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VERTICALLY COUNTABLE SPHERES AND THEIR WILD SETS

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A 2-sphere S in E^3 is said to have vertical order n if the intersection of each vertical line with S contains no more than n points. It is shown that $S \cup \operatorname{Int} S$ is a 3-cell that is locally tame from $\operatorname{Ext} S$ modulo a 0-dimensional set if S has vertical order 5. A subset X of E^3 is said to have countable (finite) vertical order if the intersection of X with each vertical line consists of countably (finitely) many points. A 2-sphere in E^3 with countable vertical order can have a wild set of dimension no larger than one.

For each 2-sphere S in E^3 there is a homeomorphism $h: E^3 \to E^3$ such that each vertical line intersecting $h(S)$ does so in a 0-dimensional set [2, Theorem 10.1]; thus the condition that a 2-sphere be "vertically 0-dimensional" imposes no restriction on the wildness of the 2-sphere. A study of vertically finite 2-spheres (spheres with finite vertical order) was begun in [10] where it was proven that a 2-sphere in E^3 having vertical order 3 is tame. Even though there are wild 2-spheres having vertical order 4, it is known that $S \cup \operatorname{Int} S$ is a 3-cell if S has vertical order 5 [11]. We extend this result to show that the set $W(S)$ of points where the 2-sphere S fails to be locally tame must be 0-dimensional if S has vertical order 5. An example is given at the end of the paper to show that 5 is the largest integer for which this result is true. We also show that the wildness of a vertically countable sphere is limited to a 1-dimensional set.

In the remainder of the paper we use $\pi: E^3 \to E^2$ to denote the vertical projection of E^3 onto the horizontal plane E^2. For convenience, we always assume that E^2 is located vertically below the sphere or cube under investigation. We use $L(x)$ to denote the vertical line containing the point x.

A vertical line L is said to pierce a subdisk D of a 2-sphere S if there is an interval I in L such that $I \cap S$ is a point $p \in D$ and I intersects both $\operatorname{Int} S$ and $\operatorname{Ext} S$. We say that L links the boundary $\operatorname{Bd} D$ of a disk D if L intersects every disk bounded by $\operatorname{Bd} D$.

2. Spheres having countable vertical order.

THEOREM 2.1. *If S is a 2-sphere in E^3 having countable vertical order, then $W(S)$ contains no open subset of S.*

Proof. Suppose that $W(S)$ contains a disk D in S. We shall produce a contradiction by exhibiting a vertical line L whose intersection with D contains a Cantor set.

Assertion A. If D' is a subdisk of D, then there is an open subset U of E^3 such that $\pi(U) \subset \pi(D')$.

To prove Assertion A it suffices to show that $\pi(D')$ is not one-dimensional. This follows from [9, Theorem VI.7, p. 91] since the map $\pi \mid D' : D' \rightarrow \pi(D)$ is closed.

Assertion B. If D' is a subdisk of D and U is an open subset of E^3 such that $\pi(U) \subset \pi(D')$, then there exist disjoint disks D_1 and D_2 in D' and an open subset N of U such that each vertical line through $\mathrm{cl}\,(N)$ intersects both D_1 and D_2.

In order to select the disks D_i in Assertion B we first show the existence of a vertical line L containing two points r and t in D' and containing two sequences $\{u_i\}$ and $\{l_i\}$ of points such that

(1) $\{u_i\}$ converges to r from above,
(2) $\{l_i\}$ converges to r from below,
(3) there is a component V_1 of $E^3 - S$ containing every u_i, and
(4) $E^3 - (S \cup V_1) = V_2$ contains every l_i.

