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THE NUMBER OF MULTINOMIAL COEFFICIENTS DIVISIBLE BY A FIXED POWER OF A PRIME

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In this paper some results of L. Carlitz and the writer concerning the number of binomial coefficients divisible by p^j but not by p^{j+1} are generalized to multinomial coefficients. In particular $\theta_j(k; n)$ is defined to be the number of multinomial coefficients $n!/n_1!, \cdots, n_k!$ divisible by exactly p^j, and formulas are found for $\theta_j(k; n)$ for certain values of j and n. Also the generating function technique used by Carlitz for binomial coefficients is generalized to multinomial coefficients.

1. Introduction. Let p be a fixed prime and let n and j be nonnegative integers. L. Carlitz [2], [3] has defined $\theta_j(n)$ as the number of binomial coefficients

$$\binom{n}{r} \qquad (r = 0, 1, \cdots, n)$$

divisible by exactly p^j and he has found formulas for $\theta_j(n)$ for certain values of j and n. In particular, if we write

$$n = a_0 + a_1 p + \cdots + a_s p^s \qquad (0 \leqq a_i < p) \tag{1.1}$$

then

$$\theta_0(n) = (a_0 + 1)(a_1 + 1) \cdots (a_s + 1)$$

$$\theta_1(n) = \sum_{i=0}^{s-1} (a_0 + 1) \cdots (a_{i-1} + 1)(p - a_i - 1)a_{i+1}(a_{i+2} + 1) \cdots (a_s + 1) .$$

The writer [5], [6] has also considered the problem of evaluating $\theta_j(n)$.

The purpose of this paper is to consider the analogous problem for multinomial coefficients and to generalize some of the formulas developed by Carlitz and the writer. Thus we define $\theta_j(k; n)$ as the number of multinomial coefficients

$$(n_1, \cdots, n_k) = \frac{n!}{n_1! \cdots n_k!} (n_1 + \cdots + n_k = n)$$

divisible by exactly p^j. In this definition the order of the terms $n_1, \cdots, n_k$ is important. We are distinguishing, for example, between (1, 2, 3) and (2, 1, 3). Clearly $\theta_j(2; n) = \theta_j(n)$.

In this paper we find formulas for $\theta_0(k; n)$, $\theta_1(k; n)$, and $\theta_2(k; n)$. We also show how the generating function method used by Carlitz can be generalized to multinomial coefficients, and we evaluate $\theta_j(k; n)$ for special values of j and n.

Throughout this paper we assume p is a fixed prime number and k is a fixed positive integer, $k > 1$.

2. Preliminaries. Let $E(n_1, \cdots, n_k)$ denote the largest value of w such that p^w divides $(n_1, \cdots, n_k)$. To determine $E(n_1, \cdots, n_k)$ we shall make use of an analogue [4] of Kummer's famous theorem for binomial coefficients:

LEMMA 2.1. *Let n have expansion* (1.1), *let* $n = n_1 + \cdots + n_k$ *and let*

$$n_i = a_{i,0} + a_{i,1}p + \cdots + 2_{i,s}p^s \qquad (0 \leqq a_{i,r} < p) \tag{2.1}$$

for $i = 1, \cdots, k$. *If*

$$\begin{aligned} a_{1,0} + \cdots + a_{k,0} &= \varepsilon_0 p + a_0 \\ \varepsilon_0 + a_{1,1} + \cdots + a_{k,1} &= \varepsilon_1 p + a_1 \\ &\cdots\cdots \\ \varepsilon_{s-1} + a_{1,s} + \cdots + a_{k,s} &= a_s \end{aligned}$$

where each $\varepsilon_i = 0, 1, \cdots,$ *or* $k - 1$, *then*

$$E(n_1, \cdots, n_k) = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{s-1} .$$

