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THE LOCAL COMPACTNESS OF υX

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Necessary and sufficient conditions are given for the local compactness of the Hewitt realcompactification υX of a completely regular Hausdorff space X; the conditions are expressed in terms of the space X alone. In addition, the local compactness of other extensions is considered.

Introduction. There has been much recent interest in determining conditions on a completely regular Hausdorff space X that are equivalent to the local compactness of its Hewitt realcompactification υX. This interest stems primarily from the fact that the seemingly artificial hypothesis "υX is locally compact" enters quite naturally into the examination of the relation $\upsilon X \times \upsilon Y = \upsilon(X \times Y)$. Apparently the only known condition equivalent to the local compactness of υX is one discussed by Comfort in [1] and [2]. As remarked by Comfort, the condition is not on X alone, but involves υX essentially in its statement.

In the present paper a condition on X is given which is equivalent to the local compactness of υX (Theorem 2.7) and a number of known results are obtained as corollaries of this characterization theorem. Another characterization (Theorem 2.3) is given of the local compactness of υX in terms of real maximal ideals.

It was shown by Comfort in [1] and [2] that the local pseudocompactness of X plays an important role in connection with the local compactness of υX. The precise role is established below, where it is shown that the local pseudocompactness of X is equivalent to the local compactness of the extension ηX of X constructed by Johnson and Mandelker in [9]. In addition a characterization is given of those spaces for which the extension ψX constructed by Johnson and Mandelker is locally compact.

Our attention will be restricted entirely to completely regular Hausdorff spaces. The terminology and notation of [4] will be used without further comment.

Given $f \in C(X)$ the symbols $N(f)$ and $S(f)$ represent respectively $\{x \in X: f(x) \neq 0\}$ and $\mathrm{cl}_X \{x \in X: f(x) \neq 0\}$; these sets are called the *cozero set* and the *support* of f. If A and B are subsets of X, write $A \ll B$ if A is completely separated from $X - B$. We shall frequently apply [4, 1.15] to construct additional separating zero sets when $A \ll B$.

The symbol $\boldsymbol{M}^p$ will denote the maximal ideal in $C(X)$ which corresponds to the point p of βX, and $\mathscr{M}^p$ will denote the corresponding z-ultrafilter (written $\boldsymbol{A}^p$ in [4]). Similarly $\boldsymbol{O}^p$ represents the ideal defined in [4, 7.12] and $\mathscr{O}^p$ the corresponding z-filter.

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2. **The local compactness of υX.** The family $C_\psi(X)$ of functions with pseudocompact support, discussed at length in [9] and [11], plays the major role in our condition for local compactness of υX. This is to be expected since by 2.1(d) below the isomorphism $f \to f^\upsilon$ is an isomorphism of $C_\psi(X)$ with $C_K(\upsilon X)$. We write $\mathscr{C}_\psi(X)$ for the corresponding collection of zero sets.

The following results are either found in [4] or may be established using results from [4].

2.1. (a) $Z(f^\upsilon) = \mathrm{cl}_{\upsilon X} Z(f)$.

(b) $N(f^\upsilon) = \mathrm{int}_{\upsilon X} (\upsilon X - Z(f))$.

(c) $S(f^\upsilon) = \mathrm{cl}_{\upsilon X} S(f) = \mathrm{cl}_{\upsilon X} N(f)$.

(d) $S(f^\upsilon)$ is compact if and only if $S(f)$ is pseudocompact.

The following result from [11] is frequently useful.

2.2. Every support in a pseudocompact space is pseudocompact.

Since the isomorphism $f \to f^\upsilon$ induces a bijection between real maximal ideals in $C(X)$ and fixed ideals in $C(\upsilon X)$, the following result is an immediate consequence of [4, 4D3].

THEOREM 2.3. *The space υX is locally compact if and only if $C_\psi(X)$ is not contained in any real maximal ideal.*

We turn now toward a condition expressible in terms of X alone. For each $\varepsilon > 0$ and each $f \in C(X)$, define $U_\varepsilon(f) = \{x \in X : |f(x)| \geqq \varepsilon\}$; this is a zero set in X. The following results are essential for our characterization theorem.

2.4. (a) If $\varepsilon > \delta > 0$ then $N(f) \gg U_\delta(f) \gg U_\varepsilon(f.)$

(b) If $p \in \beta X$ and $f \in C^*(X)$ then $f^\beta(p) = 0$ if and only if $U_\varepsilon(f) \notin \mathscr{M}^p$ for each $\varepsilon > 0$. (Also $\mathscr{M}^p$ may be replaced by $\mathscr{O}^p$ in this condition).