Notice that some vertical line L' intersects D' in more than two points [7, Theorem 2.3], so we may choose two points r' and t' in $L' \cap D'$. Let B be an open ball centered at r' such that $B \cap S \subset D'$. If r' does not satisfy the four conditions above relative to L', it must be because some interval I in $L' \cap B$ has r' as its midpoint and lies, except for r', in a single component, say V_1, of $E^3 - S$. Let B_1 and B_2 be disjoint round open balls of equal radius centered at points of L' above and below r', respectively such that $B_1 \cup B_2 \subset V_1 \cap B$. Now close to r' and vertically between B_1 and B_2, there must exist a point e of V_2. Then $L = L(e)$ intersects V_2 between its two intersections with $V_1 \cap (B_1 \cup B_2)$, so L intersects D' at least twice. Let r be the lowest point of the component of $L \cap (S \cup V_1)$ containing $L \cap B_1$, and choose t to be some other point of $L \cap S$. Since S has countable vertical order it is clear that r is a limit point of $L \cap V_1$ from above and of $L \cap V_2$ from below. Thus conditions (1), (2), (3), and (4) are satisfied.

Choose a disk D_1 in D' such that $r \in \mathrm{Int}\, D_1$ and $t \notin D_1$. We claim that there is an open set U_1 containing r such that every vertical line through U_1 intersects D_1. Suppose there is no such open set, and for each i let E_i be a horizontal disk centered at l_i and lying in V_2. There must be a sequence $\{x_i\}$ such that $x_i \in E_i$, for each i, no $L(x_i)$ intersects D_1, and $\{L(x_i)\}$ converges to $L(r)$. For erch i let y_i be the first point of S above x_i on $L(x_i)$ (such a point will exist for sufficiently large integers i since u_i and l_i are different components of $E^3 - S$), and let I_i be the vertical interval $[x_i, y_i]$ in $S \cup V_2$. Since some subsequence of $\{y_i\}$ converges, we assume for notational convenience that $\{y_i\}$ converges to a point y. Of course $y \in L(r) \cap S$. It is clear that y is not above r on $L(r)$ because $\{r, y\} \subset \liminf I_i \subset S \cup V_2$ whereas $\{u_i\} \to r$ and $u_i \in V_1$. Nor is y below r on $L(r)$ because $\{l_i\} \to r$, $\{l_i, x_i\} \subset E_i$, and x_i lies vertically below y_i. Thus $\{y_i\}$ converges to r, and we have the contradiction that most of the y_i's must belong to D' while $L(y_i) \cap D'$ was supposed to be empty. The existence of U_1 is established.

Now choose a disk D_2 such that $D_1 \cap D_2 = \emptyset$, $t \in \operatorname{Int} D_2$, $D_2 \subset D'$, and $\pi(D_2) \subset \pi(U_1)$. From Assertion A there is an open set U_2 such that every vertical line through U_2 intersects D_2. Such a line will also intersect U_1 and hence D_1. Choose N to be any open subset of U such that $\pi(\text{cl}(N) \subset \pi(U_1) \cap \pi(U_2))$.

Now that the two assertions have been proven it might be clear how to proceed inductively to produce a vertical line containing uncountably many points of S; nevertheless, we give a brief outline. From Assertion A there is an open set U such that every vertical line through U intersects D. Now we apply Assertion B to obtain an open set U_1, whose closure lies in U, and two disjoint disks D_1 and D_2 in D such that every vertical line through $\text{cl}\,(U_1)$ intersects both D_1 and D_2. This ends the first step in the construction. Assertion B can now be applied to D_1 to obtain two disjoint disks D_{11} and D_{12} in D_1 and an open set N_1 such that vertical lines through $\text{cl}\,(N_1)$ intersect both D_{11} and D_{12}. Now B is applied to D_2 and N_1 so that at the completion of step 2 we have an open set U_2 whose closure lies in U_1 and four disjoint disks D_{11}, D_{12}, D_{21}, and D_{22} in D where each vertical line through $\text{cl}\,(U_2)$ intersects each of the four disks. When the construction is finished it is clear that a vertical line through $\bigcap_1^\infty \text{cl}\,(U_i)$ will intersect each of the 2^n disks at the nth step. Thus such a line contains an uncountable set of points of S. This contradiction establishes the theorem.

