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POLYNOMIALS AND HAUSDORFF MATRICES

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If f is a function from the rational numbers in $[0, 1]$ to the complex plane and c is a complex sequence, then the Hausdorff matrix $H(c)$ for c and a sequence $L(f, c)$ are defined:

$$H(c)_{np} = \binom{n}{p} \sum_{q=0}^{n-p} (-1)^q \binom{n-p}{q} c_{p+q} = \binom{n}{p} \Delta^{n-p} c_p$$

$$L(f, c)_n = \sum_{p=0}^{n} H(c)_{np} f(p/n) .$$

THEOREM. **If f is a function from the rationals in $[0, 1]$ to the plane and $L(f, c)$ converges for each complex sequence c, then f is a subset (contraction) of a polynomial.**

J. S. MacNerney [2, p. 56] and A. Jakimovski [1] have shown:

THEOREM A. *If f is a polynomial and c is a complex sequence then $L(f, c)$ converges.*

Our theorem is a converse to Theorem A and an improvement of Theorem 7 of [3] where we proved that if f is a *continuous* function from $[0, 1]$ to the complex plane and $L(f, c)$ converges for each complex sequence c, then f is a subset of a polynomial.

LEMMA 1. *Suppose that M is an infinite complex matrix and M transforms each sequence to a bounded sequence. Then there is a positive integer P such that if p is an integer exceeding P then $M_{np} = 0$, $n = 0, 1, \cdots$.*

Lemma 2, which follows, is not necessary for our result. It is included because it gives an interesting way of proving Lemma 3. The proof for Lemma 3 given here is due to the referee. It is simpler and more direct than the author's argument which uses Lemma 2. The author is grateful to the referee for the pretty proof of Lemma 3 which appears in this paper.

LEMMA 2. *Suppose that n is a positive integer, each of $\{a_p\}_1^n$ and $\{b_p\}_1^n$ is an increasing sequence of nonnegative integers, M is an $n \times n$ matrix such that*

$$M_{mp} = \binom{a_m}{b_p} \qquad (m, p = 1, 2, \cdots, n)$$

and $a_k \geqq b_k$, $k = 1, 2, \cdots, n$. *Then the determinant of* M *is positive.*

LEMMA 3. *Suppose that* f *is a function from the rationals in* $[0, 1]$ *to the complex plane,* P *is a nonnegative integer, and*

$$\sum_{q=0}^{p} (-1)^q \binom{p}{q} f(q/n) = 0$$

for each integer-pair $\{n, p\}$ *satisfying* $P < p \leqq n$. *Then* f *is a subset of a polynomial.*

Lemma 1 is known. Also, its proof is not difficult.

Lemma 2 is Theorem 2 of [4]. (This present paper was the motivation for [4].)

Proof of Lemma 3. Suppose that each of p and n is an integer and $P < p \leqq n$. For each number x let $Q(x)$ be

$$\sum_{k=0}^{P} \binom{x}{k} \sum_{q=0}^{k} (-1)^{k+q} \binom{k}{q} f(q/n)$$

(here $\binom{x}{0} = 1$ and $\binom{x}{k+1} = (x-k)/(k+1)\binom{x}{k}$, $k = 0, 1, 2, \cdots$), and let $H_n(x)$ be $Q(nx)$. H_n is a polynomial of degree P, at most, and

$$H_n(j/n) = f(j/n) \qquad (j = 0, 1, \cdots, n) .$$

Now, if m is a multiple of n, $H_m = H_n$, since each of H_m and H_n is a polynomial of degree P, at most, $P < n$, and H_m and H_n agree at $0, 1/n, 2/n, \cdots, n/n$.

Consequently, if n and m are integers exceeding P, $H_n = H_{m \cdot n} = H_m$. Now, let n be $P + 1$ and let s be a positive integer and let r be an integer in $[0, s]$. Then

$$H_n\left(\frac{r}{s}\right) = H_n\left(\frac{rn}{sn}\right) = H_{sn}\left(\frac{rn}{sn}\right) = f\left(\frac{rn}{sn}\right) = f\left(\frac{r}{s}\right) ,$$

and f is a subset of the polynomial H_n.

Proof of Theorem. Let c be a complex sequence. For each positive integer n,

$$L(f, c)_n = \sum_{p=0}^{n} c_p \binom{n}{p} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} f(q/n) .$$

Define

$$M_{np} = \binom{n}{p} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} f(q/n) \qquad (n = 1, 2, \cdots; p = 0, 1, \cdots, n)$$

and $M_{np} = 0$ if $n = 0$ or $p > n$. By hypothesis M transforms each sequence to a convergent sequence, so, by Lemma 1, there is a positive integer P such that if p is an integer exceeding P then $M_{np} = 0$, $n = 0, 1, \cdots$.

Thus, by Lemma 3, f is a subset of a polynomial.

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