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SIERPINSKI CURVES IN FINITE 2-COMPLEXES

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In this note certain one-dimensional continua are defined for finite 2-complexes. These continua, called S-curves, are a generalization of the Sierpinski plane universal curve. By a 2-complex is meant a finite connected 2-dimensional euclidean polyhedron which has a triangulation such that every 1-simplex is the face of at least one 2-simplex. It is shown that any two S-curves in a 2-complex are homeomorphic. In addition, it is established that two 2-complexes (with the property that every 1-simplex in a triangulation is the face of two or more 2-simplexes) are homeomorphic if and only if the corresponding S-curves are homeomorphic.

In 1916 Sierpinski [4] described a one-dimensional continuum that is known as the Sierpinski plane universal curve. In 1958 Whyburn [7] defined the notion of an S-curve in a 2-sphere and established that an S-curve in a 2-sphere is homeomorphic to the Sierpinski plane universal curve. In 1966 Borsuk [1] defined an S-curve in a surface. He established that any two S-curves in a given surface are homeomorphic and that two surfaces are homeomorphic if and only if the corresponding S-curves are homeomorphic. In this paper the same type of theorems are established for certain 2-complexes.

In order to define an S-curve in a 2-complex, it is necessary to introduce some terminology from Whittlesey [5] or [6]. A point x in a 2-complex K is a *regular point* if it has a neighborhood in K homeomorphic to the plane (euclidean 2-dimensional space). The *regular part* of K is the collection of all regular points in K. The points of K which are not regular are called *singular*; the collection of all singular points in K constitute the *singular graph* of K. Let $D_1, D_2, \cdots$ be a sequence of mutually disjoint closed discs contained in the regular part of K. Then $\mathrm{A}(K) = K - \bigcup_{i=1}^{\infty} \mathrm{Int}\, D_i$ (Int = interior in the sense of manifolds) is said to be an *S-curve in K* provided that $\bigcup_{i=1}^{\infty} D_i$ is dense in K and the diameters of the D_i converge to zero. Note that if the 2-complex is also a surface, then this definition is precisely that of Borsuk [1, pp. 81–82].

LEMMA 1. *Let K be a 2-complex and Σ an upper semi-continuous decomposition of K with the property that every nondegenerate element of Σ is contained in the regular part of K and each nondegenerate element has arbitrarily small neighborhoods (in K) homeomorphic with the plane. Then the decomposition space K_Σ is homeomorphic to K.*

Proof. It follows from results of Whittlesey [5, p. 843] that there exists a finite collection of bounded surfaces (compact, connected 2-manifolds with nonempty boundary) $M_1, \cdots, M_j$ such that K is an identification space of their topological sum $M_1 + \cdots + M_j$. The identification takes place on the boundaries of the surfaces. More precisely, if $f: M_1 + \cdots + M_j \to K$ is the identification map, then f restricted to the manifold interiors of the surfaces is a homeomorphism.

Consider the following diagram:

$$\begin{array}{ccc} M_1 + \cdots + M_j & \xrightarrow{f} & K \\ p \downarrow & & \downarrow q \\ M_{\Sigma(1)} + \cdots + M_{\Sigma(j)} & \xrightarrow[f_*]{} & K_\Sigma . \end{array}$$

The upper semi-continuous decomposition Σ of K induces an upper semi-continuous decomposition $\Sigma(i)$ of M_i, $i = 1, \cdots, j$. $\Sigma(i)$ has as nondegenerate elements those sets B such that $B = f^{-1}(b)$ where b is a nondegenerate element of Σ. Let p_i be the identification map of M_i onto the decomposition space $M_{\Sigma(i)}$, $i = 1, \cdots, j$. Let p denote the identification map induced by the identification maps p_i, $i = 1, \cdots, j$, and let q denote the identification map for the decomposition Σ of K. The map f is a relation-preserving continuous map that is an identification. Hence, the induced map f_* is continuous and is also an identification [2, Theorem 4.3, p. 126].

