By Theorem 4, (1) implies (2). Conversely, if φ is the o-homomorphism of M into the free f-module L over M, the positive cone of M is a subset of $Q = \{x \mid \varphi(x) \geqq 0\}$, which is an isolated order. Thus, (2) implies (1) by Theorem 2.

Note that φ is an o-isomorphism of M into L if and only if M is isolated.

It is now easy to construct the free f-module over an arbitrary set E. Let M be the free module generated by E, and trivially order M by $P = \{0\}$. The free f-module L generated by M is a free f-module over E, with obvious definitions.

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