If n has expansion (1.1) and if $\nu(n)$ is the largest value of w such that p^w divides $n!$, then it is familiar [1, *p.* 55] that

$$\nu(n) = \frac{n - S(n)}{p - 1}$$

where $S(n) = a_0 + a_1 + \cdots + a_s$. Thus we have

LEMMA 2.2. *If* $n = n_1 + \cdots + n_k$ *then*

$$E(n_1, \cdots, n_k) = \frac{S(n_1) + \cdots + S(n_k) - S(n)}{p - 1} .$$

Furthermore, if $E_t(n_1, \cdots, n_k)$ *is the largest value of* w *such that* p^w *divides*

$$(n + t) \cdots (n + 1) \qquad (n_1, \cdots, n_k) ,$$

then

$$E_t(n_1, \cdots, n_k) = \frac{S(n_1) + \cdots + S(n_k) - S(n + t) + t}{p - 1} .$$

Compositions, or ordered partitions, are important in evaluating $\theta_j(k; n)$. We define a composition of a nonnegative integer u into r parts to be an ordered sequence of r nonnegative integers whose sum is u. This is more general than the usual definition of composition in that we allow 0 to be one or more of the parts. See [7, pp. 124–125] for example.

Throughout this paper we shall let $C(u)$ denote the number of compositions of u into exactly k parts, with no part larger than $p-1$. We define $C(u) = 0$ if $u < 0$.

LEMMA 2.3. *$C(u)$ is the coefficient of x^u in the expansion of*

$$(1 + x + x^2 + \cdots + x^{p-1})^k = \left[\sum_{i=0}^{\infty} \binom{k+i-1}{i} x^i\right](1 - x^p)^k .$$

It is clear from Lemma 2.3 that if $0 \leqq a < p$ and if $0 \leqq b$, then

$$C(a + bp) = \sum_{i=0}^{b} (-1)^i \binom{k}{i} \binom{k-1+a+(b-i)p}{k-1} . \tag{2.2}$$

In particular, for $0 \leqq a < p$,

$$C(a) = \binom{k-1+a}{k-1} ,$$

$$C(a+p) = \binom{k-1+a+p}{k-1} - k\binom{k-1+a}{k-1} ,$$

$$C(a+2p) = \binom{k-1+a+2p}{k-1} - k\binom{k-1+a+p}{k-1} + \binom{k}{2}\binom{k-1+a}{k-1} .$$

3. Evaluation of $\theta_0(k; n)$, $\theta_1(k; n)$, $\theta_2(k; n)$.

THEOREM 3.1. *If n has expansion* (1.1) *then*

$$\theta_0(k; n) = C(a_0)C(a_1) \cdots C(a_s) .$$

Proof. We use Lemma 2.1. If $E(n_1, \cdots, n_k) = 0$ then we must have

$$\sum_{i=1}^{k} a_{i,r} = a_r \qquad (r = 0, \cdots, s) .$$

For a given r, the total number of ways we can have this equality is equal to $C(a_r)$.

Note that by Lemma 2.3 we have

$$C(a_r) = \binom{a_r + k - 1}{k-1} \qquad (r = 0, \cdots, s) .$$

THEOREM 3.2. *If n has expansion* (1.1) *then*

$$\theta_1(k; n) = \sum_{i=0}^{s-1} C(a_0) \cdots C(a_{i-1})C(a_i + p)C(a_{i+1} - 1)C(a_{i+2}) \cdots C(a_s) .$$

Proof. Using Lemma 2.1, we see that if $E(n_1, \cdots, n_k) = 1$ then we must have exactly one $\varepsilon_i = 1$, $0 \leqq i < s$. So for some i we have

$$\begin{aligned} a_{1,i} + \cdots + a_{k,i} &= a_i + p , \\ a_{1,i+1} + \cdots + a_{k,i+1} &= a_{i+1} - 1 , \\ a_{1,r} + \cdots + a_{k,r} &= a_r \qquad (r \neq i, i + 1) . \end{aligned}$$

