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THE HANF NUMBER OF OMITTING COMPLETE TYPES

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It is proved in this paper that the Hanf number m^C of omitting complete types by models of complete countable theories is the same as that of omitting not necessarily complete type by models of a countable theory.

Introduction. Morley [3] proved that if L is a countable first-order language, T a theory in L, p is a type in L, and T has models omitting p in every cardinality $\lambda < \beth_{\omega_1}$, *then* T has models omitting p in every infinite cardinality. He also proved that the bound $\beth_{\omega_1}$ cannot be improved, in other words the Hanf number is $\beth_{\omega_1}$. He asked what is the Hanf number m^C when we restrict ourselves to complete T and p. Clearly $m^C \leqq \beth_{\omega_1}$. Independently several people noticed that $m^C \geqq \beth_\omega$ and J. Knight noticed that $m^C > \beth_\omega$.

Malitz [2] proved that the Hanf number for complete $L_{\infty,\omega}$-theories with one axiom $\psi \in L_{\omega_1,\omega}$ is $\beth_{\omega_1}$. We shall prove

THEOREM 1. $m^C = \beth_{\omega_1}$.

NOTATION. Natural numbers will be i, j, k, l, m, n, ordinals α, β, δ; cardinals λ, μ. $|A|$ is the cardinality of A, $\beth_\alpha = \sum_{\beta<\alpha} 2^{\beth_\beta} + \aleph_0$.

M will be a model with universe $|M|$, with corresponding countable first-order language $L(M)$. For a predicate $R \in L(M)$, the corresponding relation is R^M or $R(M)$, and if there is no danger of confusion just R. Every M will have the one place predicate P and individual constants c_n such that $P = P^M = \{c_n: n < \omega\}$, $n \neq m \Rightarrow c_n \neq c_m$ (we shall not distinguish between the individual constants and their interpretation). A type p in L is a set of formulas $\varphi(x_0) \in L$; p is complete for T in L if it is consistent and for no $\varphi(x_0) \in L$ both $T \cup p \cup \{\varphi(x_0)\}$ and $T \cup p \cup \{\neg\varphi(x_0)\}$ are consistent.

An element $b \in |M|$ realizes p if $\varphi(x_0) \in p$ implies $M \models \varphi[b]$ ($\models$ -satisfaction sign), and M realizes p if some $a \in |M|$ realizes it. A complete theory in L is a maximal consistent set of sentences of L. For every permutation θ of P, model M, and sublanguage L of $L(M)$ we define an Ehrenfeucht game $EG(M, L, \theta)$ between player I and II with ω moves as follows: in the nth move first player I chooses $i \in \{0, 1\}$ and $a_n^i \in |M|$ and secondly player II chooses $a_n^{1-i} \in |M|$. Player II wins if the extension θ^* of θ defined by $\theta^*(a_n^0) = a_n^1$ preserves all atomic formulas of L. That is if $R(x_1, \cdots, x_n)$ is an atomic formula in L, $\theta^*(b_i)$ is defined then $M \models R[b_1, \cdots, b_n]$ iff $M \models R[\theta^*(b_1), \cdots, \theta^*(b_n)]$.

REMARK. So if I chooses $a_n^i \in P$, II should choose $a_n^1 = \theta(a_n^0)$.

Define $\Gamma(n_0) = \{\theta\colon \theta$ a permutation of P, $n < n_0 \Rightarrow \theta(c_n) = c_n$ and only for finitely many n $\theta(c_n) \neq c_n\}$.

$M \mid L$ is the reduct of M to the language $L \subseteq L(M)$, that is $M \mid L$ is M without the relations R^M, $R \in L(M)$, $R \notin L$, and constants $c_n \in L(M)$, $c_n \notin L$.

THEOREM 2. *For every ordinal $\alpha < \omega_1$ there is a countable first-order language L_α a complete theory T_α in L, such that*

(i) $p = \{P(x_0)\} \cup \{x_0 \neq c_n\colon n < \omega\}$ *is a complete type for* T_α.

(ii) T_α *has a model of cardinality* $\beth_\alpha$ *omitting* p.

(iii) T_α *has no model of cardinality* $> \beth_\alpha$ *omitting* p.

