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REAL ANALYTIC OPEN MAPS

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Let R and C be the real and complex fields, respectively, and for $\zeta \in C$ let $\mathscr{R}(\zeta)$ be the real part of ζ. If $f: M^{p+1} \to N^p$ is real analytic and open with $p \geqq 1$, then there is a closed subspace $X \subset M^{p+1}$ such that $\dim f(X) \leqq p-2$ and, for every $x \in M^{p+1} - X$, there is a natural number $d(x)$ with f at x locally topologically equivalent to the map

$$\phi_{d(x)}: C \times R^{p-1} \longrightarrow R \times R^{p-1}$$

defined by $\phi_{d(x)}(z, t_1, \cdots, t_{p-1}) = (\mathscr{R}(z^{d(x)}), t_1, \cdots, t_{p-1})$.

In [7] Nathan proved: If $f: M^2 \to N^1$ is real analytic and open, then for every $x \in M^2$ there is a natural number $d(x)$ with f at x locally topologically equivalent to the map $\phi_{d(x)}: C \to R$ defined by $\phi_{d(x)}(z) = \mathscr{R}(z^{d(x)})$. This is the case $p = 1$ of the above theorem, but our proof is not a generalization of his.

Examples (see (3.3)) show that "topologically equivalent" cannot be replaced by "analytically equivalent" or even "C^1 equivalent", f real analytic cannot be replaced by f C^∞ (but see (3.1)), an exceptional set X with $\dim f(X) \geqq p-2$ is needed, and $\dim X$ may be $p-1$.

CONVENTIONS 1.2. We must assume that the reader has [2] at hand, and we follow its conventions. In particular we need [2; (2.2), (2.4), (2.5), (2.6), (2.8), (2.9), (3.1), and (3.9)]. For $f: M^n \to N^p$, B_f is the set of x in M^n at which f is not locally topologically equivalent to the projection map $\rho: R^n \to R^p$. The symbol $\approx$ is read "is diffeomorphic to".

DEFINITIONS 1.3. C-analytic sets are defined in [2]. A C-analytic set is called *C-irreducible* [9, p. 155] if it is not the sum of two C-analytic subsets distinct from itself. Whitney and Bruhat [9, p. 155, Proposition 11] prove that any C-analytic set V is uniquely the (countable) locally finite union of C-irreducible C-analytic subsets V_m, no one of which contains another. The V_m are called the *C-irreducible components* of V. Conversely, any locally finite union of C-analytic sets is a C-analytic set [9, p. 154].

DEFINITIONS 1.4. Let V be a complex analytic set of dimension v. There is a complex analytic subset $S \subset V$ such that $\dim S < v$ and $V - S$ is a complex analytic v-manifold [8, p. 500]. (The points of

$V - S$ are called *simple* or *regular*.) Let M be a complex analytic manifold, and let $T(M, p)$ for $p \in M$ be the tangent plane of M at p. Suppose that for each $p \in M$, v-plane T, and sequence $\{q_i\} \subset V - S$ with $q_i \to p$ and $T(V, q_i) \to T$, we have $T(M, p) \subset T$; then V is said to be *a-regular over M*. If V and M also satisfy another property (b-regular), then V is said to be *regular over M* [8, p. 540].

LEMMA 1.5. (Whitney [7, p. 540, Lemma 19.3].) *Suppose that V and W are complex analytic sets, and* $\dim V > \dim W$. *Then there is a complex analytic subset S of W such that* $\dim S < \dim W$, *each point of $W - S$ is simple, and V is regular over the complex analytic manifold $W - S$.*

2. Analytic sets and maps.

LEMMA 2.1. *Let $f: M^n \to N^p$ be C^ω, and let $V \subset M^n$ be a non-empty C-analytic subset of M^n with dimension v. Then*

(a) (Whitney and Bruhat [9, p. 156, Proposition 13]), *there is a C-analytic subset $S \subset V$ such that* $\dim S \leqq v - 1$ *and $V - S$ is a v-dimensional C^ω submanifold of M^n;*

(b) *there is a C-analytic subset $E \subset V$ such that $V - E$ is a v-dimensional C^ω submanifold of M^n, $f \mid (V - E)$ has constant rank r, and* $\dim f(E) \leqq v - 1$;

(c) $\dim f(V) \leqq \max\{v - 1, r\} \leqq v$; *and*

(d) *if V is C-irreducible* (1.3), *then* $\dim f(V) \leqq r$.