(c) For each $f \in C(X)$, $U_\varepsilon(f^\upsilon) = \mathrm{cl}_{\upsilon X} U_\varepsilon(f)$.

Proof. The proofs of (a) and (c) are straightforward.

(b) For any $\varepsilon > 0$, if $U_\varepsilon(f) \in \mathscr{M}^p$, then $p \in \mathrm{cl}_{\beta X} U_\varepsilon(f)$ and $f(p) \geqq \varepsilon$, a contradiction. Hence $U_\varepsilon(f) \notin \mathscr{M}^p$.

Conversely let $U_\varepsilon(f) \notin \mathscr{O}^p$ for each $\varepsilon > 0$. For every $\varepsilon > 0$ we have $\{x \in X : |f(x)| \leqq \varepsilon\} \in \mathscr{M}^p$ by [4, 7.12(b)]; hence $f^\beta(p) \leqq \varepsilon$. Thus $f^\beta(p) = 0$.

Let $U(X)$ represent the set of units in $C(X)$; by [4, 1.12], these are the functions with empty zero set. Clearly the image of $U(X)$

under the isomorphism $f \to f^{\upsilon}$ is $U(\upsilon X)$.

A *scale* on X is a function ε: $f \to \varepsilon(f)$ from $U(X)$ to the positive real numbers. If ε is a scale, put $\mathscr{E}(\varepsilon) = \{U_{\varepsilon(f)}(f): f \in U(X)\}$.

THEOREM 2.5. *A z-ultrafilter is real if and only if it contains $\mathscr{E}(\varepsilon)$ for some scale ε.*

Proof. Let ε be a scale on X and let $\mathscr{M}^p$ be a hyper-real z-ultrafilter. By [4, 8.8] there is a bounded unit f in $C(X)$ with $f^{\beta}(p) = 0$; hence $U_{\varepsilon(f)}(f) \notin \mathscr{M}^p$. Thus $\mathscr{M}^p$ does not contain $\mathscr{E}(\varepsilon)$.

Let $p \in \upsilon X$. For any $f \in U(X)$ put $\varepsilon(f) = |f^{\upsilon}(p)|$. Since f^{υ} is a unit of $C(\upsilon X)$, $\varepsilon(f) > 0$, and thus ε is a scale on X. Since $p \in U_{\varepsilon(f)}(f^{\upsilon}) = \mathrm{cl}_{\upsilon X} U_{\varepsilon(f)}(f) \subset \mathrm{cl}_{\beta X} U_{\varepsilon(f)}(f)$, it follows that $U_{\varepsilon(f)}(f) \in \mathscr{M}^p$. Thus $\mathscr{E}(\varepsilon) \subset \mathscr{M}^p$.

COROLLARY 2.6. *A filter $\mathscr{F}$ on X is contained in a real z-ultrafilter if and only if every member of $\mathscr{F}$ meets every member of $\mathscr{E}(\varepsilon)$ for some scale ε.*

THEOREM 2.7. *The space υX is locally compact if and only if X satisfies the following condition*: (RL). *For every scale ε there are $f_1, \cdots, f_k \in U(X)$ and $g \in C_{\psi}(X)$ such that $Z(g) \cap (\bigcap_{i=1}^{k} U_{\varepsilon(f_i)}(f_i)) = \phi$.*

Proof. Let X satisfy (RL). For any $p \in \upsilon X$, by 2.5 there is a scale ε on X such that $\mathscr{E}(\varepsilon) \subset \mathscr{M}^p$. By (RL), $\mathscr{C}_{\psi}(X) \not\subset \mathscr{M}^p$; thus υX is locally compact by 2.3.

Suppose υX is locally compact and ε is a scale on X. By 2.3, $\mathscr{C}_{\psi}(X) \not\subset \mathscr{M}^p$ for any $p \in \upsilon X$. Thus, by 2.5, $\mathscr{C}_{\psi}(X) \cup \mathscr{E}(\varepsilon)$ lacks the finite intersection property; that is, condition RL is satisfied.

REMARK 2.8. It is clear that we need consider in condition RL only those scales for which the family $\mathscr{E}(\varepsilon)$ has the finite intersection property, since the condition is trivially fulfilled when some finite subfamily of $\mathscr{E}(\varepsilon)$ has empty intersection. The condition is also fulfilled trivially when $\mathscr{C}_{\psi}(X)$ contains a unit of $C(X)$, and this occurs precisely when X is pseudocompact.

