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ON NORMAL SUBGROUPS OF DIFFERENTIABLE HOMEOMORPHISMS

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The algebraic side of the group of all homeomorphisms of an interval or a circle has been studied exhaustively. In this paper the objects of study are the homeomorphisms with local polynomial approximations at each point. The algebraic side of the group of all such homeomorphisms is examined, particularly the minimal normal subgroup and the commutator subgroup. The results are like those in the topological case.

The normal subgroups of the group $H(X)$ of all homeomorphisms from a locally euclidean manifold X onto itself have been studied by various authors for nearly forty years. The minimal normal subgroup of $H(X)$ has long been known to be the group $SH(X)$ generated by the members of $H(X)$ that are the identity outside some euclidean ball, and the manner of proof shows us also that $SH(X)$ has no proper normal subgroups of its own. When these same questions are asked of the group $CD_n(X)$ of all n-times continuously differentiable homeomorphisms, then the answers are much harder to come by. Recently D. B. A. Epstein [1] showed that the minimal normal subgroup and the commutator subgroup C_n of $SCD_n(X)$ are one and the same and that C_1 is dense in $SCD_1(X)$. It is now known that $C_n = SCD_n(X)$ when X is any euclidean space from some work of W. Thurston to appear.

In this paper we shall take X to be 1-dimensional and study not $CD_n(X)$ but the group $E_n(X)$ of all homeomorphisms with local polynomial approximations of degree n at each point of X. The minimal normal subgroups and the commutator subgroups of $E_n(X)$ and $SE_n(X)$ are exhibited, and $SE_n(X)$ is shown to have no proper normal subgroups of its own. The results are just like those in [2] for $H(X)$ and $SH(X)$.

2. Diffeomorphisms. Let X be the line E^1 or the circle S^1 with local coordinate systems, and $H(X)$ the group of all homeomorphisms from X onto itself. We shall single out certain subgroups of $H(X)$, writing h' for the first and $h^{(n)}$ for the nth derivative of some h in $H(X)$. For each integer $n \geqq 1$ we put

$$D_n(X) = \{h \in H(X): h'(x) \neq 0 \text{ and } h^{(n)}(x) \text{ exists for each } x \text{ in } X\}\ ,$$

$$CD_n(X) = \{h \in D_n(X): h^{(n)} \text{ is continuous}\}\ ,$$

$$D_\infty(X) = \bigcap \{D_n(X): n \geqq 1\} = \bigcap \{CD_n(X): n \geqq 1\} .$$

Evidently $D_n(X)$, $CD_n(X)$, and $D_\infty(X)$ are groups.

A somewhat different family $E_n(X)$ consists of those h in $H(X)$ which can be written as

$$h(p + x) = h(p) + \sum_{i=1}^{n} a_i x^i + x^n e(x) , \qquad x \in U_p , \tag{1}$$

where p is any point in X, U_p is a neighborhood of 0, $a_1 \neq 0$, and $e(x) \to 0$ as $x \to 0$. The coefficients a_i will depend upon p. Evidently $CD_n(X) \subset E_n(X)$. To see that $E_n(X)$ is a group, we will need the help of a lemma.

LEMMA 1. *Given the integer $n \geqq 1$ and the polynomial $P(x) = a_1x + a_2x^2 + \cdots + a_nx^n$ with $a_1 \neq 0$, we can find a polynomial $Q(y) = b_1y + b_2y^2 + \cdots + b_ny^n$ with $b_1 \neq 0$ satisfying*

$$\lim_{x \to 0} \frac{x - Q(P(x))}{x^n} = 0 .$$

Proof. The equation $y = P(x)$ defines x as an implicit function of y in some neighborhood V of $y = 0$, and we can compute its Taylor series with remainder. Thus

$$x = b_0 + b_1y + b_2y^2 + \cdots + b_ny^n + y^nR(y) ,$$

where $R(y) \to 0$ as $y \to 0$. Now $b_0 = 0$ and

$$1 = P'(x)x'(y) = P'(0)x'(0) = a_1x'(0) ,$$

so that $b_1 \neq 0$ and we can solve for $x^{(i)}(0)$, hence also b_i, by repeated differentiation. If we put $Q(y) = b_1y + b_2y^2 + \cdots + b_ny^n$, then

$$\begin{aligned}\lim_{x \to 0} \frac{x - Q(P(x))}{x^n} &= \lim_{x \to 0} \frac{(P(x))^n R(P(x))}{x^n} \\ &= a_1^n \lim_{x \to 0} R(P(x)) = 0\end{aligned}$$

which is the result that we seek.

LEMMA 2. *The family $E_n(X)$ is a group.*

Proof. Given some h in $H(X)$ satisfying (1), we put $h(p) = q$ and $h(p + x) = q + y$. Thus (1) reads

$$y = \sum_{i=1}^{n} a_i x^i + x^n e(x) = P(x) + x^n e(x) , \qquad x \in U_p .$$

We let $Q(y)$ be the polynomial from Lemma 1 and define $d(y)$ by the equation

$$x = Q(y) + y^n d(y) , \qquad y \in V_q ,$$

where V_q is a neighborhood of 0 inside the V from Lemma 1 and so small that x lies in U_p. Now

$$\begin{aligned}\lim_{y\to 0} d(y) &= \lim_{y\to 0}\frac{x - Q(y)}{y^n} = \lim_{y\to 0}\frac{x - Q(P(x) + x^n e(x))}{y^n} \\ &= \lim_{x\to 0}\frac{x^n}{y^n}\cdot\frac{x - Q(P(x)) - b_1 x^n e(x) - x^n S(x)}{x^n} \\ &= b_1^n \lim_{x\to 0}\Big(\frac{x - Q(P(x))}{x^n} - b_1 e(x) - S(x)\Big) = 0 ,\end{aligned}$$

where $S(x) \to 0$ as $x \to 0$. This means that h^{-1} belongs to $E_n(X)$. Suppose next that g belongs to $E_n(X)$ and

$$(2) \qquad g(q + y) = g(q) + \sum_{i=1}^{n} c_i y^i + y^n \bar{e}(y) , \qquad y \in V_q ,$$

where q is any point in X, V_q is a neighborhood of 0, $c_1 \neq 0$, and $\bar{e}(y) \to 0$ as $y \to 0$. If we put $g(q) = r$ and $g(q + y) = r + z$, then (2) reads

$$z = \sum_{i=1}^{n} c_i y^i + y^n \bar{e}(y) = R(y) + y^n \bar{e}(y) , \qquad y \in V_q .$$

We define $\bar{d}(x)$ by the equation

$$z = R(P(x)) + x^n \bar{d}(x) , \qquad x \in W_r ,$$

where W_r is a neighborhood of 0 inside U_p and so small that y lies in V_q. Now

$$\begin{aligned}\lim_{x\to 0}\bar{d}(x) &= \lim_{x\to 0}\frac{z - R(P(x))}{x^n} \\ &= \lim_{x\to 0}\frac{R(P(x) + x^n e(x)) + y^n \bar{e}(y) - R(P(x))}{x^n} \\ &= \lim_{x\to 0}\frac{c_1 x^n e(x) + x^n T(x)}{x^n} + a_1^n \lim_{y\to 0}\bar{e}(y) = 0 ,\end{aligned}$$

where $T(x) \to 0$ as $x \to 0$. This means that gh belongs to $E_n(X)$, and $E_n(X)$ is a group.