Proof. We use induction on v; if $v = 0$, then V is discrete and the results are trivial.

Let V_m be the C-irreducible components of V (1.3). According to [2, p. 22, (3.1)] there is a C-analytic subset E_m of V_m such that $\dim E_m < \dim V_m$, $V_m - E_m$ is a C^ω submanifold of V_m with dimension $\dim V_m$, and $f \mid (V_m - E_m)$ has constant rank r_m. Let r be the maximum r_m for those m with $\dim V_m = v$.

If (1) $\dim V_m < v$, or (2) $\dim V_m = v$ and $r_m < r$, let $F_m = V_m$; (3) otherwise, let $F_m = E_m$. Since the V_m are locally finite, the F_m are also. Let $S \subset V$ be the C-analytic subset given by (a). Then by inductive hypothesis (c), $\dim f(S) < v$ and $\dim f(F_m) < v$ in cases (1) and (3). In case (2) $\dim f(E_m) < v$ also, and, from the Rank Theorem [1, p. 155] applied to $f \mid (V_m - E_m)$, $\dim(f(V_m - E_m)) \leqq r_m < r \leqq v$. Since each of E_m and $(V_m - E_m)$ is the countable union of compact sets, $\dim f(V_m) < v$. Let $E = S \cup (\bigcup_m F_m)$. Then $\dim f(E) < v$; (b) results from the local finiteness of the F_m and (1.3); and (again from the Rank Theorem) (c) is a corollary.

Now suppose that V is C-irreducible. Let W be the set E of (b), let W_m be its C-irreducible components, and let E_m be as given by (b) for W_m. If each $f|(W_m - E_m)$ has rank at most r, then $\dim f(W_m) \leqq r$ by inductive hypothesis, and (d) follows. Thus we may suppose that for some W_m and E_m, call them W and E, $f|(W - E)$ has rank greater than r.

Let M^*, N^*, f^*, V^*, W^*, and $(W - E)^*$ be complexifications (see e.g. [2, (2.4), (2.5), (2.6)]), where M^* is small enough that V^* is irreducible in M^* [9, p. 155, Proposition 11 and p. 151, Corollary 2]. Let $E' \subset V^*$ be as given by [2, (3.1)] for V^* and f^*, so that $f^*|(V^* - E')$ has constant rank k. By definition of r, V has a simple point x at which $f|V$ has rank r; thus $f^*|V^*$ has rank r at x also, so that $k \geqq r$. Since $\dim E' < \dim V^* = v$ [9, p. 155, Proposition 12], $\dim (E' \cap M^n) < v$; thus $k = r$.

Let S^* be the analytic subset of $(W - E)^*$ given by (1.5) such that V^* is regular over the manifold $X^* = (W - E)^* - S^*$ and let $q \in X^*$. Since V^* is irreducible, the simple points of V^* are dense in V^* [5, p. 68, Corollary 2]. Thus $V^* = Cl[V^* - E']$, so there exist $q_i \in V^* - E'$ with $q_i \to q$. Let T_i and T be the tangent planes of $V^* - E'$ at q_i and of X^* at q, respectively. Since the Grassman manifold G of v-planes in C^n is compact, there are $T' \in G$ and a subsequence $T_{i(j)} \to T'$, and since V^* is a-regular over X^*, $T \subset T'$. Now $f|(V^* - E')$ has rank r, while $f^*|X^*$ has rank greater than r, and a contradiction results.

Substantially the same proof yields the complex analog, where C-analytic is replaced by analytic. There is a unique minimal set E satisfying (b), viz. the intersection of all sets E satisfying (b).