Certainly if $\mathscr{C}_{\psi}(X)$ lacks the countable intersection property then it is not contained in a real z-ultrafilter. It will now be shown that $\mathscr{C}_{\psi}(X)$ lacks the property precisely when υX is locally compact and σ-compact. Our condition is shown to be related to one given by Hager in [7].

THEOREM 2.9. *The following are equivalent for a space X.*

(a) *υX is locally compact and σ-compact.*

(b) *$\mathscr{C}_\psi(X)$ lacks the countable intersection property.*

(c) *(Hager) $X = \bigcup_{n=1}^\infty A_n$, where each A_n is pseudocompact and $A_n \ll A_{n+1}$ for each n.*

Proof. (a) implies (c). If υX is locally compact and σ-compact then [3, XI, 7.2] $\upsilon X = \bigcup_{n=1}^\infty U_n$, where each U_n is open and has compact closure and $\mathrm{cl}_{\upsilon X} U_n \subset U_{n+1}$ for each n. By [4, 3.11(a)], $U_n \ll U_{n+1}$. Setting $A_n = \mathrm{cl}_X (U_n \cap X)$ it follows from [2, 4.1] that each A_n is pseudocompact.

(c) implies (b). Let $X = \bigcup_{n=1}^\infty A_n$, with A_n pseudocompact and $A_n \ll A_{n+1}$ for each n. Choose for each n a function f_n such that $A_n \subset N(f_n) \subset S(f_n) \subset A_{n+1}$. Then, by [2.2] each $f_n \in C_\psi(X)$, and clearly $\bigcap_{n=1}^\infty Z(f_n) = \phi$.

(b) implies (a). Let $\bigcap_{n=1}^\infty Z(f_n) = \phi$, where $f_n \in C_\psi(X)$ for each n. Then, by 2.1(a) and [4, 8.7], $\bigcap_{n=1}^\infty Z(f_n^\upsilon) = \phi$, and thus $\bigcup_{n=1}^\infty S(f_n^\upsilon) = \upsilon X$. By 2.1(d) each $S(f_n^\upsilon)$ is compact, thus υX is σ-compact. By 2.3, υX is locally compact.

Comfort [2, 4.6] gives another condition (C) which is equivalent to the local compactness of υX. A direct proof of the equivalence of (C) with the condition of Theorem 2.3 will now be given.

2.10. $\mathscr{C}_\psi(X)$ is not contained in any real z-ultrafilter if and only if: (C) For each $p \in \upsilon X$ there exist pseudocompact subsets A and B of X such that $p \in \mathrm{cl}_{\upsilon X} A$ and $A \ll B$.

Proof. If X satisfies condition (C) and $\mathscr{M}^p$ is a real maximal ideal then there are pseudocompact sets A and B and functions $f, g \in C(X)$ such that $p \in \mathrm{cl}_{\upsilon X} A$ and $A \subset Z(f) \subset N(g) \subset B$. It follows from 2.2 that $g \in C_\psi(X)$. Since $p \in \mathrm{cl}_{\upsilon X} A$ then $f \in M^p$, and thus $g \notin M^p$. Thus $C_\psi(X) \not\subset M^p$.

Conversely, for any $p \in \upsilon X$ there is $f \in C_\psi(X)$ and $g \in \mathscr{M}^p$ such that $Z(f) \cap Z(g) = \phi$; thus $Z(g) \ll N(f)$ and there exist $h, k \in C(X)$ such that $Z(g) \subset N(k) \subset Z(h) \subset N(f)$. Put $A = S(k)$ and $B = S(f)$. Since $A \subset S(f)$, it follows from 2.2 that A is pseudocompact. Also $p \in \mathrm{cl}_{\beta X} Z(g)$, since $g \in \mathscr{M}^p$, so $p \in \mathrm{cl}_{\upsilon X} A$. Finally, $A \subset Z(h)$ and $X - B \subset Z(f)$, with $Z(h) \cap Z(f) = \phi$, so $A \ll B$. Thus condition (C) is satisfied.

3. The local pseudocompactness of X. The space X is *locally pseudocompact* if every point has a pseudocompact neighborhood. Locally pseudocompact spaces are discussed in [1] and [2]. The results in this section clarify the relationship between the local pseudocompactness of X and the local compactness of υX.

3.1. The space X is locally pseudocompact if and only if $C_\psi(X)$ is not contained in any fixed maximal ideal.

Proof. Let X be locally pseudocompact. Then any point x in X has a pseudocompact neighborhood A. Therefore, there is $f \in C(X)$ with $x \in N(f) \subset A$. Thus $x \notin Z(f)$ and by 2.2, the set $S(f)$ is pseudocompact, so $f \in C_\psi(X)$. Therefore, $C_\psi(X)$ is not contained in any fixed maximal ideal. Conversely suppose $C_\psi(X)$ is contained in no fixed maximal ideal. Then for each $x \in X$ there is $f \in C_\psi(X)$ with $x \in N(f)$, and thus $S(f)$ is a pseudocompact neighborhood of x.

