PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 1, 1974

ISOMORPHISMS OF GROUP EXTENSIONS

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To my parents

Let $0 \to G \to E \to \Pi \to 1$ and $0 \to G \to E' \to \Pi \to 1$ be two crossed product extensions given by the crossed product groups $E = [G, \varphi, f, \Pi]$ and $E' = [G, \varphi', f', \Pi]$ respectively. A homomorphism $\Gamma: E \to E'$ is *stabilizing* if the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \Pi & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \Gamma} & & \| & & \\ 0 & \longrightarrow & G & \longrightarrow & E' & \longrightarrow & \Pi & \longrightarrow & 1 \end{array}$$

commutes. In this paper, a necessary and sufficient condition for the existence of a stabilizing homomorphism (hence isomorphism) between any two crossed product extensions is obtained.

The result is applied to obtain a necessary and sufficient condition for the existence of an automorphism $\Phi: E \to E$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \Pi & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \sigma} & & \\ 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \Pi & \longrightarrow & 1 \end{array}$$

commutative, given $(\sigma, \tau) \in \operatorname{Aut} \Pi \times \operatorname{Aut} G$.

NOTATION. In general, we use the notation in [3]. Throughout the paper, G and Π denote two fixed groups. G will be written in additive notation and Π in multiplicative notation. $\operatorname{Aut} G$, $\operatorname{Out} G$, and ZG are the automorphism group, the outer automorphism group, and the center of G, respectively. For any element $a \in G$, $\mu(a)$ denotes the inner automorphism $\mu(a)(g) = a + g - a$ given by conjugation with a. When X is a group the natural image of an element $x \in X$ in a quotient group of X is denoted $\bar{x}$. When φ is a map, $\bar{\varphi}$ denotes the map $\bar{\varphi}(x) = \overline{\varphi(x)}$.

Given groups G, Π, and functions $\varphi: \Pi \to \operatorname{Aut} G$, $f: \Pi \times \Pi \to G$ satisfying

$$(1) \qquad \varphi(x)f(y, z) + f(x, yz) = f(x, y) + f(xy, z) ,$$

$$(2) \qquad \varphi(x)\varphi(y) = \mu[f(x, y)]\varphi(xy) ,$$

and the normalization conditions $\varphi(1) = 1$, $f(x, 1) = 0 = f(1, y)$, the set $G \times \Pi$ under the sum defined by

$$(3) \qquad (g, x) + (h, y) = (g + \varphi(x)h + f(x, y), xy)$$

is a group. The group so constructed is called a *crossed product group*, and is denoted $[G, \varphi, f, \Pi]$, or simply E. With the homomorphism $G \to E$ defined by $g \mapsto (g, 1)$, and $E \to \Pi$ defined by $(g, x) \mapsto x$, we have an extension of G by Π

$$0 \longrightarrow G \longrightarrow E \longrightarrow \Pi \longrightarrow 1 .$$

The extension is called a *crossed product extension*.

Results. Let $E = [G, \varphi, f, \Pi]$ and $E' = [G, \varphi', f', \Pi]$ be two crossed product groups. Define a stabilizing homomorphism $\Gamma: E \to E'$ as in the abstract. Notice that, by the "5 lemma" for groups, a stabilizing homomorphism is an isomorphism. Clearly, a homomorphism $\Gamma: E \to E'$ is stabilizing if and only if Γ is of the form

$$(4) \qquad \Gamma(g, x) = (g + \gamma(x), x) ,$$

and

$$(5) \quad \varphi(x)g + f(x, y) + \gamma(xy) = \gamma(x) + \varphi'(x)[g + \gamma(y)] + f'(x, y) .$$

Because of the normalization conditions, $\gamma(1) = 0$. Setting $y = 1$ in (5), we obtain

$$(6) \qquad \varphi(x)g + \gamma(x) = \gamma(x) + \varphi'(x)g .$$

Setting $g = 0$ in (5), we obtain

$$(7) \qquad f(x, y) + \gamma(xy) = \gamma(x) + \varphi'(x)\gamma(y) + f'(x, y) .$$