LEMMA 2.2. *Let $f: K^k \times R^{p-1} \to R \times R^{p-1}$ ($p \geqq 1$) be a C^ω layer map* (i.e., $f(K^k \times \{t\}) \subset R \times \{t\}$), *let $f_t: K^k \to R$ be defined by $(f_t(x), t) = f(x, t)$, and let $\Gamma \subset R_{p-1}(f)$ be a C-analytic subset with* $\dim \Gamma \leqq p - 1$. *Then there is a C-analytic subset $\Delta \subset \Gamma$ such that* $\dim f(\Delta) \leqq p - 2$ *and*

$$\dim ((\Gamma - \Delta) \cap (K^k \times \{t\})) \leqq 0$$

for each $t \in R^{p-1}$.

Proof. Let $E \subset \Gamma$ and r be as given by (2.1(b)). If $r < p - 1$, then let $\Delta = \Gamma$; if $r = p - 1$, let $\Delta = E$. In either case, $\dim f(\Delta) \leqq p - 2$ (2.1(c)). If $\Gamma - \Delta \neq \emptyset$, it is a $C^\omega(p - 1)$-manifold, and $f|(\Gamma - \Delta)$ has rank $p - 1$. Since $\Gamma \subset R_{p-1}(f)$, $R_{p-1}(f) \cap (K^k \times \{t\}) = R_0(f_t)$, and $\dim (f_t(R_0(f_t))) \leqq 0$ (by Sard's Theorem [1, p. 156]), $\Gamma - \Delta$ is transverse to each $K^k \times \{t\}$ at each point of intersection. In other words, the inclusion map $i: \Gamma - \Delta \to K^k \times R^{p-1}$ is transverse regular on $K^k \times \{t\}$, so by Thom's Transversality Theorem [1, p. 165] $i^{-1}(K^k \times \{t\}) =$

$(\Gamma - \Delta) \cap (K^k\{t\})$ is a 0-dimensional manifold.

LEMMA 2.3. *If* $f: R^2 \times R^{p-1} \to R \times R^{p-1}$ *is an open* C^ω *layer map, then there is a closed subset* $X \subset R^2 \times R^{p-1}$ *such that* $\dim f(X) \leqq p-2$ *and* $\dim((B_f - X) \cap f^{-1}(y, t)) \leqq 0$ *for each* $(y, t) \in R \times R^{p-1}$.

Proof. By the Rank Theorem [1, p. 155] $B_f \subset R_{p-1}(f)$. (*) It suffices to prove the theorem locally, i.e., to show that for each $(x, t) \in R_{p-1}(f)$, there are neighborhoods $P \approx R^2$ of x and $Q \approx R^{p-1}$ of t such that $f \mid P \times Q$ satisfies the conclusion.

Now $R_{p-1}(f)$ is a C-analytic set [2, (2.9)], and since $\dim(f(R_{p-1}(f))) \leqq p-1$ by Sard's Theorem [1, p. 156] and f is open, $\dim(R_{p-1}(f)) \leqq p$. It is the union of its C-irreducible components V_m with dimension v_m; let E_m and r_m be as given by (2.1(b)) (or [2, (3.1)]). Let E be the union of the C-analytic subset $S \subset R_{p-1}(f)$ given by (2.1(a)), the V_m for $v_m = r_m = p-1$, and the E_m for $v_m = p$ and $r_m = p-1$, and let F be the union of the V_m with $r_m \leqq p-2$. Let $G \subset E$ be the C-analytic subset Δ given by (2.2) for $\Gamma = E$. Then $\dim(f(F \cup G)) \leqq p-2$ (2.1(d)), so we may define X to contain $F \cup G$. Thus (see (*)) we may consider only neighborhoods $P \times Q$ disjoint from $F \cup G$, i.e., it suffices to prove the lemma in case $F = G = \varnothing$. By (2.2) $\dim(E \cap (R^2 \times \{t\})) \leqq 0$ for each $t \in R^{p-1}$, so (see (*)) it suffices to prove the lemma at the points of $R_{p-1}(f) - E$, i.e., to assume $E = \varnothing$. Thus $R_{p-1}(f)$ is a p-manifold (or is $\varnothing$) and $f \mid R_{p-1}(f)$ has rank $p-1$.