For any space Y denote by $L(Y)$ the set of all points of Y that have a compact neighborhood in Y; i.e., $L(Y) = Y - R(Y)$, where $R(Y)$ is as defined in [8, p. 87]. Clearly $L(Y)$ is locally compact. For any space X define $\kappa X = \{p \in \beta X\colon \mathscr{C}_\psi(X) \not\subset \mathscr{M}^p\}$; equivalently, $\kappa X = \beta X - \theta(\mathscr{C}_\psi(X))$, where $\theta(\mathscr{C}_\psi(X))$ is as defined in [4, 70].

THEOREM 3.2. *For each space* X, $\kappa X = L(\upsilon X) = \operatorname{int}_{\beta X} \upsilon X$, *and thus* κX *is locally compact.*

Proof. The relation $L(\upsilon X) = \operatorname{int}_{\beta X} \upsilon X$ follows from [4, 3.15(b)]. By [9, 3.1], $\beta X - \kappa X = \theta(\mathscr{C}_\psi(X)) = \operatorname{cl}_{\beta X}(\beta X - \upsilon X)$, so $\kappa X = \operatorname{int}_{\beta X} \upsilon X$.

COROLLARY 3.3. *The space* X *is locally pseudocompact if and only if* $X \subset \kappa X$. *In this case* κX *is the largest locally compact space between* X *and* υX.

The following result is due to Comfort ([1] and [2]).

COROLLARY 3.4. *The space* X *is locally pseudocompact if and only if there is a locally compact space* Y *between* X *and* υX.

4. **Functions with small support.** Another ideal in $C(X)$ plays an important role in connection with local compactness. Before discussing this ideal, the class of *small* sets will be examined, where a set $A \subset X$ is *small* if any zero set contained in A is compact.

4.1. The set A is small if and only if every zero set that intersects $X - A$ in a compact set is compact.

Proof. Certainly in the latter condition holds then A is small. Now suppose A is small and Z is a zero set such that $Z \cap (X - A)$ is compact. If α is a cover of Z by cozero sets then finitely many of the cozero sets cover $Z \cap (X - A)$. Their union is a cozero set $N(g)$ and $Z(g) \cap Z$ is compact, since it is a zero set. Thus, finitely many additional members of α can be chosen to complete the choice of a finite subcover of Z.

4.2. The finite union of small cozero sets is small.

Proof. Let $N(f)$ and $N(g)$ be small, and let $Z(h) \subset N(f) \cup N(g)$. Then $Z(h) \cap (X - N(g)) = Z(h) \cap Z(g) \subset N(f)$. Since $N(f)$ and $N(g)$ are small it follows from 4.1 that $Z(h)$ is compact.

A function $f \in C(X)$ has *small support* if and only if $N(f)$ is small. Equivalently, according to [4, 4E2], the function f belongs to every free maximal ideal in $C(X)$. It is clear from this latter characterization that the collection $C_s(X)$ of functions with small support is an ideal; this can also be shown directly from 4.2.

REMARK 4.3. The term small *support* may be misleading; the condition applies to $N(f)$ and not $S(f)$. The ideal $C_s(X)$ contains the ideal $C_K(X)$ [4, 4D5 and 4E2]. Spaces for which $C_K(X) = C_s(X)$ are called *μ-compact* and are fully discussed in [9] and [11]; in [9] the ideal $C_s(X)$ is called $I(X)$.

The following result should be compared with [4, 4D1 and 4D3], as well as with Theorem 2.3.

THEOREM 4.4. *The space X is locally compact if and only if $C_s(X)$ is not contained in any fixed maximal ideal.*

Proof. If X is locally compact then $C_K(X)$ is not contained in any fixed maximal ideal; since $C_K(X) \subset C_s(X)$ then $C_s(X)$ is not contained in any fixed maximal ideal.

Now if $C_s(X)$ is not contained in any fixed maximal ideal then for each $x \in X$ there is $f \in C_s(X)$ such that $x \in N(f)$. Thus, there is a zero set neighborhood of x such that $Z \subset N(f)$, and it follows that Z is compact. Thus X is locally compact.