Clearly the total number of ways we can have these equalities is

$$C(a_0) \cdots C(a_{i-1})C(a_i + p)C(a_{i+1} - 1)C(a_{i+2}) \cdots C(a_s) .$$

To simplify the formula for $\theta_2(k; n)$ we introduce the following notation. Let

$$\begin{aligned} A_i &= \left[\prod_{t=0}^{s} C(a_t)\right] \Big/ \left[C(a_i)C(a_{i+1})C(a_{i+2})\right] , \\ B_i &= \left[\prod_{t=0}^{s} C(a_t)\right] \Big/ \left[C(a_i)C(a_{i+1})\right] , \\ H_{i,r} &= \left[\prod_{t=0}^{s} C(a_t)\right] \Big/ \left[C(a_i)C(a_{i+1})C(a_r)C(a_{r+1})\right] . \end{aligned}$$

THEOREM 3.3. *If n has expansion* (1.1) *then*

$$\begin{aligned} \theta_2(k; n) = &\sum_{i=0}^{s-2} C(p + a_i)C(p + a_{i+1} - 1)C(a_{i+2} - 1)A_i \\ &+ \sum_{i=0}^{s-1} C(2p + a_i)C(a_{i+1} - 2)B_i \\ &+ \sum_{r=i+2}^{s-1} \sum_{i=0}^{s-3} C(p + a_i)C(a_{i+1} - 1)C(p + a_r)C(a_{r+1} - 1)H_{i,r} . \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.2. We determine the number of ways we can have exactly two of the ε's equal to 1 or exactly one ε equal to 2, and all other ε's equal to 0.

For example, let $p = 5$, $k = 3$, and $n = 278 = 3 + 5^2 + 2 \cdot 5^3$. We have

$$\begin{aligned} \theta_0(3; 278) &= C(3)C(0)C(1)C(2) = 180; \\ \theta_1(3; 278) &= C(3)C(5)C(0)C(2) + C(3)C(0)C(6)C(1) = 1650 , \\ \theta_2(3; 278) &= C(8)C(4)C(0)C(2) + C(3)C(5)C(5)C(1) \\ &\quad + C(3)C(0)C(11)C(0) = 11,100 . \end{aligned}$$

In each example we have used (2.2) to evaluate $C(u)$.

4. Generating functions for $\theta_j(k; n)$. Let $\psi_{t,j}(k; n)$ denote the

number of products $(n+t)\cdots(n+1)(n_1, \cdots, n_k)$, $n_1+\cdots+n_k=n$, divisible by exactly p^j. Clearly

$$\psi_{t,j}(k; n) = \theta_{j-r}(k; n) \tag{4.1}$$

if p^r is the highest power of p dividing $(n+t)\cdots(n+1)$.
Also

$$\psi_{t,j}(k; n) = 0$$

if p^{j+1} divides $(n+t)\cdots(n+1)$. We introduce the following generating functions:

$$F_0(x, y) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \theta_j(k; n)x^n y^j ,$$

$$F_t(x, y) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \psi_{t,j}(k; n)x^n y^j \qquad (t > 0) .$$

Using an argument analogous to that of Carlitz [3], we obtain

$$F_0(x, y) = \sum_{t=0}^{m} y^t f_t(x) F_t(x^p, y) \tag{4.2}$$

where m is the integer such that

$$mp \leqq k(p-1) < (m+1)p \tag{4.3}$$

and

$$f_t(x) = \sum_{a=tp}^{tp+p-1} C(a)x^a \qquad (0 \leqq t < m) ,$$

$$f_m(x) = \sum_{a=mp}^{kp-k} C(a)x^a .$$

Comparing coefficients of $x^n y^j$ on both sides of (4.2), we have, for $0 \leqq a < p$,

$$\theta_j(k; a+bp) = C(a)\theta_j(k; b) + \sum_{t=1}^{m} C(a+tp)\psi_{t,j-t}(k; b-t) . \tag{4.4}$$