REMARK. Clearly Theorem 2 implies Theorem 1.

Proof. We shall define by induction on $\alpha < \omega_1$ models M_α such that

(1) $\|M_\alpha\|$, the cardinality of $|M_\alpha|$, is, $\beth_\alpha$, and of course $P = P(M_\alpha) = \{c_n\colon n < \omega\}$ and except for the c_n's $L(M_\alpha)$ has only predicates.

(2) There is no model elementarily equivalent to M_α of cardinality $> \beth_\alpha$ which omits p.

(3) If $(\exists\beta)(\alpha = \beta + 2)$ then $Q_\alpha \in L(M_\alpha)$ and $|Q_\alpha(M_\alpha)| = \beth_\alpha$

(4) For every finite sublanguage L of $L(M_\alpha)$ there is $n_L = n(L) < \omega$, such that for every permutation $\theta \in \Gamma(n_L)$ player II has a winning strategy in $EG(M_\alpha, L, \theta)$.

(5) In (4) if $(\exists\beta)(\alpha = \beta + 2)$ then in the winning strategy of II, if I chooses $a_n^i \in Q_\alpha(M_\alpha)$ then II chooses $a_n^{1-i} = a_n^i$.

The induction will go as follows. First we define M_0, M_1, and M_2; later we define $M_{\alpha+1}$ by M_α when $(\exists\beta)(\alpha = \beta + 2)$; last for limit ordinal δ we define M_δ, $M_{\delta+1}$, $M_{\delta+2}$ by M_α $\alpha < \delta$.

But before defining the M_α's, let us show how this will finish the proof. We choose $L_\alpha = L(M_\alpha)$. T_α is the set of sentences of L_α that M_α satisfies. Clearly (ii), (iii) are satisfied. To prove (i) let $\varphi(x_0) \in L_\alpha$, so for some finite sublanguage L of L_α $\varphi(x_0) \in L$. By possibly interchanging $\varphi(x_0)$ and $\neg\varphi(x_0)$ we can assume $M_\alpha \models \varphi[c_{n(L)}]$. For $k \geqq n(L)$ let θ_k be the permutation of P interchanging $c_{n(L)}c_k$, and leaving the other elements fixed.

Clearly $\theta \in \Gamma(n_L)$, hence player II has a winning strategy in $EG(M_\alpha, L, \theta)$. By Ehrenfeucht [1] this implies $c_{n(L)}$ and $c_k = \theta(c_{n(L)})$ satisfy the same formulas of L. Hence $M_\alpha \models \varphi[c_{n(L)}] \equiv \varphi[c_k]$, hence $M_\alpha \models \varphi[c_k]$. As this holds for any $k \geqq n(L)$ $M_\alpha \models (\forall x)[P(x) \land \bigwedge_{i<n(L)} x \neq c_i \rightarrow \varphi(x)]$. Hence $T_\alpha \cup p \cup \{\neg\varphi(x_0)\}$ is inconsistent. So p is complete

(for T_α, L_α) and we finish.

So let us define

Case I. $\alpha = 0, 1, 2$

(A) Let us define M_0:

$|M_0| = P$, and its only predicate is P (and of course the individual constants c_n, which we will not mention in later cases). Clearly (1), (2) are immediate. (3) and (5) are satisfied vacuously. As for (4), let $n_L = \max\{n+1 : c_n \in L\}$. Clearly θ is an automorphism of $M_0 \mid L$ (the reduct of M_0 to L).

So player II will play by the automorphism: if I chooses a_n^0, II will choose $a_n^1 = \theta(a_n^0)$, and if I chooses a_n^1, II will choose $a_n^0 = \theta^{-1}(a_n^1)$.

(B) $|M_1| = |M_0| \cup P_1(M_1)$, where $P_1(M_1) = \mathscr{P}(|M_0|)$, where $\mathscr{P}(A) =$ the power set of $A = \{B : B \subseteqq A\}$.