We now apply [2, (3.9)] to each component Γ of $R_{p-1}(f)$. Since f is open, each $k(\Gamma)$ is odd and B_f is contained in the at most $(p-1)$-dimensional analytic set $A = \bigcup_\Gamma A(\Gamma)$. Let $\Delta \subset A$ be the C-analytic subset given by (2.2). We may take $\Delta = X$, and the conclusion results.

3. Proof of the theorem.

THEOREM 3.1. ([3, (1.1) and (4.1)].) *Let* $f: M^n \to N^p$ *be a* C^3 *open map with* $p \geqq 1$, *and let* $\dim(B_f \cap f^{-1}(y)) \leqq 0$ *for each* $y \in N^p$. *Then there is a closed set* $X \subset M^{p+1}$ *such that* $\dim f(X) \leqq p-2$ *and, for every* $x \in M^{p+1} - X$, *there is a natural number* $d(x)$ *with* f *at* x *locally topologically equivalent to the map*

$$\phi_{d(x)}: C \times R^{p-1} \longrightarrow R \times R^{p-1}$$

defined by $\phi_{d(x)}(z, t_1, \cdots, t_{p-1}) = (\mathscr{R}(z^{d(x)}), t_1, \cdots, t_{p-1})$.

Proof of (1.1) 3.2. Let $X = X(f)$ be the complement of the set on which f has the desired structure; then $X \subset B_f$ is closed. We suppose that $\dim f(X) \geqq p - 1$, and will obtain a contradiction.

Since f is C^3, $\dim(f(R_{p-2}(f))) \leqq p - 2$ [1, p. 156]. If, for every $x \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$, there is an open neighborhood $U_x \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ of x with $\bar{U}_x$ compact and $\dim(f(U_x \cap X)) \leqq p - 2$, it follows from the fact that $\{U_x\}$ has a countable subcover that $\dim(f(X)) \leqq p - 2$. Thus, there is an $\bar{x} \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ such that, for every open neighborhood $U \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ of $\bar{x}$, $\dim(f(U \cap X)) \geqq p - 1$.

By [1, p. 156, Layering Lemma] there are open neighborhoods U of $\bar{x}$ and V of $f(\bar{x})$ and C^r diffeomorphisms $\sigma: R^2 \times R^{p-1} \approx U$ and $\rho: V \approx R \times R^{p-1}$ such that $\rho \circ f \circ \sigma = g$ is a C^r layer map and $\sigma(\bar{x}) = (0, 0)$. Thus $\dim g(X(g)) \geqq p - 1$. By (2.3) there is a closed set $Y \subset R^2 \times R^{p-1}$ such that $\dim g(Y) \leqq p - 2$ and $\dim((B_g - Y) \cap g^{-1}(y)) \leqq 0$ for each $y \in R \times R^{p-1}$.

Let h be the restriction $g \mid [(R^2 \times R^{p-1}) - Y]$; then $X(h) = X(g) - Y$, $\dim h(X(h)) = p - 1$, and $\dim(B_h \cap h^{-1}(y)) \leqq 0$ for each $y \in R \times R^{p-1}$, contradicting (3.1).

EXAMPLES 3.3. Open maps $f: M^2 \rightarrow R$ with $\dim(B_f \cap f^{-1}(y)) = 1$ are given in [4, p. 341] and [6, p. 329]; the latter example may be assumed to be C^∞ except on one point inverse, and thus [1, p. 151] may be assumed to be C^∞. As a result, "f real analytic" may not be replaced by "$f C^\infty$" in (1.1).

The maps f and g defined by $f(z) = \mathscr{R}(z)$ and $g(z) = (\mathscr{R}(z))^3$ are locally topologically equivalent at 0, but are not locally C^1 equivalent, since g has rank 0 at the origin.

There are examples [2, (4.7)(b)] with $X = B_f$, $\dim B_f = p - 1$, and $\dim f(B_f) = p - 2$.

REMARK 3.4. A real analytic open map $f: M^p \rightarrow N^p$ is light [2, p. 28, (4.2)], and thus for $p \geqq 2$ satisfies a structure theorem [1, p. 155] similar to (1.1).

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