LEMMA 3. *If p is a limit point of the set $K(h)$ of all fixed points of h, then $P(x) = x$.*

Proof. We are given a sequence $\{x_m\}$ converging to 0, where

$x_m \neq 0$, $h(p + x_m) = p + x_m$ for each m, and of course $h(p) = p$. Putting $x = x_m$ in (1), we have

$$x_m = a_1 x_m + a_2 x_m^2 + \cdots + a_n x_m^n + x_m^n e(x_m) .$$

If we divide out by x_m and let $m \to \infty$, then we get $a_1 = 1$. If we repeat this process, then we get $a_2 = \cdots = a_n = 0$. Therefore, $P(x) = x$.

LEMMA 4. *If $P(x) = x$ for every p in X, then h is a translation of the line or a rotation of the circle.*

Proof. We have

$$h'(p) = \lim_{x \to 0} \frac{h(p + x) - h(p)}{x} = \lim_{x \to 0} \frac{x + x^n e(x)}{x} = 1$$

for each p in X.

Evidently $E_n(X) \supset E_{n+1}(X)$, for we can write

$$\begin{aligned} h(p + x) &= h(p) + \sum_{i=1}^{n+1} a_i x^i + x^{n+1} e(x) \\ &= h(p) + \sum_{i=1}^{n} a_i x^i + x^n (a_{n+1} x + x e(x)) , \end{aligned}$$

where $d(x) = a_{n+1} x + x e(x) \to 0$ as $x \to 0$. We put $E_\infty(X) = \bigcap \{E_n(X)\colon n \geqq 1\}$ and note that $E_\infty(X) \supset D_\infty(X)$.

LEMMA 5. *Suppose that*

$$\begin{aligned} h(p + x) &= h(p) + \sum_{i=1}^{n} a_i x^i + x^n e(x) \\ &= h(p) + \sum_{i=1}^{n+1} \bar{a}_i x^i + x^{n+1} \bar{e}(x) , \qquad x \in U_p . \end{aligned}$$

Then $a_i = \bar{a}_i$ for $1 \leqq i \leqq n$.

Proof. We have

$$\sum_{i=1}^{n} a_i x^i + x^n e(x) = \sum_{i=1}^{n+1} \bar{a}_i x^i + x^{n+1} \bar{e}(x) , \qquad x \in U_p .$$

If we divide out by x and let $x \to 0$, then we get $a_1 = \bar{a}_1$. If we repeat this operation, then we get $a_2 = \bar{a}_2, \cdots, a_n = \bar{a}_n$.

Lemma 5 shows us the form of a member h of $E_\infty(X)$. In some neighborhood of each point p we can write h as a partial sum of a certain power series with a remainder that goes to zero faster than the highest power of x retained, and the neighborhood depends on this power.

From the simple fact that $D_1(X) = E_1(X)$ we are led to enquire whether $D_n(X) = E_n(X)$ for other values of n. To see that this is not so, we choose h in $H(X)$ so that $h(p_0) = p_0$ at some point p_0 in X,

$$h(p_0 + x) = p_0 + x + x^3 \sin 1/x$$

for x in a sufficiently small neighborhood to the right of 0, $h(p) = p$ for p in some neighborhood to the left of 0, and h beyond these neighborhoods has derivatives of all orders with first derivative that is never zero. Evidently h belongs to $E_2(X)$. But $h'(p_0) = 1$,

$$h'(p_0 + x) = 1 - x \cos 1/x + 3x^2 \sin 1/x ,$$

and $h''(p_0)$ is not defined. Thus h does not belong to $D_2(X)$, and $E_2(X)$ does not lie in $D_2(X)$. In the same way we can show that $E_{2n}(X)$ does not lie in $D_{n+1}(X)$ when we put

$$h(p_0 + x) = p_0 + x + x^{2n+1} \sin 1/x .$$

If we consider the homeomorphism g given in part by

$$g(p_0 + x) = p_0 + x + x^2 \sin 1/x ,$$

then we find that g belongs to $E_1(X)$ but not to $CD_1(X)$, and $E_1(X)$ does not lie in $CD_1(X)$. In the same way we can show that $E_{2n-1}(X)$ does not lie in $CD_n(X)$ when we put

$$g(p_0 + x) = p_0 + x + x^{2n} \sin 1/x .$$

Finally, if we choose the homeomorphism f given in part by

$$f(p_0 + x) = p_0 + x + e^{-1/x} \sin e^{1/x} ,$$

then f evidently belongs to $E_\infty(X)$. But

$$f'(p_0 + x) = 1 - \frac{\cos e^{1/x}}{x^2} + \frac{e^{-1/x} \sin e^{1/x}}{x^2}$$

shows us that $f'(p_0 + x)$ has no limit as $x \to 0$. Thus f does not belong to $CD_1(X)$, and $E_\infty(X)$ does not lie in $CD_1(X)$. In particular, $D_\infty(X)$ is a proper subset of $E_\infty(X)$.

3. Minimal normal subgroups. We say that a member h of $H(X)$ has compact support if $X - K(h)$ lies in a compact set different from X. For any subset G of $H(X)$, we let SG be the group generated by those members of G with compact support. If G is a group, then SG is evidently a normal subgroup of G. We note that if U, V are open subsets of X, and U lies in a compact set different from X, then some member of $SD_\infty(X)$ maps U into V.

THEOREM 1. *If G is a subgroup of $H(X)$, $SE_n(X)$ lies in the normalizer of G, and G has more than one member, then $G \supset SE_n(X)$.*