Conversely, (5) can immediately be derived from (6) and (7). Summarizing, we have

LEMMA. *If $E = [G, \varphi, f, \Pi]$ and $E' = [G, \varphi', f', \Pi]$ are two crossed product groups, then $\Gamma: E \to E'$ is a stabilizing isomorphism if and only if Γ is of the form* (4), *where the map $\gamma: \Pi \to G$ satisfies* (6) *and* (7).

In particular, when $\varphi = \varphi'$, and $f = f'$, we see that by (6), $\gamma(x) \in ZG$, and by (7), $\gamma \in Z^1(\Pi, ZG)$. $Z^1(\Pi, ZG)$ is the group of normalized 1-cocycles, and the Π-module structure on ZG is given by φ.

COROLLARY. *If $E = [G, \varphi, f, \Pi]$ is a crossed product group, then $\Gamma: E \to E$ is a stabilizing automorphism if and only if Γ is of the form* (4), *where $\gamma \in Z^1(\Pi, ZG)$.*

We remark that both the lemma and the corollary are well-known. See for instance, [5, p. 127], [2, 17.1 Satz, p. 119], [4].

It is obvious from (6) that, as homomorphisms from Π to Out G,

$$\bar{\varphi} = \bar{\varphi}' . \tag{8}$$

Trivially, (7) implies that

$$k(x, y) = -f(x, y) + \gamma(x) + \varphi'(x)\gamma(y) + f'(x, y) - \gamma(xy) \tag{9}$$

is equal to 0.

Conversely, given crossed product groups $E = [G, \varphi, f, \Pi]$ and $E = [G, \varphi', f', \Pi]$, we can ascertain the existence of a stabilizing isomorphism from E to E' by the following procedure. First, we decide whether condition (8) is fulfilled. If not, the question is settled. If (8) is satisfied, then there is a function $\gamma: \Pi \rightarrow G$ such that (6) holds, and ZG acquires a well-defined Π-module structure with operators $xc = \varphi(x)c(=\varphi'(x)c)$, for $c \in ZG$. We set $\gamma(1) = 0$. It is now meaningful to speak of the group $Z^2(\Pi, ZG)$ (resp., $B^2(\Pi, ZG)$) of the 2-dimensional normalized cocycles (resp., coboundaries) of Π with values in ZG. Define $k(x, y)$ by (9). We claim that $k(x, y) \in Z^2(\Pi, ZG)$. Trivially, $k(x, 1) = 0 = k(1, y)$. To see $k(x, y) \in ZG$, we merely observe that conjugating $\varphi'(xy)g$ with $\gamma(xy)$ and conjugating $\varphi'(xy)g$ with $-f(x, y) + \gamma(x) + \varphi'(x)\gamma(y) + f'(x, y)$ give the same result $\varphi(xy)g$. $k(x, y)$, being the difference of these two elements, is therefore in ZG. To verify the identity

$$xk(y, z) - k(xy, z) + k(x, yz) - k(x, y) = 0 ,$$

we observe that

$$\begin{aligned} k(x, yz) &- k(xy, z) \\ &= -f(x, yz) + \gamma(x) + \varphi'(x)\gamma(yz) + f'(x, yz) - f'(xy, z) \\ &\quad - \varphi'(xy)\gamma(z) - \gamma(xy) + f(xy, z) \\ &= -f(x, yz) + \gamma(x) + \varphi'(x)[\gamma(yz) - f'(y, z) - \varphi'(y)\gamma(z)] \\ &\quad + f'(x, y) - \gamma(xy) + f(xy, z) \\ &= -f(x, yz) + \gamma(x) - \varphi'(x)k(y, z) - \varphi'(x)f(y, z) - \gamma(x) \\ &\quad + f(x, y) + k(x, y) + f(xy, z) \\ &= k(x, y) - xk(y, z). \end{aligned}$$

We made use of the identities (1), (2), (9), (6), and (1), in that order.

