PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 2, 1974

WHITEHEAD GROUPS OF TWISTED FREE ASSOCIATIVE ALGEBRAS

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Let R be an associative ring with identity and X a set of noncommuting variables $\{x_\lambda\}_{\lambda \in \Lambda}$. Let $R\{X\}$ be the free associative algebra on X over R. Then S. Gersten has shown that if $K_1R \rightarrow K_1R[t]$ is an isomorphism, where $R[t]$ is the polynomial extension of R, then $K_1R \rightarrow K_1R\{X\}$ is an isomorphism.

The purpose of this paper is to extend the result of Gersten to twisted free associative algebras.

Let R be an associative ring with identity and X a set of noncommuting variables $\{x_\lambda\}_{\lambda \in \Lambda}$ and $\mathfrak{a} = \{\alpha_\lambda\}_{\lambda \in \Lambda}$ a set of automorphisms α_λ of R. The $\mathfrak{a}$-twisted free associative algebra on X over R, denoted by $R_\mathfrak{a}\{X\}$, is defined as follows:

Additively, $R_\mathfrak{a}\{X\} = R\{X\}$ so that its elements are finite linear combinations of words $w(x_\lambda)$ in x_λ with coefficients in R.

If $w(x_\lambda) = x_{\lambda_1} \cdots x_{\lambda_k}$ is a word in x_λ, we denote the automorphism $\alpha_{\lambda_1} \cdots \alpha_{\lambda_k}$ by $w(\alpha_\lambda)$.

Multiplication in $R_\mathfrak{a}\{X\}$ is given by:

$$(rw(x_\lambda))(r'w'(x_\lambda)) = rw(\alpha_\lambda)^{-1}(r')w(x_\lambda)w'(x_\lambda) ,$$

for any $rw(x_\lambda), r'w'(x_\lambda) \in R_\mathfrak{a}\{X\}$.

In particular, if $X = \{t\}$ and $\mathfrak{a} = \{\alpha\}$, then $R_\mathfrak{a}\{X\}$ is just the α-twisted polynomial ring $R_\alpha[t]$.

We shall consider $R_\mathfrak{a}\{X\}$ as an R-ring with augmentation ε_X: $R_\mathfrak{a}\{X\} \rightarrow R$ defined by $\varepsilon_X(x_\lambda) = 0$ for each $x_\lambda \in X$. Denoted by $\bar{K}_1R_\mathfrak{a}\{X\}$ the cokernel of the homomorphism $i_*: K_1R \rightarrow K_1R_\mathfrak{a}\{X\}$ induced by the inclusion $i: R \rightarrow R_\mathfrak{a}\{X\}$. Note that the augmentation ε_X induces a homomorphism $\varepsilon_{X*}: K_1R_\mathfrak{a}\{X\} \rightarrow K_1R$ which splits i_*.

Let $W(X)$ be the set of all the words $w(x_\lambda)$ in x_λ. For each $w(x_\lambda)$ in $W(X)$, let β_w be the automorphism $w(\alpha_\lambda)$, h_{β_w} the homomorphism of $R_{\beta_w}[t]$ into $R_\mathfrak{a}\{X\}$ defined by $h_{\beta_w}(t) = w(x_\lambda)$ and $\bar{h}_{\beta_w}$ the homomorphism of $\bar{K}_1R_{\beta_w}[t]$ into $\bar{K}_1R_\mathfrak{a}\{X\}$ induced by h_{β_w}. Then our main result is:

THEOREM 1. *The group $\bar{K}_1R_\mathfrak{a}\{X\}$ is generated by the homomorphic images of $\bar{K}_1R_{\beta_w}[t]$ under $\bar{h}_{\beta_w}$ and $w(x_\lambda)$ runs over $W(X)$.*

As a consequence, we have:

THEOREM 2. (*Twisted Case of Gersten's Theorem*). *If $K_1R \rightarrow K_1R_{\beta_w}[t]$ is an isomorphism for each β_w, then $K_1R \rightarrow K_1R_\mathfrak{a}\{X\}$ is an* *isomorphism.*

Now, let A be an invertible matrix over $R_\alpha\{X\}$. By Higman's trick (cf. [4]), we can make A equivalent in $K_1R_\alpha\{X\}$ to

$$B = B_0 + B_1x_1 + \cdots + B_nx_n ,$$

where $x_1, \cdots, x_n$ are distinct elements of X and $B_i(i = 0, 1, \cdots, n)$ are $m \times m$ matrices over R for some integer m. By applying the homomorphism ε_{X^*} to B, we deduce that B_0 is invertible. Hence A can be made equivalent in $\bar{K}_1R_\alpha\{X\}$ to

$$N = I + N_1x_1 + \cdots + N_nx_n , \tag{1}$$

where $N = B_0^{-1}B$ and $N_i = B_0^{-1}B_i(i = 1, \cdots, n)$.