The predicates of M_1 are those of M_0, P_1 and ε_1

$$\varepsilon_1(M_1) = \{\langle c, A\rangle : c \in |M_0|, A \in P_1, c \in A\} .$$

As in (A) it is clear that M_1 satisfies the induction conditions, as if $\theta \in \Gamma(n_L)$ $L \subseteqq L(M_1)$, L finite, then θ can be extended to an automorphism of M_1 by

$$\theta(A) \stackrel{\text{def}}{=} \{\theta(c) : c \in A\} .$$

(C) Let us define an equivalence relation E_1 on $P_1(M_1)$: AE_1B iff for some $\theta \in \Gamma(0)$ $A = \theta(B)[= \{\theta(c) : c \in B\}]$.

This is an equivalence relation, as $\Gamma(0)$ is a group of permutations, and as $|\Gamma(0)| = \aleph_0$, each equivalence class is countable. Define

$$\begin{aligned} |M_2| &= |M_1| \cup Q_2(M_2) \\ Q_2(M_2) &= \{S : S \subseteqq P_1(M_1), A, B \in P_1, AE_1B \Rightarrow A \in S \longleftrightarrow B \in S\} \\ \varepsilon_2(M_2) &= \{\langle A, S\rangle : A \in P_1, S \in Q_2, A \in S\} . \end{aligned}$$

The relations of M_2 will be the relations of M_1, and Q_2, ε_2. By the definition of Q_2, each $\theta \in \Gamma(n_L)$ [L a finite sublanguage of $L(M_2)$] can be extended to an automorphism θ^* of $M_2 \mid L$, which is the identity over Q_2. As before (1), (2), (4) hold, and as θ^* is the identity over Q_2, also (5) holds. As for (3) each E_1-equivalence class is countable, and $|P_1(M_1)| = 2^{|P|} = 2^{\aleph_0}$, the number of E_1-equivalence classes is $\beth_1$, so $|Q_2| = 2^{\beth_1} = \beth_2$.

Case II. We define $M_{\alpha+1}$, where M_α is defined, $(\exists\beta)(\alpha = \beta + 2)$. Let

$$|M_{\alpha+1}| = |M_\alpha| \cup \mathscr{P}(Q_\alpha(M_\alpha)) .$$

The relations of $M_{\alpha+1}$ will be those of M_α and in addition $Q_{\alpha+1}(M_{\alpha+1}) = \mathscr{P}(Q_\alpha(M_\alpha))$

$$\varepsilon_{\alpha+1}(M_{\alpha+1}) = \{\langle a, A\rangle : a \in Q_\alpha(M_\alpha), A \in Q_{\alpha+1}(M_{\alpha+1}), a \in A\} .$$

Clearly Conditions (1), (2), (3) are satisfied. As for (4), (5) the winning strategy of player II in $EG(M_{\alpha+1}, L, \theta)[\theta \in \Gamma(n_L)]$ will be as follows: when I chooses elements in $|M_\alpha|$ he will pretend all the game is in $|M_\alpha|$ and play accordingly; and if player I chooses $a_n^i \in Q_{\alpha+1}(M_{\alpha+1})$, then player II will choose $a_n^{1-i} = a_n^i$. As M_α satisfies (5) this is a winning strategy, and trivially it satisfies (5).

Case III. δ a limit ordinal, M_α is defined for $\alpha < \delta$; and we shall define M_δ, $M_{\delta+1}$, $M_{\delta+2}$.

PART A. By changing, when necessary, names of elements and relations, we can assume that for $\alpha < \beta < \delta$,

$$|M_\alpha| \cap |M_\beta| = P, \quad \text{and} \quad L(M_\alpha) \cap L(M_\beta) = \{P, c_n : n < \omega\} ,$$

but that if $(\exists\beta)(\alpha = \beta + 2)$ then still $Q_\alpha \in L(M_\alpha)$. Choose an increasing sequence of ordinals α_n $n < \omega$, $\delta = \bigcup_{n<\omega} \alpha_n$ and $(\exists\beta)(\alpha_n = \beta + 2)$. Define M_δ as follows

$$|M_\delta| = \bigcup_{n<\omega} M_{\alpha_n} .$$

The relations of M_δ will be those of M_{α_n} for each $n < \omega$ and $R_\delta^{M_\delta}$

$$R_\delta^{M_\delta} = \{\langle c, a\rangle : c = c_n \in P, a \in (M_{\alpha_n} - P)\} .$$