COROLLARY 2.2. *If S is a 2-sphere in E^3 having countable vertical order, then S is locally tame modulo a 1-dimensional subset.*

3. Spheres of vertical order order 5.

The following four lemmas are used to establish the main result (Theorem 3.5).

LEMMA 3.1. *If S has vertical* 5, *then S is locally tame at each point of S that is vertically above or below a point of* $\operatorname{Int} S$; *that is,* $\pi(\operatorname{Int} S) \cap \pi(W(S)) = \emptyset$.

Proof. Let p be a point of S such that $L(p) \cap \operatorname{Int} S \neq \emptyset$. Thus

$L(p)$ must link the boundaries of each of two disjoint disks D_1 and D_2 in S. Let B be a ball lying in $\operatorname{Int} S$ such that each vertical line through B links both $\operatorname{Bd} D_1$ and $\operatorname{Bd} D_2$. If $p \notin D_1 \cup D_2$, then there is a disk D_3 in S such that $p \in \operatorname{Int} D_3$, $D_3 \cap (D_1 \cup D_2) = \varnothing$, and $\pi(D_3) \subset \pi(B)$. Then each vertical line intersecting D_3 also intersects both D_1 and D_2. Since D has vertical order 5 it is clear that D_3 has vertical order 3. Thus D is locally tame at p [7, Theorem 2.3] and so is S.

We may now assume that $p \in \operatorname{Int} D_1$. Let D_1' be a subdisk of D_1 such that $\pi(D_1') \subset \pi(B)$, and, for each $\xi > 0$, let X^ξ be the union of all vertical intervals of diameter no less than ξ in $S \cup \operatorname{Int} S$ that intersect D_1'. It is an exercise to see that X^ξ is closed, and it follows from [6, Theorem 5] that X^ξ is a *-taming set. Now consider a point q in D_1' but not in $X^{1/i}$ for any i. It follows that q lies in no vertical interval in $S \cup \operatorname{Int} S$. Thus $L(q)$ does not pierce D_1' at q, and $L(q)$ must pierce D_1' at some other point t by the choice of B. Let D be a disk in D_1' with t in its interior such that $q \notin D$ and $L(q)$ links $\operatorname{Bd} D$. Then there is a disk D_q in $D_1' - D$ such that $q \in \operatorname{Int} D_q$ and each vertical line through D_q links $\operatorname{Bd} D$. Thus such a line intersects both D and D_2. This means that D_q has vertical order 3 and is tame [7, Theorem 2.3]. Now we see that each point of D_1' either lies in the interior of a tame disk in D_1' or lies in $\bigcup_1^\infty X^{1/i}$. Since a tame disk is a *-taming set and a countable number of tame disks suffice to cover $D_1' - \bigcup_1^\infty X^{1/i}$, we see that D_1' lies in a *-taming set of the form $(\bigcup_1^\infty X^{1/i}) \cup$ (a countable collection of tame disks) in $S \cup \operatorname{Int} S$ [5, Theorem 3.7 and Corollary 3.8]. Thus S is locally tame at p from $E^3 - (S \cup \operatorname{Int} S)$ by the definition of a *-taming set. Since S is locally tame from $\operatorname{Int} S$ [11], it follows that S is locally tame at p.

LEMMA 3.2. *If M is a continuum in $W(S)$ and S is a 2-sphere having vertical order* 5, *then M is tame.*

Proof. We may assume that M is nondegenerate since singleton sets always lie on tame spheres. From the previous lemma it is clear that $\pi(M) \subset \operatorname{Bd} \pi(\operatorname{Int} S)$. Let $U = \operatorname{Int} S$ and let X be the component of $\operatorname{Bd} \pi(U)$ containing $\pi(M)$. We shall show the existence of a space homeomorphism $H: E^3 \rightarrow E^3$ such that $\pi(H(M))$ is either an arc or a simple closed curve. Then $H(M)$ is clearly tame since it lies in $\pi^{-1}(\pi(H(M)))$.