Finally, if $k(x, y) \in B^2(\Pi, ZG)$, then $k(x, y) = x\beta(y) - \beta(xy) + \beta(x)$. The function $\gamma': \Pi \rightarrow G$ defined by $\gamma'(x) = \gamma(x) - \beta(x)$ satisfies (6) and (7). Therefore, the map Γ defined by (4), using γ' instead of γ, is a stabilizing isomorphism. If $k(x, y) \notin B^2(\Pi, ZG)$, then there could not exist a stabilizing isomorphism $\Gamma: E \rightarrow E'$. For, if there were to exist such an isomorphism, the discussion leading up to the above lemma would show that $\Gamma(g, x) = (g + \gamma'(x), x)$, with γ' satisfying (6) and (7).

Since γ and γ' both satisfy (6), $\beta(x) = \gamma(x) - \gamma'(x) \in ZG$. By (7) we have $k(x, y) = x\beta(y) - \beta(xy) + \beta(x)$ showing $k(x, y) \in B^2(\Pi, ZG)$. This discussion also shows that $\overline{k(x, y)}$ in $H^2(\Pi, ZG)$ is independent of the choice of γ. These results may now be summarized as follows.

THEOREM 1. *Let $E = [G, \varphi, f, \Pi]$ and $E' = [G', \varphi', f', \Pi]$ be two crossed product groups. Then there exists a stabilizing isomorphism $\Gamma: E \to E'$, if and only if*

(A) $\bar{\varphi} = \bar{\varphi}'$, *and*

(B) $\bar{k} = 0$ *in* $H^2(\Pi, ZG)$,

where $k(x, y)$ is defined as above.

We note that Theorem 1 is well-known (and is easily seen to be true) in the case where $\varphi = \varphi'$ [3, Theorem 8.8, p. 128, Lemma 8.2].

An application. Let $0 \to G \to E \to \Pi \to 1$ be a group extension. Call an automorphism of E taking G onto G an automorphism *over* G. Clearly, any automorphism of E over G induces automorphisms τ on G and σ on Π. It is easy to see that, in general, not every pair $(\sigma, \tau) \in \operatorname{Aut} \Pi \times \operatorname{Aut} G$ can be so induced by an automorphism of E over G. In [4], Charles Wells defined an exact sequence which gives a necessary and sufficient condition for a pair $(\sigma, \tau) \in \operatorname{Aut} \Pi \times \operatorname{Aut} G$ to be inducible by an automorphism of E over G. We now apply Theorem 1 to prove a similar result. We hope our method will also help clarify the nature of the map $C \to H^2_\alpha(\Pi, ZG)$ as defined in [4].

Let $0 \to G \to E \to \Pi \to 1$ be a group extension. We may (and do) assume that E is of the form $E = [G, \varphi, f, \Pi]$ with homomorphisms $G \to E$, $E \to \Pi$ of the form as defined in the definition of a crossed product extension at the beginning of this paper. We say that a pair $(\sigma, \tau) \in \operatorname{Aut} \Pi \times \operatorname{Aut} G$ is *inducible* if there exists an automorphism $\Phi: E \to E$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \Pi & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \Pi & \longrightarrow & 1 \end{array} \tag{10}$$

is commutative.