It is easy to check that Conditions (1), (2) are satisfied. Conditions (3) and (5) are vacuous. So let us prove Condition (4) holds. Let L be a finite sublanguage of $L(M_\delta)$; then $L \subseteqq \bigcup_{j<n_0} L_j \cup \{R\}$, where $L_j = L \cap L(M_{\alpha_j})$ is a finite sublanguage of $L(M_{\alpha_j})$. Define $n_L = \max[\{n_{L_j} : j < n_0\} \cup \{n_0\}]$. Let $\theta \in \Gamma(n_L)$. We shall describe now the winning strategy of player II in $EG(M_\delta, L, \theta)$. When player I will choose $i \in \{0, 1\}$, $a_n^i \in M_{\alpha_j}$, $j < n_0$, player II will pretend all the game is in the model M_{α_j}, and so play his winning strategy for $EG(M_{\alpha_j}, L \cap L(M_{\alpha_j}), \theta)$. If player I chooses $i \in \{0, 1\}$, $a_n^i \in M_{\alpha_j}$ $j \geqq n_0$ then player II will choose $a_n^{1-i} \in M_{\alpha_k}$ [where $i = 0 \Rightarrow k = \theta(j)$, $i = 1 \Rightarrow j = \theta(k)$] such that for any $m < n$ $a_m^i = a_n^i \Leftrightarrow a_m^{1-i} = a_n^{1-i}$.

Note that for $j \geqq n_0$, in $M_\delta \mid L$, every permutation of elements of M_{α_j} is an automorphism, as the only relation an $a \in |M_{\alpha_j}|$ satisfies is $R_\delta[c_j, a]$.

PART B. Here we define $M_{\delta+1}$. Let $A^* = \bigcup_{n<\omega} Q_{\alpha_n}(M_{\alpha_n})$, and $|M_{\delta+1}| = |M_\delta| \cup \mathscr{P}(A^*)$.

The relations of $M_{\delta+1}$ will be those of M_δ, and in addition

$$P_\delta(M_{\delta+1}) = |M_\delta|, \; P_{\delta+1}(M_{\delta+1}) = \mathscr{P}(A^*)$$
$$\varepsilon_{\delta+1}(M_{\delta+1}) = \{\langle b, B\rangle : b \in A^*, B \in \mathscr{P}(A^*), b \in B\} .$$

It is easy to see that Conditions (1), (2) are satisfied, and (3), (5) are vacuous. So let us prove (4) — let L be a finite sublanguage of $L(M_{\delta+1})$. So

$$L \subseteqq \bigcup_{i<n_0} L_i \cup \{R_\delta, P_\delta, P_{\delta+1}, \varepsilon_{\delta+1}\}, \; L_i = L \cap L(M_{\alpha_i}) .$$

Define again

$$n_L = \max[\{n_{L_j} : j < n_0\} \cup \{n_0\}] .$$

Let $\theta \in \Gamma(n_L)$ and we should describe player II's winning strategy in $EG(M_{\delta+1}, L, \theta)$. When player I chooses an element in M_{α_j} $j < n_0$, player II will ignore all elements chosen outside M_{α_j}, and play by his winning strategy in $EG(M_{\alpha_j}, L_j, \theta)$. In the other cases player II will play so that the following conditions are satisfied for every n

P (1) $a_n^0 \in P_{\delta+1}(M_{\delta+1}) \Leftrightarrow a_n^1 \in P_{\delta+1}(M_{\delta+1})$
P (2) if $c_j = \theta(c_k)$, then $a_n^0 \in |M_{\alpha_k}| \Leftrightarrow a_n^1 \in |M_{\alpha_j}|$
P (3) if $m < n$ then $a_m^0 = a_n^0 \Leftrightarrow a_m^1 = a_n^1$
P (4) if $m, l \leqq n$ and $a_m^0 \in A^*$, $a_l^0 \in P_{\delta+1}$ then $a_m^0 \in a_l^0 \Leftrightarrow a_m^1 \in a_l^1$
P (5) if $a_m^0 \in P_{\delta+1}$, $l < \omega$, $c_l = \theta(c_l)$ then $a_m^0 \cap Q_{\alpha_l}(M_{\alpha_l}) = a_m^1 \cap Q_{\alpha_l}(M_{\alpha_l})$
P (6) if $c_j = \theta(c_k) j \neq k < \omega$, *then* $\langle a_m^0 : m \leqq n, a_m^0 \in P_{\delta+1}\rangle$ and $\langle a_m^1 : m \leqq n, a_m^1 \in P_{\delta+1}\rangle$ genarate corresponding finite Boolean algebras of subsets of $Q_{\alpha_k}(M_{\alpha_k})$ and $Q_{\alpha_j}(M_{\alpha_j})$ correspondingly; then the corresponding atoms in those algebras are both infinite, or have the same power.