The continuum X can be shown locally connected as in [7, Part 0.2]. Notice that $\pi(U)$ is open and connected. We let U' be the component of $E^2 - X$ containing $\pi(U)$ and for convenience in what follows we assume that U' is bounded. Notice that $\operatorname{cl}(U') = X \cup U'$ since every point of S is accessible from $\operatorname{Int} S$. Let $B^2 = \{(x, y) \mid x^2 + y^2 \leqq 1\} \subset E^2$. There is a continuous function $f: B^2 \rightarrow \operatorname{cl}(U')$ such that

$f \mid \operatorname{Int} B^2$ is a hemeomorphism of $\operatorname{Int} B^2$ onto U' and $f^{-1}(x)$ is a totally disconnected subset of $S^1 = \operatorname{Bd} B^2$ for each $x \in X$ (see [12, p. 186]). Now we follow [7, §§ 2.1, 2.2, 2.3, and 2.4] to find a homeomorphism H of E^3 onto E^3 such that $\pi(H(\pi^{-1}(X) \cap S))$ is a simple closed curve. Thus $\pi(H(M))$ is either an arc or a simple closed curve since $\pi(H(M)) \subset \pi(H(\pi^{-1}(X) \cap S))$.

In the case where U' is not bounded the map f above takes $E^2 - \operatorname{Int} B^2$ onto $\operatorname{cl}(U')$ and causes some notational difficulties when we try to follow [7] as above. However, [7] still serves as an outline and we leave the details to the reader.

LEMMA 3.3. *If M is a nondegenerate continuum in $W(S)$ and S is a 2-sphere having vertical order* 5, *then each point of M is a limit point of $W(S) - M$.*

Proof. Suppose some point $p \in M$ is not a limit point of $W(S) - M$, and choose a disk D on S such that $p \in \operatorname{Int} D$, $\operatorname{Bd} D$ is tame [3], and $D \cap W(S) \subset M$. Let $X = M \cup (\operatorname{Bd} D)$, and let S' be a 2-sphere containing $M \cup D$ that is locally tame modulo X[1]. From Lemma 3.2 we see that X is a taming set [4, Theorem 8.1.6, p. 320]. Thus S' is tame. This is a contradiction and the result follows.

LEMMA 3.4. *If D is a disk in a 2-sphere S, S has vertical order* 5, *$p \in \operatorname{Int} D$, and V is an open subset of E^3 such that $p \in V$ and, for each vertical line L piercing D at a point in V, $L \cap \operatorname{Int} S$ has exactly one component whose closure intersects D, then D is locally tame at p.*

Proof. If $L(p)$ intersects $\operatorname{Int} S$, then the conclusion of Lemma 3.4 follows from Lemma 3.1. Thus we now assume $L(p) \cap \operatorname{Int} S = \varnothing$. Choose a 2-sphere H in the shape of a right circular cylinder such that $p \in \operatorname{Int} H$, $H \cap S \subset D$, $\operatorname{Bd} D \subset \operatorname{Ext} H$, $[L \cap (\operatorname{Int} H)] \cap S = \{p\}$, the top and bottom disks T and D of H lie in $\operatorname{Ext} S$, and each vertical line intersecting H also intersects V.

Let X be a component of $(\operatorname{Int} S) \cap H$, and let $K = \operatorname{Bd} X$. We shall show that $X \cup K$ is a disk by showing that K is a simple closed curve. To show that K is connected it suffices to prove that each simple closed curve J in X bounds a disk in X. Such a curve J cannot be essential on the annulus $H - D \cup T$ since J would link $L(p)$ while $L(p) \subset (\operatorname{Ext} S) \cup S$ and $J \subset \operatorname{Int} S$. Thus J must bound a disk E in $H - D \cup T$. From the hypothesis of Lemma 3.4 it is clear that $E \subset X$. Thus K is connected. The fact that K has vertical order 5 insures that K is arcwise accessible from both its complementary domains in H, and this implies that K is a simple closed curve.