The inverse of this matrix N can be written explicitly in the ring of formal power series. Since this inverse exists in $R_\alpha\{X\}$, all but a finite number of the power series coefficients are zero. That is, if

$$M = M_0 + M_1x_1 + \cdots + M_nx_n + \sum_{i,j=1}^{n} M_{i,j}x_ix_j + \cdots$$

is a matrix over $R_\alpha\{X\}$, where all $M_i, M_{i,j}, \cdots$ are matrices over R, such that $MN = NM = I$, then there is an integer $K > 0$ such that $M_{i_1,i_2,\cdots,i_k} = 0$ for all $k > K$, where $i_1, i_2, \cdots, i_k$ run over $1, \cdots, n$ respectively. From $NM = I$, we get, by equating coefficients of monomials in the x's, the following relations:

$$\begin{aligned}
&M_0 = I; \\
&M_i = - N_i && (i = 1, \cdots, n) ; \\
&M_{i,j} = N_i\alpha_i^{-1}(N_j) && (i, j = 1, \cdots, n) ; \\
&\vdots \\
&M_{i_1,i_2,\cdots,i_l} = (-1)^l N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}) \\
&\qquad (i_1, i_2, \cdots, i_l = 1, \cdots, n) .
\end{aligned}$$

Hence, for all $k > K$,

$$N_{i_1}\alpha_{i_1}^{-1}(N_{i_1}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{k-1}}^{-1})(N_{i_k}) = 0 . \tag{2}$$

Let us call a matrix P over R β-twisted nilpotent (β is any automorphism of R) if there exists an integer $k > 0$ such that

$$P\beta^{-1}(P) \cdots \beta^{-(k-1)}(P) = 0 .$$

Hence, it follows from (2) that each $N_i(i = 1, \cdots, n)$ in (1) is α_i-twisted nilpotent.

Our next lemma is the key to the main result:

LEMMA 3. *The matrix N in* (1) *is a product of matrices of the form $I + Pw(x_1, \cdots, x_n)$, where P is an $w(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent matrix over R. ($w(x_1, \cdots, x_n)$ denotes a word in $x_1, \cdots, x_n$.)*

Proof. Recall from (1) and (2) that each $N_i (i = 1, \cdots, n)$ in (1) is α_i-twisted nilpotent. Consider

$$I + Q = (I - N_1 x_1) \cdots (I - N_n x_n) N .$$

Then Q is of the form $\sum_j Q_j s_j$, where s_j is a monomial of degree at least two in the $x_1, \cdots, x_n$. In fact, if $s_j = x_{i_1} x_{i_2} \cdots x_{i_l} (l \geqq 2)$, then

$$(3) \qquad Q_j = \pm N_{i_1} \alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}) .$$

Hence, for $k > K/2$,

$$Q_j \beta^{-1}(Q_j) \cdots \beta^{-(k-1)}(Q_j) = 0 ,$$

for each j, where β is an automorphism obtained by replacing the x_i in s_j by α_i respectively. That is, Q_j is $s_j(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent for each j. Now, consider

$$I + Q' = \prod_j (I - Q_j s_j)(I + Q) .$$

Then Q' is of the form $\sum_\sigma Q'_\sigma y_\sigma$, where each y_σ is a monomial of degree at least four in the $x_1, \cdots, x_n$ and for $l \geqq 4$, Q'_σ is of the form as given on the right hand side of (3). Thus, for $k > K/4$,

$$Q'_\sigma \gamma^{-1}(Q'_\sigma) \cdots \gamma^{-(k-1)}(Q'_\sigma) = 0 ,$$

for each σ, where γ is an automorphism obtained by replacing the x_i in y_σ by α_i respectively. That is, Q'_σ is $y_\sigma(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent for each σ.

Left multiplying $I + Q'$ by $\prod_\sigma (I - Q'_\sigma y_\sigma)$, and repeating the above argument, we will finally arrive, after a finite steps (because of the finite bound K and condition (2)), at the conclusion that

$$\prod (I + Pw(x_1, \cdots, x_n)) \cdot N = I ,$$

where P is an $w(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent matrix over R and $w(x_1, \cdots, x_n)$ is a word in $x_1, \cdots, x_n$.

This completes the proof.

The above discussions are modifications of those given in [3] and ([1], p. 647) for (untwisted) free associative algebras; and the following result is already contained in the above proof (also cf. [2]).

LEMMA 4. *For any automorphism β of R, $\bar{K}_1 R_\beta[t]$ is generated by the elements of the form $I + Pt$, where P is an β-twisted nilpotent matrix over R.*

Proof of Theorem 1. It follows immediately from Lemmas 3 and 4.

REFERENCES

1. H. Bass, *Algebraic K-Theory*, W. A. Benjamin, Inc., New York, 1968.
2. H. Bass, A. Heller, and R. G. Swan, *The Whitehead group of a polynomial extension*, Publ. I. H. E. S. No. 22, (1964), 61–79.
3. S. Gersten, *Whitehead groups of free associative algebras*, Bull. Amer. Math. Soc., **71** (1965), 157–159.
4. G. Higman, *The units of group rings*, Proc. London Math. Soc., **46** (1940), 231-248.

Received October 18, 1972. This research was supported in part by a postgraduate fellowship of the National Research Council of Canada. It contains parts of the results from the author's doctoral thesis at the University of British Columbia written under the direction of Professor E. Luft. The author is most indebted to Professor E. Luft, and to Professor K. Y. Lam for their valuable suggestions and encouragement during the preparation of the thesis.

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