Proof. Suppose that g_0 belongs to G and is not the identity. Then g_0 moves some point p_0 in X, and we can find a neighborhood U_0 of p_0 which does not meet $g_0(U_0)$. We put

$$h_0(p_0 + x) = p_0 + x - e^{-1/x} = p_0 + x + S(x) , \quad 0 \leqq x \leqq u ,$$

where u is chosen so small that $h_0'(p) > 0$ for $p_0 < p < p_0 + u = p_1$ and $(p_0, p_1 + u) \subset U_0$. Since $h_0(p) < p$ there, the sequence $\{p_k\}$ given by $p_{k+1} = h_0^k(p_1)$ will converge monotonely to p_0. We can define h_0 on the rest of X so as to be supported on $(p_0, p_1 + u)$ and belong to $E_n(X)$, for $S(x)/x^n \to 0$ as $x \to +0$ tells us that h_0 satisfies (1) to the right of $p = p_0$ with $P(x) = x$. Evidently $g_0^{-1}h_0^{-1}g_0h_0 = g_1$ agrees with h_0 on U_0 and belongs to G. Suppose that f is any member of $SE_n(X)$, and $X - K(f) = U$ lies in a compact set different from X. If we choose f_0 in $SD_\infty(X)$ so that $f_0(U) \subset (p_2, p_1)$, then $f_0 f f_0^{-1} = f_1$ is supported on (p_2, p_1) and belongs to $SE_n(X)$. If we can show that f_1 belongs to G, then f must also belong to G, and $G \supset SE_n(X)$. We first note that

$$g_1^k([p_2, p_1]) = [p_{k+2}, p_{k+1}] , \qquad k = 0, 1, 2, \cdots .$$

We then define

$$h(p) = \begin{cases} g_1^k f_1 g_1^{-k}(p) \text{ for } p \text{ in } [p_{k+2}, p_{k+1}], & k = 0, 1, 2, \cdots , \\ p \text{ for } p \text{ in } X - (p_0, p_1] . \end{cases}$$

Lemma 3 tells us that we can join two pieces of h together at $p = p_k$ so as to satisfy (1) there, for $e(x)$ may be defined in different ways on either side of p. Thus (1) holds at every point p of X except possibly $p = p_0$. To see that h satisfies (1) at $p = p_0$, we observe that if p lies in $[p_{k+2}, p_{k+1}]$, then so does $h(p)$. If we put $p_k = p_0 + x_k$, then

$$\begin{aligned} \lim_{x\to 0} \frac{|h(p_0 + x) - p_0 - x|}{x^n} &\leqq \lim_{k\to\infty} \frac{p_{k+1} - p_{k+2}}{(p_{k+2} - p_0)^n} \\ &= \lim_{k\to\infty} \frac{p_{k+1} - h_0(p_{k+1})}{(h_0(p_{k+1}) - p_0)^n} = \lim_{k\to\infty} \frac{-S(x_{k+1})}{(x_{k+1} + S(x_{k+1}))^n} \\ &= \lim_{k\to\infty} \frac{-S(x_{k+1})/x_{k+1}^n}{(1 + S(x_{k+1})/x_{k+1})^n} = 0 . \end{aligned}$$

This means that h satisfies (1) at $p = p_0$ with $P(x) = x$, and so h belongs to $SE_n(X)$. Now $g_1 h g_1^{-1}$ is supported on (p_0, p_2) and agrees with h there, for if p lies in $[p_{k+2}, p_{k+1}]$ and $k \geqq 1$, then

$$g_1 h g_1^{-1}(p) = g_1 g_1^{k-1} f_1 g_1^{-k+1} g_1^{-1}(p) = g_1^k f_1 g_1^{-k}(p) = h(p) .$$

Therefore, $hg_1h^{-1}g_1^{-1} = f_1$ must lie in G, and our proof is complete.

COROLLARY 1. *Theorem* 1 *remains true if we replace* n *by* ∞.

Proof. We have already seen that h_0 satisfies (1) to the right of $p = p_0$ with $P(x) = x$ for every value of n. Hence, we can define h_0 on the rest of X so as to be supported on $(p_0, p_1 + u)$ and belong to $E_\infty(X)$. The rest of the proof is the same as before with n replaced by ∞.

COROLLARY 2. $SE_n(X)$ *and* $SE_\infty(X)$ *are simple groups.*

COROLLARY 3. $SD_1(X)$ *is a simple group.*

Proof. Evidently $E_1(X) = D_1(X)$.

COROLLARY 4. *The commutator subgroup of* $SE_n(X)$ *is* $SE_n(X)$, *and the same for* $SE_\infty(X)$.

Proof. Some commutators are evidently not the identity.

For any subset G of $H(X)$, we write G^+ for the family of all orientation-preserving members of G, and G^- for $G - G^+$. If G is a group, then G^+ is evidently a normal subgroup of G with index one or two.

COROLLARY 5. *If* $X = S^1$, *then* $E_n^+(X)$ *and* $E_\infty^+(X)$ *are simple groups.*

Proof. It is enough to show that $E_n^+(X) = SE_n(X)$. To see this, we use the same method as for the case $H^+(X) = SH(X)$. Given h in $E_n^+(X)$, we can piece together a member g of $E_n^+(X)$ that agrees with h on a small open subset of X and agrees with the identity on another small open subset of X. Then g and $g^{-1}h$ both belong to $SE_n(X)$, and so does $g(g^{-1}h) = h$. The same argument works for $E_\infty^+(X)$.

4. Behavior at the endpoints. From now on we shall deal only with the space $X = [-1, +1]$ and think of E^1 as the subset $(-1, +1)$ of X. For any subset G of $H(X)$ we put

$$S_-G = \{h \in G\colon K(h) \supset [-1, p) \text{ for some } p > -1\},$$

$$S_+G = \{h \in G\colon K(h) \supset (p, +1] \text{ for some } p < +1\}.$$

Evidently $S_-G \cap S_+G = SG$, and both are normal subgroups of G

provided that G is a subgroup of $H^+(X)$. Thus $S_-E_n(X)$, $S_+E_n(X)$ are normal subgroups of $E_n^+(X)$, and the same is true if we replace n by ∞. The normal subgroups of $H^+(X)$ are few in number, namely $S_-H(X)$, $S_+H(X)$, and $SH(X)$. Those of $H(X)$ are even fewer, namely $H^+(X)$ and $SH(X)$. Proofs are given in [2]. On the other hand, those of $E_n^+(X)$, $E_\infty^+(X)$, $E_n(X)$, and $E_\infty(X)$ are very numerous. Our aim in the rest of this paper is to show how some of them may be constructed.

The behavior of a member h of $E_n(X)$ at $p = -1$ or $+1$ is described by (1), namely

$$h(-1 + x) = h(-1) + P_-(x) + x^n d(x) , \qquad x \in U_{-1} ,$$

$$h(+1 + x) = h(+1) + P_+(x) + x^n e(x) , \qquad x \in U_{+1} ,$$

where $d(x), e(x) \to 0$ as $x \to 0$, $U_{-1} = [0, +u)$, $U_{+1} = (-u, 0]$, and $u > 0$. Let $\Delta_{n-}(X)$ be the family of those h in $E_n(X)$ for which $P_-(x) = x$, $\Delta_{n+}(X)$ the family for which $P_+(x) = x$, and $\Delta_{n-}(X) \cap \Delta_{n+}(X) = \Delta_n(X)$. Evidently all three are normal subgroups of $E_n^+(X)$. To see that $\Delta_n(X)$ is also a normal subgroup of $E_n(X)$, we have to check that $f\Delta_n(X)f^{-1} \subset \Delta_n(X)$ for each f in $E_n^-(X)$. Now $f(+1) = -1 = f^{-1}(+1)$, and Lemma 2 says that f satisfies (1) with $P_-(P_+(x)) = x = P_+(P_-(x))$. Thus fhf^{-1} belongs to $\Delta_n(X)$ whenever h does, and $\Delta_n(X)$ is normal in $E_n(X)$. Lemma 3 tells us that if h belongs to $E_n(X)$, and the derived set of $K(h)$ includes $+1$, then h belongs to $\Delta_{n+}(X)$, whereas if it includes -1, then h belongs to $\Delta_{n-}(X)$. These results are still valid when we replace n by ∞, where $P_+(x)$ and $P_-(x)$ are now power series in x.