It is easy to see that this can by done, and it is a winning strategy.

PART C. Here we define $M_{\delta+2}$.

Define equivalence relations $E_{\delta+1}, E_{\delta+1}^n$ on $P_{\delta+1}(M_{\delta+1})$: if $A, B \in P_{\delta+1}(M_{\delta+1})$, then $A, B \subseteqq A^* = \bigcup_{n<\omega} Q_{\alpha_n}(M_{\alpha_n})$; define $AE_{\delta+1}^n B$ iff $A \cap [\bigcup_{\omega>m>n} Q_{\alpha_m}(M_{\alpha_m})] = B \cap [\bigcup_{\omega>m>n} Q_{\alpha_m}(M_{\alpha_m})]$; $AE_{\delta+1}B$ iff for some n $AE_{\delta+1}^n B$.

Clearly each $E_{\delta+1}^n$ is an equivalence relation, $E_{\delta+1}^n$ refines $E_{\delta+1}^{n+1}$, hence $E_{\delta+1}$ is an equivalence relation.

It is clear that

$$|P_{\delta+1}(M_{\delta+1})| = \beth_{\delta+1}$$

but for every $n < \omega$, $A \in P_{\delta+1}(M_{\delta+1})$

$$| \{B: B \in P_{\delta+1}(M_{\delta+1}),\ BE^n_{\delta+1}A\} | \leqq | \mathscr{P}(\bigcup_{m \leqq n} Q_{\alpha_m}(M_{\alpha_m})) |$$
$$= 2^{\beth_{\alpha_n}} = \beth_{\alpha_n+1} \leqq \beth_\delta$$

hence

$$| \{B: B \in P_{\delta+1}(M_{\delta+1}),\ BE_{\delta+1}A\} | \leqq \sum_{n<\omega} \beth_\delta = \beth_\delta \,.$$

So each $E_{\delta+1}$ – equivalence class has cardinality $\leqq \beth_\delta$, hence there are $\beth_{\delta+1}$ $E_{\delta+1}$-equivalence classes.

Define $M_{\delta+2}$:

$$| M_{\delta+2} | = | M_{\delta+1} | \cup Q_{\delta+2}(M_{\delta+2})$$

where

$$Q_{\delta+2}(M_{\delta+2}) = \{S: S \subseteqq P_{\delta+1}(M_{\delta+1}),\ A,\ B \in S,\ AE_{\delta+1}B \Longrightarrow A \in S \longleftrightarrow B \in S\} \,.$$

Clearly $| Q_{\delta+2}(M_{\delta+2}) | = \beth_{\delta+2}$.

The relations of $M_{\delta+2}$ will be those of $M_{\delta+1}$, and $Q_{\delta+2}$, and

$$\varepsilon_{\delta+2}(M_{\delta+2}) = \{\langle A, S\rangle: A \in P_{\delta+1}(M_{\delta+1}),\ S \in P_{\delta+2}(M_{\delta+2}),\ A \in S\} \,.$$

It is easy to prove all conditions are satisfied as in Case II, if we notice that by Condition P (5) if for any instance of any game $EG(M_{\delta+1}, L, \theta)[\theta \in \Gamma(n_L)]$ in which player II plays his strategy, if a^i_n, a^{1-i}_n are chosen for some n and they belong to $P_{\delta+1}(M_{\delta+1})$ then they are $E_{\delta+1}$-equivalent (as $\{n: \theta(c_n) \neq n\}$ is finite).

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