Thus the closure of each component of $(\operatorname{Int} S) \cap H$ is a spanning disk for the 3-cell $C = S \cup \operatorname{Int} S$. There can be at most a countable collection $\{D_1, D_2, \cdots\}$ of these spanning disks since their interiors are pairwise disjoint. The fact that D has vertical order 5 insures that $\{D_i\}$ is a null sequence. We use these spanning disks to construct a 2-sphere S' containing p and lying in $D \cup (\bigcup_1^\infty D_i)$ and in $H \cup \operatorname{Int} H$. From the hypothesis on D we see that the interior of S' is vertically connected; thus S' is tame [7, Main Theorem]. This means that D is locally tame at p.

THEOREM 3.5. *If a 2-sphere S in E^3 has vertical order* 5, *then* $S \cup \operatorname{Int} S$ *is a 3-cell and S is locally tame from* $\operatorname{Ext} S$ *modulo a 0-dimensional set.*

Proof. That $C = S \cup \operatorname{Int} S$ is a 3-cell follows from [11]. It remains to show that the set W of wild points of S is 0-dimensional. Suppose to the contrary that there is a nondegenerate continuum M lying in W. Since C is a 3-cell there is an embedding $g: M \times [0, 1] \to C$ such that $G = g(M \times [0, 1]) \subset \operatorname{Int} S$ and $g(m, 0) = m$ for every $m \in M$. We let $F = g(M \times [0, 1])$, and we note that it follows from Lemma 3.1 that $\pi(M)$ lies in the boundary of $\pi(F)$ in E^2. For the same reason, $\pi(G) \cap \pi(M) = \emptyset$. Let U be a disk in E^2 and let p' be a point of $\operatorname{Int} U$ such that $U \cap (\pi(\operatorname{Bd} F)) \subset \pi(M)$ and $p' \in \pi(M)$. Choose a point p in $M \cap \pi^{-1}(p')$. In the next paragraph we show the existence of a disk E in S with $p \in \operatorname{Int} E$ and $\pi(E) \subset U \cap \pi(F)$.

The difficulty in choosing E is the requirement that $\pi(E) \subset \pi(F)$. If no such E exists there must exist a sequence $\{p_i\}$ of points of $\operatorname{Int} S$ converging to p such that $\pi(p_i) \in U - \pi(F)$ for each i. Using the 0-ULC of $\operatorname{Int} S$ it is easy to select a point $g \in G \subset \operatorname{Int} S$ close enough to p and an integer N large enough that g and p_N are the end points of an arc A in $\operatorname{Int} S$ where $\pi(A) \subset U$. Now $\pi(A)$ contains an arc with one end point a in $\pi(G)$ and the other end point b in $U - \pi(F)$. If this arc is traversed from b to a, then there is a first point f of $\pi(F)$ encountered. This point f clearly belongs to $\operatorname{Bd} \pi(F)$. This contradiction establishes the existence of E.

Now that the existence of E is clear we proceed by using Lemma 3.3 to pick a point q in $E \cap (W - M)$. Let V be an open ball centered at q such that $V \cap S \subset E$ and $V \cap F = \emptyset$. Since $L(q) \cap \operatorname{Int} S = \emptyset$ (see Lemma 3.1) there are open balls B_1 and B_2 centered at points above and below q, respectively, that lie in $(\operatorname{Ext} S) \cap V$. We choose a disk D in $V \cap S$ with $q \in \operatorname{Int} D$ vertically between B_1 and B_2 such that $\pi(D) \subset \pi(B_1) \cap \pi(B_2)$. We shall show that D is locally tame at q to obtain a contradiction to $q \in W$.

In order to apply Lemma 3.4 we must show that if a vertical line L pierces D at a point of V, then $L \cap \operatorname{Int} S$ has exactly one component whose closure intersects D. Suppose to the contrary that for some such line L there are two components X and Y of $L \cap \operatorname{Int} S$ whose closures intersect D. Now $X \cup Y \subset V$ since D lies between B_1 and B_2. Since $L \cap \operatorname{Int} S = \emptyset$ and $\pi(D) \subset \pi(F)$, we see that $L \cap G \neq \emptyset$. Thus $L \cap (\operatorname{Int} S)$ has a third component Z, different from both X and Y because Z lies either above B_1 or below B_2. Now the only way to avoid there being 6 points in $L \cap S$ is for X and Y to share an end point x. In this case there is a point e of $\operatorname{Ext} S$ close enough to x to insure that there are three components of $L(e) \cap \operatorname{Int} S$ with pairwise disjoint closures. Now $L(e) \cap S$ contains 6 points contrary to the hypothesis.