Other normal subgroups of $E_n(X)$ can also be constructed. Choose a sequence $\{p_k\}$ converging to -1, where $p_k = -1 + x_k$, $0 < x_k < +1$, $x_{k+1}/x_k \to L$ as $k \to \infty$, and $0 \leqq L < 1$. Then choose a sequence $\{q_k\}$ converging to -1, where $q_k = p_k + y_k$, $0 < y_k < +1$, and $y_k/x_k^m \to 0$ as $k \to \infty$ for every m. We define a function g in $H(X)$ so that $X - K(g) = \bigcup \{(p_k, q_k)\}$, g satisfies (1) at every point p in X except possibly $p = -1$, and $P(x) = x$ at $p = p_k$ or q_k. To see that (1) holds at $p = -1$, we note that if x lies in (p_k, q_k), then so does $g(x)$. Thus

$$\lim_{x \to +0} \frac{|g(-1 + x) - g(-1) - x|}{x^n} \leqq \lim_{k \to \infty} \frac{y_k}{x_k^n} = 0 ,$$

and (1) holds with $P_-(x) = x$. Since $P_+(x) = x$, we have g in $\Delta_n(X)$. The normal subgroup G of $E_n(X)$ generated by g consists of products of conjugates in $E_n(X)$ of g, and G is clearly larger than $SE_n(X)$. To see that G is a proper subgroup of $\Delta_n(X)$, we will show that for every member f of G, the derived set of $K(f)$ includes -1. The same argument shows that it includes $+1$ as well. If h belongs to $E_n(X)$

and satisfies (1) at $p = -1$ with $P(x) = P_-^*(x) = a_1 x + \cdots$, then we shall put $h(p_k) = h(-1) + x_k^*$ and $h(q_k) = h(p_k) + y_k^*$. We have

$$\lim_{k\to\infty} \frac{x_{k+1}^*}{x_k^*} = \lim_{k\to\infty} \frac{P_-^*(x_{k+1}) + x_{k+1}^n e^*(x_{k+1})}{P_-^*(x_k) + x_k^n e^*(x_k)} = \lim_{k\to\infty} \frac{x_{k+1}}{x_k} = L\,,$$

$$(3)\quad \lim_{k\to\infty} \frac{y_k^*}{(x_k^*)^n} = \lim_{k\to\infty} \frac{P_-^*(x_k+y_k) - P_-^*(x_k) + (x_k+y_k)^n e^*(x_k+y_k) - x_k^n e^*(x_k)}{(P_-^*(x_k) + x_k^n e^*(x_k))^n}$$

$$= \lim_{k\to\infty} \frac{a_1 y_k}{a_1^n x_k^n} = 0\,.$$

Suppose that f is the product of j conjugates $g_i = h_i g h_i^{-1} (1 \leqq i \leqq j)$ of g, where $f = g_j \cdots g_1$. Our work would be made easier if any h_i lay in $E_n^-(X)$, for then $K(g_i) = h_i(K(g))$ would include a neighborhood of -1. So we shall assume that h_i lies in $E_n^+(X)$ for $1 \leqq i \leqq j$. We put

$$p_k^{(i)} = h_i(p_k) = h_i(-1) + x_k^{(i)}\,,$$
$$q_k^{(i)} = h_i(q_k) = h_i(p_k) + y_k^{(i)}\,, \qquad 1 \leqq i \leqq j\,.$$

From (3) we know that

$$\lim_{k\to\infty} x_{k+1}^{(i)}/x_k^{(i)} = L,\ \lim_{k\to\infty} y_k^{(i)}/(x_k^{(i)})^n = 0\,.$$

Given any number $u > 0$, we choose k_0 so large that $k \geqq k_0$ implies

$$x_{k+1}^{(i)}/x_k^{(i)} \leqq L^* < 1\,,\quad y_k^{(i)}/x_k^{(i)} \leqq (1 - L^*)/2j\,, \qquad 1 \leqq i \leqq j\,,$$

and put

$$k(i) = \sup\{k\colon x_k^{(i)} > u \text{ or } k = k_0\}\,, \qquad 1 \leqq i \leqq j\,.$$

From $K(g_i) = h_i(K(g))$ and $K(f) \supset \bigcap \{K(g_i)\colon 1 \leqq i \leqq j\}$ we see that

$$(4)\quad (X - K(f)) \cap [-1, -1 + u) \subset \bigcup \{(p_k^{(i)}, q_k^{(i)})\colon k > k(i), 1 \leqq i \leqq j\}\,.$$

Now the sum of the lengths of all the intervals on the right side of (4) is

$$\sum_{i=1}^{j} \sum_{k=k(i)+1}^{\infty} y_k^{(i)} \leqq \sum_{i=1}^{j} \sum_{k=k(i)+1}^{\infty} \frac{1 - L^*}{2j} x_k^{(i)}$$
$$\leqq \frac{1 - L^*}{2j} \sum_{i=1}^{j} \sum_{m=0}^{\infty} x_{k(i)+1}^{(i)} (L^*)^m = \frac{1 - L^*}{2j} \sum_{i=1}^{j} \frac{x_{k(i)+1}^{(i)}}{1 - L^*}$$
$$\leqq \frac{1}{2j} \sum_{i=1}^{j} u = \frac{u}{2}\,.$$

These intervals could not possibly cover $(-1, -1 + u)$, for otherwise a finite chain of them would reach from $-1 + u/5$ to $-1 + 4u/5$, which is impossible. Hence, $X - K(f)$ does not contain $(-1, -1 + u)$ for any $u > 0$, and the derived set of $K(f)$ includes -1. Therefore, G

is not $\Delta_n(X)$. Evidently the same argument also shows that we can make g belong to $\Delta_\infty(X)$, and the normal subgroup of $E_\infty(X)$ generated by g is neither $SE_\infty(X)$ nor $\Delta_\infty(X)$. We shall now sum up these results.

THEOREM 2. *There are normal subgroups of $E_n(X)$ that lie between $SE_n(X)$ and $\Delta_n(X)$. The same is true of normal subgroups of $E_\infty(X)$ between $SE_\infty(X)$ and $\Delta_\infty(X)$.*

We let e be the identity element of $H(X)$ and call a member h of $H(X)$ an involution if $h^2 = e$. Evidently an involution different from e has just one fixed point in $(-1, +1)$.