For $(\sigma, \tau) \in \operatorname{Aut} \Pi \times \operatorname{Aut} G$, let $\varphi_\sigma(x) = \varphi(\sigma x)$, $f_\sigma(x, y) = f(\sigma x, \sigma y)$; $\varphi_\tau(x) = \tau\varphi(x)\tau^{-1}$, $f_\tau(x, y) = \tau f(x, y)$. If $\bar{\varphi}_\sigma = \bar{\varphi}_\tau$, there exists a map $\gamma: \Pi \to G$ such that $\varphi_\tau(x) + \gamma(x) = \gamma(x) + \varphi_\sigma(x)$. Choose γ so that $\gamma(1) = 0$. In this case, define

$$k_{\sigma,\tau}(x, y) = -f_\tau(x, y) + \gamma(x) + \varphi_\sigma(x)\gamma(y) + f_\sigma(x, y) - \gamma(xy) . \tag{11}$$

THEOREM 2. *The pair $(\sigma, \tau) \in \operatorname{Aut} \Pi \times \operatorname{Aut} G$ is inducible if and* *only if*

(A) $\bar{\varphi}_\sigma = \bar{\varphi}_\tau$, *and*

(B) $\bar{k}_{\sigma,\tau} = 0$ *in* $H^2(\Pi, ZG)$,

where $k_{\sigma,\tau}(x, y)$ *is defined as in* (11), *and the* Π*-module structure on* ZG *is induced by the homomorphism* $\bar{\varphi}_\sigma = \bar{\varphi}_\tau$.

Proof. Let $0 \to G \to E \to \Pi \to 1$ be a group extension. Set $E = [G, \varphi, f, \Pi]$. Let $\varphi_\sigma, f_\sigma, \varphi_\tau, f_\tau, k_{\sigma,\tau}$ be as defined in the paragraph preceding Theorem 2. Let $E_\tau = [G, \varphi_\tau, f_\tau, \Pi]$, $E_\sigma = [G, \varphi_\sigma, f_\sigma, \Pi]$. Define $T: E \to E_\tau$ by $T(g, x) = (\tau g, x)$, and $\Sigma: E_\sigma \to E$ by $\Sigma\,(g, x) = (g, \sigma x)$. It is a straightforward matter to check that T and Σ are both group homomorphisms and that the following two diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \Pi & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & G & \longrightarrow & E_\tau & \longrightarrow & \Pi & \longrightarrow & 1 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & E_\sigma & \longrightarrow & \Pi & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \sigma} & & \\ 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \Pi & \longrightarrow & 1 \end{array}$$

commute.

If there is a stabilizing isomorphism $\Gamma: E_\tau \to E_{\sigma'}$ then (σ, τ) is clearly inducible.

Conversely, if (σ, τ) is inducible, then there exists an automorphism $\Phi: E \to E$ such that the diagram (10) is commutative. Such an automorphism is necessarily of the form $\Phi(g, x) = (\tau g + \gamma(x), \sigma x)$, where γ satisfies (6) and (7) with φ replaced by φ_τ and φ' replaced by φ_σ. Therefore, $\Gamma = \Sigma^{-1}\Phi T^{-1}$ is a stabilizing isomorphism from E_τ to E_σ. By Theorem 1, (σ, τ) is inducible if and only if (A) and (B) are satisfied.

Condition (A) of Theorem 2 can be stated more explicitly as follows: For any $x \in \Pi$, $\bar{\tau}\overline{\varphi(x)}\bar{\tau}^{-1} = \overline{\varphi(\sigma x)}$.

As direct consequences of Theorem 2, we have

COROLLARY 1. *If* $\operatorname{Out} G = 1$ *and* $H^2(\Pi, ZG) = 0$, *then every pair* $(\sigma, \tau) \in \operatorname{Aut} \Pi \times \operatorname{Aut} G$ *is inducible.*

COROLLARY 2. *If* $\operatorname{Out} G = 1$ *and* $ZG = 0$, *then for any group* E *such that* $G \lhd E$ *and for each* $\tau \in \operatorname{Aut} G$, *there exists* $\Phi \in \operatorname{Aut} E$ *such that restriction of* Φ *to* G *is equal to* τ.

The second corollary also follows directly from [1, Theorem 1].

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Received September 28, 1972.

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