REMARK 4.5. One sense in which Theorem 4.4 is more appropriate than the characterization [4, 4D3] of local compactness is when the generalization to T_1 spaces and T_1 compactifications is considered. In [5] the *compact small* sets of a space X are defined as those sets such that any closed set contained in A is compact. It is shown there that the spaces for which each point has a compact-small neighborhood are appropriate generalizations of locally compact completely regular spaces. It is shown in [6] that results analogous to Theorem 4.4 hold for locally compact-small spaces.

5. The local compactness of ηX and ψX. Two additional

subspaces of υX are of special interest in connection with local compactness. Mandelker defines (in [11]) a space X to be *ψ-compact* if $C_K(X) = C_\psi(X)$, and Mandelker and Johnson define (in [9]) a space X to be *η-compact* if $C_s(X) = C_\psi(X)$; they construct extensions ηX and ψX as the intersections respectively of the η- compact and the ψ-compact subspaces of βX.

The following results are shown in [9].

5.1. (a) $\eta X = X \cup \mathrm{int}_{\beta X}\, \upsilon X$.

(b) $\psi X - X = \bigcup_{f \in C_\psi(X)} [S(f^\upsilon) - S(f)]$.

The next results are immediate from 5.1(a) and Theorem 3.2.

5.2. (a) $\kappa X = \mathrm{int}_{\beta X}\, \eta X = \mathrm{int}_{\beta X}\, \psi X$.

(b) $\eta X = X \cup \kappa X$

The next theorem characterizes the local compactness of ηX. The proof is immediate from 5.2 and Corollary 3.3.

THEOREM 5.3. *The space ηX is locally compact if and only if X is locally pseudocompact.*

THEOREM 5.4. *The space ψX is locally compact if and only if X is locally pseudocompact and $\mathscr{C}_\psi(X)$ is round.*

Proof. Let ψX be locally compact. Then X is locally pseudocompact by Corollary 3.4. Also ψX is open in βX, so $\beta X - \psi X$ is closed. By [9, 5.3], $C_\psi(X) = M^{\beta X - \psi X}$ and thus $\beta X - \psi X$ is round; hence by [10, 4.2] $\mathscr{C}_\psi(X)$ is round.

Let X be locally pseudocompact and let $\mathscr{C}_\psi(X)$ be round. By 3.3 and 5.2(a), $X \subset \kappa X \subset \psi X$. Let $p \in \upsilon X - X$. Using 5.1(b) choose $f \in C_\psi(X)$ so that $p \in S(f^\upsilon)$; since $\mathscr{C}_\psi(X)$ is round there is $g \in C_\psi(X)$ with $Zg \ll Zf$. By [4, 7.14], $\mathrm{cl}_{\beta X}\, Z(f)$ is a neighborhood of $\mathrm{cl}_{\beta X}\, Z(g)$, and thus there is a compact set F with $\beta X - \mathrm{cl}_{\beta X}\, Z(f) \subset F \subset \beta X - \mathrm{cl}_{\beta X}\, Z(g)$. Since $N(f) \subset \beta X - \mathrm{cl}_{\beta X}\, Z(f)$ and (by [9, 3.1]) $\beta X - \upsilon X \subset \mathrm{cl}_{\beta X}\, Z(g)$, it follows that $p \in S(f^\upsilon) \subset F \subset \beta X - \mathrm{cl}_{\beta X}\, Z(g) \subset \upsilon X$, and hence

$$p \in \mathrm{int}_{\beta X}\, \upsilon X = \kappa X .$$

Thus $\psi X = \kappa X$ and so ψX is locally compact.

It is instructive in the use of scales to deduce directly from Condition (RL) that X is locally pseudocompact and $\mathscr{C}_\psi(X)$ is round.

5.5. If X satisfies Condition (RL) then X is locally pseudocompact and $\mathscr{C}_\psi(X)$ is round.

Proof. The first paragraph of the proof of Theorem 2.7 shows that X will be locally pseudocompact. Now suppose $f \in C_\psi(X)$. Then

$S(f)$ is pseudocompact, and it follows that every $h \in U(X)$ is bounded aways from zero on $N(f)$. Choose a scale ε so that $|h| \geqq \varepsilon(h)$ on $N(f)$, for each $h \in U(X)$. Since (RL) is satisfied, there are $h_1, \cdots, h_k \in U(X)$ and $g \in C_\psi(X)$ such that $Zg \cap (\cap U_{\varepsilon(h_i)}(h_i)) = \varnothing$. Clearly

$$Z(g) \subset \bigcup_{i=1}^{k} \{x \in X: |h_i(x)| < \varepsilon(h_i)\} \subset Z(f) .$$

It follows that $g \ll f$.

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