LEMMA 6. *Suppose that for $i = 1, 2$ there is an involution h_i of $E_n(X)$ with fixed point p_i that satisfies (1) at $p = p_i$ with $P(x) = P_i(x)$. If there is a polynomial $Q(x) = a_1x + a_2x^2 + \cdots$ with $a_1 > 0$ such that $P_1Q(x)$ and $QP_2(x)$ agree in all terms of degree $\leqq n$, then there is a member f of $S_-E_n(X)$ such that $h_1f = fh_2$. We may also choose f in $S_+E_n(X)$, and the result remains true if we replace n by ∞.*

Proof. We choose h_0 in $SE_n(X)$ so that $h_0(p_2) = p_1$ and h_0 satisfies (1) at $p = p_2$ with $P(x) = Q(x)$. We shall assume that h_i is different from e, so that h_i lies in $E_n^-(X)$. Thus $h_3 = h_0^{-1}h_1h_0$ is an involution of $E_n(X)$ with fixed point p_2 and satisfying (1) at $p = p_2$ with $P(x) = P_2(x)$. Our result will be proved if we can find g in $S_-E_n(X)$ so that $h_3g = gh_2$. We put

$$g(p) = \begin{cases} p & \text{for } -1 \leqq p \leqq p_2 \,, \\ h_3h_2(p) & \text{for } p_2 \leqq p \leqq +1 \,. \end{cases}$$

Evidently g belongs to $S_-H(X)$ and satisfies (1) at each point p of X, except possibly $p = p_2$. But $h_2^2 = e$ satisfies (1) at $p = p_2$ with $P(x) = P_2^2(x) = x$, so h_3h_2 also satisfies (1) at $p = p_2$ with $P(x) = x$. Consequently, g belongs to $S_-E_n(X)$, and

$$gh_2(p) = \begin{cases} h_3h_2h_2(p) = h_3(p) = h_3g(p) & \text{for } -1 \leqq p \leqq p_2 \,, \\ h_2(p) = h_3h_3h_2(p) = h_3g(p) & \text{for } p_2 \leqq p \leqq +1 \,. \end{cases}$$

Hence, $gh_2 = h_3g$.

5. Groups of polynomials. It is now time to study the compositions of polynomials more closely. We begin with the family Π_∞ of power series

$$P(x) = a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \,, \qquad a_1 \neq 0$$

with any coefficients whatever. By working with the partial sums of $P(x)$, we see that the composition $PQ(x) = P(Q(x))$ of two members P, Q of Π_∞ also belongs to Π_∞, that $P_0(x) = x$ is the identity element of Π_∞, and from Lemma 1 that P has an inverse P^{-1} in Π_∞ satisfying $P^{-1}P = P_0$. Since the associative law always holds for composition, we see that Π_∞ is a group. For each integer $n \geqq 1$, we define an equivalence relation $P\Xi_n Q$ between members P and Q of Π_∞ to mean that $P(x)$ and $Q(x)$ agree in all terms of degree $\leqq n$. This relation is evidently compatible with the group operation, and the quotient group Π_n we can think of as the group of all nth degree polynomials

$$P(x) = a_1x + a_2x^2 + \cdots + a_nx^n\,, \qquad a_1 \neq 0$$

under composition. If Γ is a subgroup of Π_∞, then we shall let Γ^n be the quotient group of Γ under Ξ_n. We put $\{P_0\} = \varDelta_\infty$ and use $\varDelta_n$ for the subgroup of Π_n with just one element. A normal subgroup of Π_∞ closely related to $\varDelta_n$ is the kernel $\varDelta_n^*$ of the homomorphism from Π_∞ onto Π_n induced by the relation Ξ_n. It consists of all members of Π_∞ whose first n coefficients are $1, 0, \cdots, 0$. We shall use Π to mean either Π_∞ or Π_n, and the same for Ξ, $\varDelta$, and $E(X)$. The family of all those P in Π with $a_1 > 0$ we will denote by Π^+ and put $\Pi - \Pi^+ = \Pi^-$. Evidently Π^+ is a normal subgroup of Π and compatible with Ξ. We define $\Gamma^+ = \Gamma \cap \Pi^+$, $\Gamma^- = \Gamma - \Gamma^+$, and note that Γ^+ is a normal subgroup of Γ with index one or two.

LEMMA 7. *The only normal subgroups Γ of Π_n that lie in $\varDelta_1^{*n}$ are $\Gamma = \varDelta_m^{*n}$ for $1 \leqq m \leqq n$.*

Proof. It is evidently sufficient to prove the following statement for each value of m:
(5) For each integer k with $0 < k < m$, if Γ contains some member P of the form $P(x) = x + a_{k+1}x^{k+1} + \cdots$ with $a_{k+1} \neq 0$, then for each R in $\varDelta_k^{*n}$, Γ contains some Q such that $Q\Xi_m R$.

Now (5) certainly holds when $m = 1$. We shall proceed by induction and suppose that (5) holds for some value $m > 1$. To verify it for $m + 1$, we first note that if $k = m$ and $P_1(x) = a_1x$, then

$$P^{-1}(x) = x - a_{k+1}x^{k+1} + \cdots\,,$$

$$P_1^{-1}PP_1(x) = x + a_1^m a_{m+1}x^{m+1} + \cdots\,,$$

$$P_1^{-1}P^{-1}P_1(x) = x - a_1^m a_{m+1}x^{m+1} + \cdots\,.$$

Evidently $P_1^{-1}PP_1$ and $P_1^{-1}P^{-1}P_1$ both belong to Γ, and by varying a_1 through positive values we get all coefficients of x^{m+1}. Thus (5) holds in this case. If $k < m$, then our induction hypothesis says that Γ

contains some member Q of the form

$$Q(x) = x + b_m x^m + b_{m+1}x^{m+1} + \cdots ,$$

where $b_m \neq 0$. If we put $P_2(x) = x + a_2 x^2$, then

$$QP_2(x) = x + a_2x^2 + b_mx^m + (b_{m+1} + ma_2b_m)x^{m+1} + \cdots ,$$

and this agrees with $P_2\bar{Q}(x)$ in all terms of degree $\leqq m+1$, where

$$\bar{Q}(x) = x + b_mx^m + (b_{m+1} + (m-2)a_2b_m)x^{m+1} + \cdots .$$

Thus $QP_2\Xi_{m+1}P_2\bar{Q}$ and $P_2^{-1}QP_2\Xi_{m+1}\bar{Q}$ while $P_2^{-1}QP_2$ lies in Γ. If we put

$$P_m(x) = x + a_mx^m , \quad P_{m+1}(x) = x + a_{m+1}x^{m+1} ,$$
$$Q_m(x) = x + c_mx^m , \quad Q_{m+1}(x) = x + c_{m+1}x^{m+1} ,$$

then evidently $P_iQ_j\Xi_{m+1}Q_jP_i$ for $i, j = m, m+1$. We must now distinguish the cases $m > 2$ and $m = 2$. In the case $m > 2$ we have