In (4.4) it is understood that $\psi_{t,j}(k; u) = 0$ if $u < 0$ and $\psi_{t,-1}(k; u) = 0$.
Also, for $t < p$,

$$F_t(x, y) = \sum_{r=1}^{h} y^r g_r(x) F_r(x^p, y)$$

where h is the integer such that

$$hp - t \leqq k(p-1) < (h+1)p - t , \tag{4.5}$$

and

$$g_0(x) = \sum_{a=0}^{p-t-1} C(a)x^a ,$$

$$g_r(x) = \sum_{a=rp-t}^{(r+1)p-t-1} C(a)x^a \qquad (r = 1, \cdots, h-1) ,$$

$$g_h(x) = \sum_{a=hp-t}^{kp-k} C(a)x^a .$$

Thus for $0 \leqq a < p - t$, $hp + a \leqq kp - k$, we have

$$\psi_{t,j}(k; a + bp) = C(a)\theta_j(k; b) + \sum_{r=1}^{h} C(a + rp)\psi_{r,j-r}(k; b - r) . \tag{4.6}$$

For $0 \leqq a < p - t$, $hp + a > kp - k$, we have

$$\psi_{t,j}(k; a + bp) = C(a)\theta_j(k; b) + \sum_{r=1}^{h-1} C(a + rp)\psi_{r,j-r}(k; b - r) . \tag{4.7}$$

For $p - t \leqq a < p$, we have

$$\psi_{t,j}(k; a + bp) = \sum_{r=1}^{h} C(a + (r-1)p)\psi_{r,j-r}(k; a - r + 1) . \tag{4.8}$$

Here again it is understood that $\psi_{r,j}(k; u) = 0$ if $u < 0$. We remark that in all of these formulas specific values for $C(u)$ can be found from formula (2.2).

Using (4.4) we can compute $\theta_j(k; n)$ for special values of n. By (4.4) and (4.1) we have, for $0 \leqq a < p$, $0 \leqq b < p$,

$$\begin{aligned} \theta_j(k; a + bp) &= C(a + jp)\theta_0(k; b - j) \\ &= C(a + jp)\ C(b - j) \quad \text{if} \quad j \leqq m , \\ &= 0 \quad \text{if} \quad j > m \end{aligned}$$

where m is defined by (4.3).

Also, if $0 \leqq a < p$,

$$\begin{aligned} \theta_j(k; a + p^2) &= C(a)C(1) \quad \text{if} \quad j = 0 , \\ &= C(a + (j-1)p)C(p - j + 1) \quad \text{if} \quad 1 \leqq j \leqq m + 1 , \\ &= 0 \quad \text{if} \quad j > m + 1 . \end{aligned}$$

If $0 \leqq a < p$, $p > 2$,

$$\begin{aligned} &\theta_j(k; a + 2p^2) \\ &= C(a + (j-2)p)\theta_1(k; 2p - j + 2) \\ &\quad + C(a + (j-1)p)\theta_0(k; 2p - j + 1) \qquad (1 < j \leqq p + 1, j \leqq m + 1) , \\ &= C(a + (j-2)p)\theta_1(k; 2p - j + 2) \qquad (j = m + 2 \leqq p + 1) , \\ &= C(a + (j-2)p)\theta_1(k; p) \qquad (j = p + 2 \leqq m + 2) , \\ &= C(a + (j-2)p)\theta_0(k; p - r + 2) \quad (j = p + r \leqq m + 2,\ 2 < r \leqq p + 2) , \\ &= 0 \quad \text{if} \quad j > m + 2 . \end{aligned}$$

If $0 \leqq a < p$, $0 \leqq b < p$,

$$\begin{aligned} &\theta_j(k; a + bp + p^2) \\ &\quad = C(a + (j-1)p)\theta_1(k; p + b - j + 1) \\ &\qquad + C(a + jp)\theta_0(k; p + b - j) && (b \geqq j;\ m + 1 > j) \\ &\quad = C(a + (j-1)p)\theta_0(k; p + b - j + 1) && (b < j \leqq p + b,\ m + 1 > j)\ , \\ &\quad = C(a + mp)\theta_1(k; p + b - m) && (j = m + 1,\ b \geqq m)\ , \\ &\quad = C(a + mp)\theta_0(k; p + b - m) && (j = m + 1,\ b < m)\ , \\ &\quad = 0 \quad \text{if} \quad j > m + 1\ . \end{aligned}$$

Some of the results in [2] can also be generalized. We use the symbols $E(n_1, \cdots, n_k)$ and $E_t(n_1, \cdots, n_k)$ as they are used in Lemma 2.2.