It follows from results of Borsuk [1, Theorem 3.1, p. 76] that M_i is homeomorphic to $M_{\Sigma(i)}$. For each i, $i = 1, \cdots, j$, the map p_i restricted to Bd M_i (Bd = boundary in the sense of manifolds) is a homeomorphism onto Bd $M_{\Sigma(i)}$. Furthermore, all the orientations of the boundaries are preserved by p_i, and so by [5, Lemma, p. 843] p_i restricted to Bd M_i can be extended to a homeomorphism h_i mapping M_i onto $M_{\Sigma(i)}$.

Next consider the diagram:

$$\begin{array}{ccc} M_1 + \cdots + M_j & \xrightarrow{f} & K \\ h \downarrow & & \downarrow h_* \\ M_{\Sigma(1)} + \cdots + M_{\Sigma(j)} & \xrightarrow[f_*]{} & K_\Sigma . \end{array}$$

The homeomorphism h is induced by the homeomorphisms h_i, $i = 1, \cdots, j$. As above, there exists a continuous mapping h_* of K onto K_Σ. Furthermore, h_* is one-to-one. Since K is compact and Hausdorff and Σ is an upper semi-continuous decomposition, it follows from [3, Theorem 3-33, p. 133] that K_Σ is Hausdorff. Thus h_* is a one-to-one continuous mapping of a compact space onto a Hausdorff space and hence is a homeomorphism.

The proof of the next result closely parallels that of Borsuk [1, pp. 82–83] but is included for completeness.

THEOREM 1. *Any two S-curves in a given 2-complex are homeomorphic.*

Proof. Let $A = K - \bigcup_{i=1}^{\infty} \operatorname{Int} D_i$ be an S-curve in a 2-complex K. Consider the upper semi-continuous decomposition Σ of K whose nondegenerate elements are the discs D_i. K is homeomorphic to K_Σ by Lemma 1. The subset of the decomposition space K_Σ consisting of points d_i corresponding to the discs D_i is countable and is contained in the regular part of K_Σ. If K has triangulation T_0, there exists a "curved" triangulation T of K_Σ isomorphic to T_0 such that no point d_i, $i = 1, 2, \cdots$, belongs to the 1-dimensional skeleton Z of T. The skeleton Z may be considered as lying in the set $K - \bigcup_{i=1}^{\infty} D_i$. Thus a triangulation T of K is obtained that is isomorphic to T_0 with the property that every disc D_i lies in the interior of a 2-simplex of K.

Similarly, if $A' = K - \bigcup_{i=1}^{\infty} \operatorname{Int} D'_i$ is another S-curve in K, it follows from the above argument that there exists another triangulation T' of K isomorphic to T such that every disc D_i lies in the interior of a 2-simplex of T'. Let Z' denote the 1-skeleton.

Since T and T' are isomorphic, there is a homeomorphism h mapping K onto K such that each 2-simplex E of T is mapped by h onto a 2-simplex E' of T'. Then $E \cap A$ and $E' \cap A'$ may be viewed as S-curves in a 2-sphere, and h as a homeomorphism mapping the outer boundary of $E \cap A$ onto the outer boundary of $E' \cap A'$. Thus by a result of Whyburn [7, p. 322], h restricted to Bd E can be extended to a homeomorphism h_E mapping $E \cap A$ onto $E' \cap A'$. The mapping h can then be extended to a homeomorphism mapping A onto A' by defining $h(x) = h_E(x)$ for $x \in A$ and x contained in the 2-simplex E of T.

Next it is established that certain 2-complexes are completely characterized by their S-curves. Let K be the union of all the proper faces of a 3-simplex and let K' be a 2-simplex. Then $A(K)$ is homeomorphic to $A(K')$ but K is not homeomorphic to K'. This example shows that extra conditions are needed on the 2-complexes for such a characterization. The sufficient conditions are stated in Theorem 2.

First, some terminology from Borsuk [1, p. 84] must be introduced. Let $A(K) = K - \bigcup_{i=1}^{\infty} \operatorname{Int} D_i$ be an S-curve associated with a 2-complex K. The set $\operatorname{Bd} A(K) = \bigcup_{i=1}^{\infty} \operatorname{Bd} D_i$ is said to be the *boundary of* $A(K)$. The set $\operatorname{Int} A(K) = A(K) - \operatorname{Bd} A(K)$ is said to be the *interior of* $A(K)$. *Singular interior points* of $A(K)$ are those interior points contained in the singular graph of K.