$$P_m^{-1}(x) = x - a_mx^m + \cdots , \quad P_{m+1}^{-1}(x) = x - a_{m+1}x^{m+1} + \cdots$$

for the first $m+1$ terms. If we put $a_m = b_m$, $c_m = b_m$, $a_{m+1} = b_{m+1}$, and $c_{m+1} = b_{m+1} + (m-2)a_2b_m$, then

$$Q\Xi_{m+1}P_mP_{m+1} , \quad \bar{Q}\Xi_{m+1}Q_mQ_{m+1} ,$$
$$Q^{-1}\bar{Q}\Xi_{m+1}P_m^{-1}Q_mP_{m+1}^{-1}Q_{m+1}\Xi_{m+1}P_{m+1}^{-1}Q_{m+1} ,$$

where

$$P_{m+1}^{-1}Q_{m+1}(x) = x + (m-2)a_2b_mx^{m+1} + \cdots .$$

Thus we get all coefficients of x^{m+1} by varying a_2, and

$$P_{m+1}^{-1}Q_{m+1}\Xi_{m+1}Q^{-1}P_2^{-1}QP_2 = R_{m+1} ,$$

where R_{m+1} lies in Γ. By our induction hypothesis, we can find a member of $\Gamma \cap \Delta_k^{*n}$ with any desired coefficients of $x^{k+1}, \cdots, x^m$. If we multiply this member on the left by R_{m+1}, then we keep the same coefficients of $x^{k+1}, \cdots, x^m$ and get any desired coefficient of x^{m+1}. This verifies (5) for $m+1$ when $m > 2$. Finally, consider the case $m = 2$. The argument for $k = m$ is still valid, so we need only consider $k = 1$. Thus our induction hypothesis says that Γ contains some member Q of the form

$$Q(x) = x + b_2x^2 + b_3x^3 + \cdots ,$$

where $b_2 \neq 0$. We can check by induction that for any integer i,

$$P_2^i(x) = x + ia_2x^2 + i(i-1)a_2^2x^3 + \cdots ,$$
$$P_3^i(x) = x + ia_3x^3 + \cdots .$$

If we put $a_2 = b_2$ and $a_3 = b_3$, then $Q\Xi_3 P_2 P_3$. Evidently $P_1^{-1}QP_1$ lies in Γ and has the form

$$P_1^{-1}QP_1(x) = x + a_1 b_2 x^2 + a_1^2 b_3 x^3 + \cdots .$$

If we put $c_2 = a_1 b_2$ and $c_3 = a_1^2 b_3$, then $P_1^{-1}QP_1\Xi_3 Q_2 Q_3$. Finally, $Q^i P_1^{-1} Q P_1 = R_3$ lies in Γ and

$$R_3 \Xi_3 P_2^i P_3^i Q_2 Q_3 \Xi_3 (P_2^i Q_2)(P_3^i Q_3) .$$

Thus R_3 has the form

$$R_3(x) = x + (i + a_1)b_2 x^2 + (i(i-1)b_2^2 + (i + a_1^2)b_3)x^3 + \cdots .$$

If we choose $a_1 = -i$, then

$$R_3(x) = x + ((b_2^2 + b_3)i^2 - (b_2^2 - b_3)i)x^3 + \cdots .$$

The coefficients $b_2^2 + b_3$ and $b_2^2 - b_3$ are not both zero, so we can choose some integer i for which the coefficient of x^3 is not zero. As in the case $k = m$ above, we can find a member of Γ with any coefficient of x^3, and the last part of the argument for $k < m$ completes the verification of (5) for the case $m = 2$. Hence, our induction step is valid in every case.

THEOREM 3. *Each normal subgroup Γ of Π_n is either Δ_m^{*n} with $1 \leqq m \leqq n$ or the inverse under the canonical mapping from Π_n onto Π_1 of a subgroup N of Π_1.*

Proof. Each member P of Γ has the form

$$P(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots , \qquad a_1 \neq 0 .$$

Evidently $N = \Gamma^1$ is a subgroup of Π_1 which we shall identify with the subgroup of Π_n consisting of all polynomials $P_1(x) = a_1 x$. Evidently Π_1 is the group of nonzero real numbers under multiplication. Now $\Gamma \cap \Delta_1^{*n}$ is a normal subgroup of Π_n, and Lemma 7 says that $\Gamma \cap \Delta_1^{*n} = \Delta_m^{*n}$ with $1 \leqq m \leqq n$. If

$$P(x) = a_1 x + a_{k+1} x^{k+1} + \cdots , \qquad a_{k+1} \neq 0 ,$$

and if we put $Q_1(x) = b_1 x$, then

$$P^{-1}(x) = (1/a_1)x - (a_{k+1}/a_1^{k+2})x^{k+1} + \cdots ,$$

$$Q_1^{-1}PQ_1(x) = a_1 x + a_{k+1} b_1^k x^{k+1} + \cdots ,$$

$$P^{-1}Q_1^{-1}PQ_1(x) = x + (a_{k+1}/a_1)(b_1^k - 1)x^{k+1} + \cdots .$$

Since $P^{-1}Q_1^{-1}PQ_1$ belongs to Γ, we get $\Delta_k^{*n} \subset \Gamma$ by taking $b_1 \neq \pm 1$.

If we put $P_1(x) = a_1x$ as above, then $P_1^{-1}P$ belongs to $\varDelta_k^{*n}$, and P_1 belongs to Γ. Thus N lies in Γ. If we put $Q_2(x) = x + b_2x$, then $Q_2^{-1}(x) = x - b_2x^2 + \cdots$ and

$$\begin{aligned} P_1^{-1}Q_2^{-1}P_1Q_2(x) &= (1/a_1)(a_1x + (a_1 - a_1^2)b_2x^2 + \cdots) \\ &= x + (1 - a_1)b_2x^2 + \cdots . \end{aligned}$$

If we suppose that N is more than the identity, then we can choose $a_1 \neq 1$, $b_2 \neq 0$, and get a member of $\varDelta_1^{*n} - \varDelta_2^{*n}$. Since $P_1^{-1}Q_2^{-1}P_1Q_2$ belongs to Γ, we must have $\varDelta_1^{*n} \subset \Gamma$. Hence, for any Q in Π_n of the form $Q(x) = a_1x + \cdots$, we know that $P_1^{-1}Q$ belongs to $\varDelta_1^{*n}$, and Q belongs to Γ.

COROLLARY 1. *Each normal subgroup Γ of Π_n^+ is either $\varDelta_m^{*n}$ with $1 \leqq m \leqq n$ or the inverse under the canonical mapping from Π_n^+ onto Π_1^+ of a subgroup N of Π_1^+.*

Proof. The proofs of Lemma 7 and Theorem 3 apply here, for nowhere was the leading coefficient of any polynomial assumed to be negative.