Let

$$\begin{aligned} F_j(n; x_1, \cdots, x_k) &= \sum_{\substack{a_1 + \cdots + a_k = n \\ E(a_1, \cdots, a_k) = j}} x_1^{a_1} \cdots x_k^{a_k}\ , \\ G_{t,j}(n; x_1, \cdots, x_k) &= \sum_{\substack{a_1 + \cdots + a_k = n \\ E_t(a_1, \cdots, a_n) = j}} x_1^{a_1} \cdots x_k^{a_k} \qquad (t > 0)\ , \\ G_{0,j}(n; x_1, \cdots, x_k) &= F_j(n; x_1, \cdots, x_k)\ . \end{aligned}$$

Note that

$$\begin{aligned} F_j(n; x, \cdots, x) &= x^n \theta_j(k; n)\ , \\ G_{t,j}(n; x, \cdots, x) &= x^n \psi_{t,j}(k; n)\ . \end{aligned}$$

By generalizing Carlitz's work in [2] in the natural way, we obtain

$$\begin{aligned} &F_j(a + bp; x_1, \cdots, x_k) \\ &\quad = \sum_{s=0}^{m} c_{sp+a}(x_1, \cdots, x_k) G_{s,j-s}(b - s; x_1^p, \cdots, x_k^p) \end{aligned} \tag{4.9}$$

where $0 \leqq a < p$, m is defined by (4.3), and

$$c_r(x_1, \cdots, x_k) = \sum_{s_1 + \cdots + s_k = r} x_1^{s_1} \cdots x_k^{s_k}\ .$$

Also, if h is defined by (4.5),

$$\begin{aligned} &G_{t,j}(a + bp; x_1, \cdots, x_k) \\ &\quad = \sum_{s=0}^{h} c_{sp+a}(x_1, \cdots, x_k) G_{s,j-s}(b - s; x_1^p, \cdots, x_k^p) \\ &\qquad\qquad (hp + a \leqq kp - k,\ 0 \leqq a < p - t)\ , \\ &\quad = \sum_{s=0}^{h-1} c_{sp+a}(x_1, \cdots, x_k) G_{s,j-s}(b - s; x_1^p, \cdots, x_k^p) \\ &\qquad\qquad (hp + a > kp - k,\ 0 \leqq a < p - t)\ , \\ &\quad = \sum_{s=1}^{h} c_{(s-1)p+a}(x_1, \cdots, x_k) G_{s,j-s}(a - s + 1; x_1^p, \cdots, x_k^p) \\ &\qquad\qquad (p - t \leqq a < p - 1)\ . \end{aligned} \tag{4.10}$$

5. Some special evaluations. If $j > \nu(n)$, where $\nu(n)$ is the exponent of the highest power of p that divides $n!$, then it is clear that $\theta_j(k; n) = 0$. For example, if $0 \leqq a < p$, $0 \leqq b < p$ then

$$\theta_j(k; a + bp) = 0 \qquad (j > b) .$$

Let n have expansion (1.1). By Lemma 2.1 it is clear that $\theta_j(k; n) = 0$ for $j > M$, where

$$\begin{aligned} M &= s(k-1) \quad \text{if} \quad k \leqq a_s + 1 \\ &= (s-1)(k-1) + a_s \quad \text{if} \quad k > a_s + 1 . \end{aligned}$$

Also,

$$\begin{aligned} &\theta_M(k; n) \\ &\quad = C(a_0 + (k-1)p)C(a_s - k + 1) \prod_{i=1}^{s-1} C(a_i - k + 1 + (k-1)p) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k \leqq a_s + 1) , \\ &\quad = C(a_0 + (k-1)p)C(a_{s-1} - k + 1 + a_s p) \prod_{i=1}^{s-2} C(a_i - k + 1 + (k-1)p) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k > a_s + 1, s > 1) , \\ &\quad = C(a_0 + a_1 p) \qquad\qquad\qquad\qquad\qquad (k > a_s + 1, s = 1) . \end{aligned}$$