4. **Examples and questions.** One can use a countably infinite null sequence of Fox-Artin [8] "feelers" whose wild points form a dense subset of an arc to see that a vertically countable 2-sphere can have an arc in its wild set. Thus Corollary 2.2 cannot be improved in this direction.

EXAMPLE 4.1. A wild 2-sphere S having vertical order 6 such that $W(S)$ is not 0-dimensional. In Figure 1 we see an embedding of

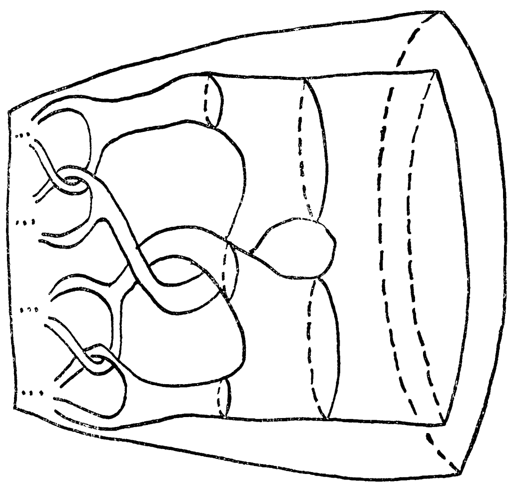

FIGURE 1.

the Alexander Horned Sphere, having vertical order 4, inside a wedge-shaped 3-cell in E^3. We attach a null sequence of such wedges to a right circular cone, as indicated in Figure 2, to obtain the desired example S. Notice that $W(S)$ is the union of a tame simple closed curve with countably infinite number of tame Cantor sets. Furthermore, every point of S is a piercing point of S.

In Example 4.1 we see that every nondegenerate continuum in $W(S)$ is tame.

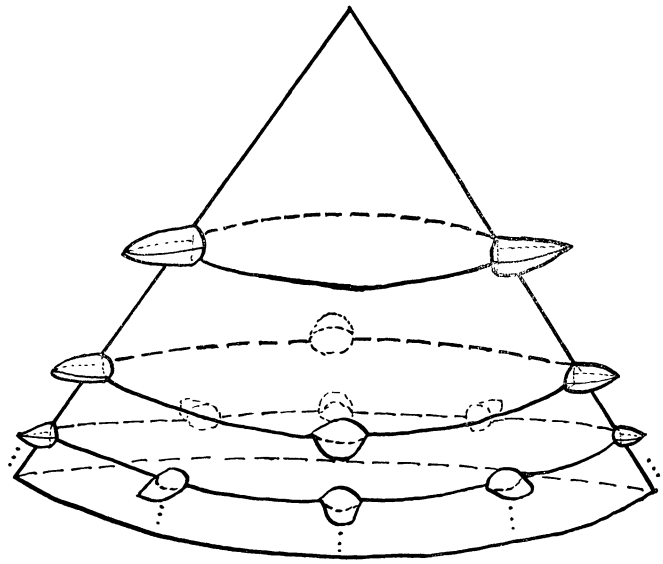

FIGURE 2.

Question 4.2. If S is a 2-sphere in E^3 having finite vertical order, then must every nondegenerate continuum in $W(S)$ be tame?

We do not know the answer to Question 4.2 even when "vertical order n" replaces "finite vertical order", unless $n \leqq 5$ where Theorem 3.5 applies. The proof of Lemma 3.2 shows an affirmative answer to Question 4.2 if it is also known that $\pi(W(S)) \cap \pi(\operatorname{Int} S) = \varnothing$.

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