Let S be a Sierpinski plane universal curve and I an arc (a space homeomorphic to the closed interval $[0, 1]$). Let Y denote the space obtained by identifying an endpoint of I with an interior point x of S. Observe that every interior point of S is interior to arbitrarily small rectangular plane neighborhoods whose boundaries lie in S. Hence Y is not embeddable in the plane. This fact will be used in the proof of the following lemma.

LEMMA 2. *Let K be a 2-complex such that every 1-simplex is the face of two or more 2-simplexes, and let $A(K)$ be the associated S-curve. A point x in $A(K)$ is a singular interior point if and only if no neighborhood of x in $A(K)$ is embeddable in the plane.*

Proof. It is clear that if x does not have a neighborhood in $A(K)$ embeddable in the plane, then x does not have such a neighborhood in K. Thus x belongs to the singular graph of K and is a singular interior point of $A(K)$.

Conversely, suppose x is a singular interior point. Then x is an element of the singular graph of K. To show that no neighborhood of x in $A(K)$ is embeddable in the plane it suffices to establish that every neighborhood of x in $A(K)$ contains a subset homeomorphic to Y (as defined above). Whittlesey has classified the singular points of a 2-complex. His definitions [5, p. 842] are used to consider the various cases.

Case 1. x is a line singularity. Then x has arbitrarily small neighborhoods in K homeomorphic to the space obtained by identifying the x-axes of n ($n \geqq 3$ by the hypothesis of the lemma) copies of the closed euclidean half-plane $y \geqq 0$. It follows that every neighborhood of x in $A(K)$ contains a subset homeomorphic to Y.

Case 2. x is a conical point. Then x has arbitrarily small neighborhoods in K homeomorphic to the set which is obtained if n copies ($n \geqq 2$) of the plane are identified at the origin. Again every neighborhood of x in $A(K)$ contains a copy of Y.

Case 3. x is a node. A node is necessarily a vertex in any triangulation of K. Let T be a triangulation of K. Then the regular part of the Star of x falls into components each of which is a cone with x at the vertex or is, topologically, an open triangle with x as a vertex and with two singular edges, both edges having x as a vertex, and the edges may be distinct or coincide. Since by hypothesis every 1-simplex is the face of two or more 2-simplexes, every neighborhood in $A(K)$ of a node will contain a copy of Y.

All possible singular interior points have been considered and the proof is completed.

THEOREM 2. *Let K and K' be 2-complexes such that every 1-simlex in a triangulation of K or K' is the face of two or more 2-simplexes. Let $A(K)$ and $A(K')$ be the S-curves associated with K and K' respectively. Then $A(K)$ is homeomorphic to $A(K')$ if and only if K is homeomorphic to K'.*

Proof. Let h mapping $A(K)$ onto $A(K')$ be a homeomorphism. Let Int D_i be an open disc in $K - A(K)$ with Bd $D_i = C$ which is contained in $A(K)$. Consider $h(C) = C'$. Then C' is a simple closed curve in $A(K')$. Next it is established that $C' = \text{Bd } D'_i$ where $A(K') = K' - \bigcup_{i=1}^{\infty} \text{Int } D'_i$.

If $x \in C$, then there exists a neighborhood of x in $A(K)$ embeddable in the plane. By Lemma 2, $h(x)$ is not a singular interior point of $A(K')$. Furthermore, if $x \in C$ then x is contained in the interior of an arc in C that does not locally decompose $A(K)$. It follows from [1, p. 84] that C' is contained in Bd $A(K')$. Hence $C' = \text{Bd } D'_i$ for some i. For each i the map h restricted to the Bd D_i can be extended to a homeomorphism h_i mapping the disc D_i onto the disc D'_i. Since the diameters of the sets $h(\text{Bd } D_i)$ converge to zero, the diameters of the discs D'_i converge to zero. Extend h to a mapping h' of K onto a subset of K' by defining $h'(x) = h(x)$ for x in $A(K)$ and $h'(x) = h_i(x)$ for x in Int D_i. Then h' is a mapping of K onto a subset of K'. But since $h(A(K)) = A(K')$, h' is also onto K'; and K is homeomorphic to K'.

The converse follows from Theorem 1.

The reader will able to make the necessary modifications to extend these results by himself to arbitrary finite 2-complexes.

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