For example, if $k = 2$ and $a_s \neq 0$ then $M = s$. This is the case for ordinary binomial coefficients. We have in this case

$$\theta_s(2; n) = (p - a_0 - 1)(p - a_1) \cdots (p - a_{s-1})a_s .$$

For $p = 2$ we can generalize the method used in [6]. Let

$$n = 2^{e_1} + \cdots + 2^{e_r} , \qquad 0 \leqq e_1 < \cdots < e_r , \tag{5.1}$$

$$n_i = 2^{e_{i,1}} + \cdots + 2^{e_{i,S(i)}} , \qquad 0 \leqq e_{i,1} < \cdots < e_{i,S(i)} . \tag{5.2}$$

Consider all the different compositions $n = n_1 + \cdots + n_k$ such that (5.1) and (5.2) hold, such that

$$S(n_1) + \cdots + S(n_k) = r + j ,$$

and such that there are a total of $r + j - t$ $e_{i,w}$'s having the property that $e_{i,w} \neq e_{x,y}$ for all x, y (except for the one case $i = x$, $w = y$). Let $b_{j,t}$ be the sum over all these compositions of the number of different ways of distributing the remaining t $e_{i,w}$'s into k distinct cells with no two identical objects in the same cell. Then for $p = 2$, $j > 0$,

$$\theta_j(k; n) = b_{j,2}k^{m+j-2} + b_{j,3}k^{m+j-3} + \cdots + b_{j,m+j} . \tag{5.3}$$

Using the convention that $e_1 - e_0 = t$ means $e_1 = t - 1$ and that $e_1 - e_0 > t$ means $e_1 > t - 1$, let

$$\begin{aligned} e_i - e_{i-1} &> 1 \quad \text{for} \quad q_1 \text{ terms } e_i\,, \\ &> 2 \quad \text{for} \quad q_2 \text{ terms } e_i\,, \\ &= 1,\ e_{i-1} - e_{i-2} = 1 \quad \text{for} \quad q_3 \text{ terms } e_i \qquad (i \neq 2)\,, \\ &= 1,\ e_{i-1} - e_{i-2} > 1 \quad \text{for} \quad q_4 \text{ terms } e_i\,, \\ &= 2 \quad \text{for} \quad q_5 \text{ terms } e_i \qquad (i \neq 1)\,, \\ &= 1 \quad \text{for} \quad q_6 \text{ terms } e_i \qquad (i \neq 1)\,. \end{aligned}$$

Then, by (5.3), for $p = 2$,

$$\begin{aligned} \theta_0(k; n) &= k^r\,, \\ \theta_1(k; n) &= q_1\binom{k}{2}k^{r-1} + q_6\binom{k}{3}k^{r-2}\,, \\ \theta_2(k; n) &= q_2\binom{k}{2}k^r + q_5\binom{k}{3}k^{r-1} \\ &\quad + \left[\binom{q_1}{2} + q_4\right]\binom{k}{2}^2 k^{r-2} \\ &\quad + \left[q_4(q_1 - 1) + q_3\right]\binom{k}{3}\binom{k}{2}k^{r-3} \\ &\quad + \left[\binom{q_4}{2} + q_3(q_3 - 1) + q_4 q_3\right]\binom{k}{3}^2 k^{r-4}\,. \end{aligned}$$

For example, let $n = 2^4 + 2^5 + 2^{20} + 2^{26} + 2^{28}$. Then $q_1 = 4$, $q_2 = 3$, $q_3 = 0$, $q_4 = 1$, $q_5 = 1$ and $q_6 = 1$. Thus

$$\begin{aligned} \theta_0(k; n) &= k^5 \\ \theta_1(k; n) &= 4\binom{k}{2}k^4 + \binom{k}{3}k^3\,, \\ \theta_2(k; n) &= 3\binom{k}{2}k^5 + \binom{k}{3}k^4 + 7\binom{k}{2}^2 k^3 + 3\binom{k}{3}\binom{k}{2}k^2\,. \end{aligned}$$

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