COROLLARY 2. *Each normal subgroup Γ of Π_n^+ is normal in Π_n.*

Proof. Evidently each subgroup N of Π_1^+ is normal in Π_1.

COROLLARY 3. *The group I generated by all involutions of Π_n consists of all members of Π_n with leading coefficient ± 1.*

Proof. Since each conjugate of each involution of Π_n is also an involution, I must be a normal subgroup of Π_n. If P belongs to I and $P(x) = a_1x + \cdots$, then $P^2 = P_0$, $a_1^2 = 1$, and $a_1 = \pm 1$. Evidently $P(x) = -x$ belongs to I, so our result follows from Theorem 3.

COROLLARY 4. *The commutator subgroup K of Π_n is $\varDelta_1^{*n}$.*

Proof. If $P(x) = a_1x + \cdots$ and $Q(x) = b_1x + \cdots$, then

$$P^{-1}Q^{-1}PQ(x) = x + \cdots$$

and K lies in $\varDelta_1^{*n}$. If we put

$$P_1(x) = a_1x\ , \quad Q_2(x) = x + b_2x^2\ ,$$

then

$$P_1^{-1}(x) = (1/a_1)x\ , \quad Q_2^{-1}(x) = x - b_2x^2 + \cdots\ ,$$

$$P_1^{-1}Q_2^{-1}P_1Q_2(x) = x + (1 - a_1)b_2x^2 + \cdots .$$

For $a_1 \neq 1$ and $b_2 \neq 0$ we get a member of $\Delta_1^{*n} - \Delta_2^{*n}$. Hence, $K = \Delta_1^{*n}$.

LEMMA 8. *Every two involutions in Π different from P_0 are conjugate.*

Proof. It is evidently sufficient to show that every involution P different from P_0 with

$$P(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is conjugate to R_∞ with $R_\infty(x) = -x$. We know that $a_1 = \pm 1$. If $a_1 = +1$, then

$$P(x) = x + a_k x^k + \cdots , \qquad a_k \neq 0 ,$$

$$x = P^2(x) = x + 2a_k x^k + \cdots$$

which is impossible. Thus $a_1 = -1$. Now suppose that for some integer $m \geqq 1$ we have found a member P_m of Π satisfying

$$(6) \qquad P_m^{-1} P P_m(x) = -x + a_j x^j + \cdots , \qquad j > m ,$$

where a_j may be zero. We shall proceed by induction and verify (6) for $m + 1$. Evidently $R_m = P_m^{-1} P P_m$ is also an involution, so we have

$$x = R_m^2(x) = x + ((-1)^j - 1)a_j x^j + \cdots .$$

Thus $a_j \neq 0$ implies that j is even. If we put $Q_j(x) = x + b_j x^j$, then

$$\begin{aligned} Q_j^{-1} R_m Q_j(x) &= Q_j^{-1}(-x + (a_j - b_j)x^j + \cdots) \\ &= -x + (a_j - b_j - (-1)^j b_j)x^j + \cdots . \end{aligned}$$

If $a_j = 0$, then $R_m = R_\infty$ and our induction step is complete. If $a_j \neq 0$, then j is even, and we can make the coefficient of x^j zero by putting $b_j = a_j/2$. Hence, our induction step is again complete with $P_{m+1} = P_m Q_j$. Now P_m agrees with P_{m+1} in all terms of degree $\leqq m$, so our induction process shows us that P is conjugate to R_∞ in Π_∞ as well as in Π_n.

COROLLARY. *The normal subgroup of Π_n generated by any involution different from P_0 is I.*

Some simple involutions of Π easily come to hand. For each integer $m \geqq 1$ we put

$$P_m(x) = \frac{x}{(1 + a_m x^m)^{1/m}} = x - (a_m/m)x^{m+1} + \cdots ,$$

$$Q_m(x) = \frac{x}{(1 + b_m x^m)^{1/m}} = x - (b_m/m)x^{m+1} + \cdots .$$

We can easily verify that

$$P_mQ_m(x) = \frac{x}{(1 + (a_m + b_m)x^m)^{1/m}},$$

$$P_m^{-1}(x) = \frac{x}{(1 - a_mx^m)^{1/m}}.$$

If we put $R_m = P_m^{-1}R_\infty P_m$, then

$$R_m(x) = \frac{-x}{(1 + (1 - (-1)^m)a_mx^m)^{1/m}}.$$

Evidently R_m is an involution. When m is even, we get $R_m = R_\infty$, and when m is odd, we get

$$R_m(x) = \frac{-x}{(1 + 2a_mx^m)^{1/m}} = -x + (2a_m/m)x^{m+1} + \cdots.$$

The terms of degree $\leqq n$ in R_m give us an involution in Π_n.

6. **Special normal subgroups.** With each member h of $E(X)$ we associate the polynomials $\pi_-(h) = P_-$ and $\pi_+(h) = P_+$. Evidently π_- and π_+ are homomorphisms from $E^+(X)$ into Π^+, and they map $E_n^+(X)$ onto Π_n^+. To see what π_- and π_+ do to $E^-(X)$, we distinguish two cases:

$$\pi_-(gh) = \pi_-(g)\pi_-(h), \quad \pi_+(gh) = \pi_+(g)\pi_+(h), \quad g \in E(X), \ h \in E^+(X),$$

$$\pi_-(gh) = \pi_+(g)\pi_-(h), \quad \pi_+(gh) = \pi_-(g)\pi_+(h), \quad g \in E(X), \ h \in E^-(X).$$

One consequence of this is that if h is in $E^-(X)$, then

$$\pi_-(h^{-1}) = (\pi_+(h))^{-1}, \quad \pi_+(h^{-1}) = (\pi_-(h))^{-1}.$$

Another consequence is that if g is in $E^+(X)$, and h is in $E^-(X)$, then

$$\begin{aligned} \pi_-(h^{-1}gh) &= \pi_+(h^{-1}g)\pi_-(h) = \pi_+(h^{-1})\pi_+(g)\pi_-(h) \\ &= (\pi_-(h))^{-1}\pi_+(g)\pi_-(h). \end{aligned} \tag{7}$$

In the same way we get

$$\pi_+(h^{-1}gh) = (\pi_+(h))^{-1}\pi_-(g)\pi_+(h). \tag{8}$$

Thus we see that if g is in $E^+(X)$, then $\pi_-(h^{-1}gh)$ is conjugate to either $\pi_-(g)$ or $\pi_+(g)$, depending on whether h is in $E^+(X)$ or $E^-(X)$. A like result holds for $\pi_+(h^{-1}gh)$. We note that π_- and π_+ both map $E_n^-(X)$ onto Π_n^-.

For any subset Γ of Π we put

$$\Gamma_-(X) = \{h \in E(X) \colon \pi_-(h) \in \Gamma\} ,$$

$$\Gamma_+(X) = \{h \in E(X) \colon \pi_+(h) \in \Gamma\} ,$$

and $\Gamma(X) = \Gamma_-(X) \cap \Gamma_+(X)$. If Γ is a normal subgroup of Π that lies in Π^+, then (7) and (8) tell us that $\Gamma(X)$ is a normal subgroup of $\Pi(X) = E(X)$ that lies in $E^+(X)$, and that $\Gamma_-(X)$, $\Gamma_+(X)$, and $\Gamma(X)$ are normal subgroups of $\Pi^+(X) = E^+(X)$. Of course all these normal subgroups include Δ or $\Delta(X)$, but we note that $\Delta(X) \supset \Delta^*(X)$.

The mapping ϕ given by $\phi(G) = (\pi_-(G), \pi_+(G))$ is a one-to-one correspondence between some of the normal subgroups G of $E_n^+(X)$ that include $\Delta_n(X)$ and all of the ordered pairs of normal subgroups from Π_n^+. For if we start with an ordered pair (B, Γ) of normal subgroups from Π_n^+, then we can assign to it the normal subgroup

$$\bar{\phi}(B, \Gamma) = B_-(X) \cap \Gamma_+(X) = G$$

of $E_n^+(X)$, where G clearly includes $\Delta_n(X)$. Since we can always find a member h of $E_n^+(X)$ with given values of $\pi_-(h)$ and $\pi_+(h)$ in Π_n^+, we have

$$\phi(\bar{\phi}(B, \Gamma)) = (\pi_-(G), \pi_+(G)) = (B, \Gamma) .$$

Thus $\bar{\phi}$ is one-to-one, ϕ is onto, and ϕ restricted to the range of $\bar{\phi}$ is one-to-one. Similarly, the mapping ψ given by $\psi(G) = \pi_+(G)$ is a one-to-one correspondence between some of the normal subgroups G of $E_n(X)$ that include $\Delta_n(X)$ but lie in $E_n^+(X)$ and all of the normal subgroups of Π_n that lie in Π_n^+. For if Γ is a normal subgroup of Π_n that lies in Π_n^+, then we put $\bar{\psi}(\Gamma) = \Gamma(X)$. Moreover, if G is a normal subgroup of $E_n(X)$ that lies in $E_n^+(X)$, then $\pi_-(G) = \pi_+(G)$. For both $\pi_-(G)$ and $\pi_+(G)$ are normal subgroups of Π_n^+, and Corollary 2 of Theorem 3 says that they are both normal in Π_n. But (7) and (8) tell us that if R lies in Π_n^-, then

$$R^{-1}\pi_-(G)R = \pi_+(G) .$$

Hence, $\pi_-(G) = \pi_+(G)$.

To see that there are other normal subgroups G of $E_n(X)$ that include $\Delta_n(X)$ besides the ones mentioned above, we choose a member g of $E_n^+(X)$ where $\pi_-(g) = \pi_+(g) = P$, and P lies in $\Pi_n^+ - \Delta_1^{*n}$. If we let Γ be the group generated by P and Δ_1^{*n}, then Γ is evidently a normal subgroup of Π_n that lies in Π_n^+. The normal subgroup G of $E_n(X)$ generated by g and $\Delta_1^{*n}(X)$ evidently lies in $E_n^+(X)$ and satisfies $\pi_-(G) = \pi_+(G) = \Gamma$. But for every member h of G we have the same leading coefficient of $\pi_-(h)$ and $\pi_+(h)$. Hence, G is much smaller than $\Gamma(X)$.

LEMMA 9. *Every two involutions in $E(X)$ different from e are conjugate in $S_-E(X)$ and also in $S_+E(X)$.*

Proof. If h_i $(i = 1, 2)$ is such an involution with fixed point p_i and satisfying (1) at $p = p_i$ with $P = P_i$, then $h_i^2 = e$, $P_i^2 = P_0$, and P_1, P_2 are involutions in Π different from P_0. From Lemma 8 we know that

$$P_2 = Q^{-1}P_1Q = P_2^{-1}Q^{-1}P_1QP_2 ,$$

where either Q or QP_2 lies in Π^+. Thus P_1 and P_2 are conjugate in Π^+, and Lemma 6 tells us that h_1 and h_2 are conjugate in $S_-E(X)$ as well as in $S_+E(X)$.

THEOREM 4. *The commutator subgroup of $E(X)$ is $E^+(X)$.*

Proof. Let h be any member of $E^+(X)$ and i be any involution of $E(X)$ different from e. Our result will follow if we can show that some member h_0 of the commutator subgroup C of $E(X)$ agrees with h in some neighborhood U_{+1} of $p = +1$ and agrees with e in some neighborhood U_{-1} of $p = -1$. For we have $h = iki$, where k is in $E^+(X)$, and some k_0 in C agrees with k in some V_{+1} and with e in some V_{-1}. Now $k_1 = ik_0i$ agrees with $iki = h$ in $i(V_{+1}) = W_{-1}$ and with $iei = e$ in $i(V_{-1}) = W_{+1}$. Thus $h_1 = h_0^{-1}k_1^{-1}h$ lies in $SE(X)$. Since C is normal in $E(X)$, k_1 must lie in C, and Theorem 1 says that h_1 lies in C. Therefore, h lies in C and $C = E^+(X)$. Now our task remains to construct h_0. We let p_0 be the fixed point of i and put

$$f(x) = \begin{cases} h(x) \text{ for } x \text{ in some } V_{+1} , \\ ih^{-1}i(x) \text{ for } x \text{ in some } V_{-1} , \end{cases}$$

and define f in the rest of X so as to make it a member of $E^+(X)$ and satisfy $f(p_0) = p_0$. Evidently i interchanges the intervals $[-1, p_0]$ and $[p_0, +1]$ while f maps each onto itself. Thus

$$ifif(x) = \begin{cases} i(ih^{-1}i)ih(x) = x \text{ for } x \text{ in some } W_{+1} , \\ ihi(ih^{-1}i)(x) = x \text{ for } x \text{ in some } W_{-1} , \end{cases}$$

and if agrees with some involution j of $E(X)$ different from e in $W_{+1} \cup W_{-1}$. Now Lemma 9 says that $j = gig^{-1}$ for some g in $E(X)$, and we may assume that g is in $E^-(X)$, for if not, then we would take jg and write $j = (jg)i(jg)^{-1}$. Finally, we choose some g_0 in $E^-(X)$ which agrees with g on some Z_{-1} and with i on some Z_{+1}. If we put

$$h_0 = (ig_0ig_0^{-1})^{-1}(igig^{-1}) ,$$

then h_0 evidently lies in C. For x in some U_{+1} we have

$$ig_0ig_0^{-1}(x) = i^4(x) = x ,$$

$$igig^{-1}(x) = ij(x) = iif(x) = f(x) = h(x) ,$$

and $h_0(x) = h(x)$. For x in some U_{-1} we have

$$ig_0ig_0^{-1}(x) = igig^{-1}(x) ,$$

and $h_0(x) = x$. This completes the proof.

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