PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 1, 1974

DETERMINING KNOT TYPES FROM DIAGRAMS OF KNOTS

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The word of knot and the characteristics of its double points, both of which may be read from the diagram of knot, are used to give necessary and sufficient conditions for two (oriented) knots to belong to the same (oriented) knot type.

1. Introduction. A knot is a circle imbedded as a polygon in 3-dimensional space $\boldsymbol{R}^3$. Two knots K and L are called *equivalent* if there is an autohomeomorphism h of $\boldsymbol{R}^3$ such that $h(K) = L$. Each equivalence class is called a *knot type*. For oriented knots a stronger equivalence may be defined: Two oriented knots K and L are called *O-equivalent* if there is an orientation preserving autohomeomorphism h of $\boldsymbol{R}^3$ such that h maps K onto L so their orientations match. In this case each equivalence class is called an *oriented knot type*.

In [8] D. E. Penney uses the diagram of a knot to define a "word" for the knot and obtain sufficient conditions for two knots to belong to the same knot type. Penney's results have been generalized by L. B. Treybig in [12] where the concept of the "boundary collection" of a knot is used to give necessary conditions for two knots to belong to the same knot type. The purpose of this paper is to use the word of a knot and the characteristics of its double points, both of which may be read from the diagram of a knot, to give necessary and sufficient conditions for two (oriented) knots to belong to the same (oriented) knot type.

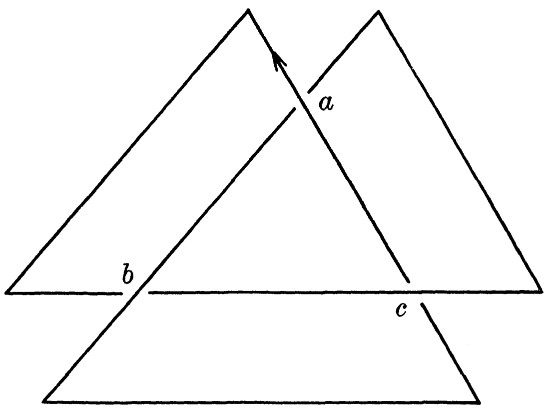
The preliminaries needed for our main results are given in §2. In §3 a relationship between the word of a knot and the characteristics of its double points is derived. This relationship is used to obtain Theorem 3.4 which yields sufficient conditions for two oriented knots to belong to the same oriented knot type. In §4 the following two principal results are obtained: Theorem 4.3 states that two prime knots K and L are equivalent iff there exists a certain finite sequence of words relating a word of K to a word of L. Theorem 4.4 states that two oriented knots K and L are O-equivalent iff there exists a certain finite sequence of words and characteristics relating the word and characteristic of K to those of L.

We remark that in each of Theorems 4.3 and 4.4 the sufficiency part follows from the results developed in §3 while the necessity is an immediate consequence of the classical work of Alexander and Briggs [1].

In §5 the group of a prime word is defined and this is used in

Theorem 5.1 to give necessary and sufficient conditions for a group to be a knot group.

2. Preliminaries and basic lemma. Let K be an oriented knot in regular position with respect to the parallel projection p from R^3 onto the xy-plane R^2. Suppose K has n double points $d_1, \cdots, d_n$ and the corresponding overcrossing (undercrossing) points are denoted by $o_1, \cdots, o_n(u_1, \cdots, u_n)$. With each $o_i(u_i)$ we associate the syllable $d_i(d_i^{-1})$. The oriented knot K determines a cyclically ordered sequence of $2n$ crossing points. By arbitrarily designating one crossing point as the first and replacing each point in the resulting sequence by its associated syllable we obtain a *word* for K (Fig. 1). Words $x_1^{a_1} \cdots x_{2m}^{a_{2m}}$ and $y_1^{b_1} \cdots y_{2n}^{b_{2n}}$ are defined to be *equivalent* if $m = n$ and there is a cyclic permutation α of $1, \cdots, 2m$ such that $x_i = x_j$ iff $y_{\alpha(i)} = y_{\alpha(j)}$ and $a_i = b_{\alpha(j)}$, $a_j = b_{\alpha(j)}$. A nonempty word of a knot is called *prime* if it contains no proper segment S such that $s \in S$ iff $s^{-1} \in S$. (For instance, $ab^{-1}ca^{-1}bc^{-1}$ is prime but $xab^{-1}ca^{-1}bc^{-1}x^{-1}$ is not.)



$ab^{-1}\, ca^{-1}\, bc^{-1}$

FIGURE 1

We remark that the results of [6] or [10] can be used to give necessary and sufficient conditions for a given word to be the word associated with the diagram of an oriented knot.

To each double point d of a diagram of an oriented knot we

$\varepsilon(d) = 1$ $\varepsilon(d) = -1$

FIGURE 2

associate an integer $\varepsilon(d) = \pm 1$, called the *characteristic* of d (see p. 18 of [9]), according as the directed underpass does or does not cross under the directed overpass from left to right so as to preserve a left-handed screw. We note that ε is independent of the orientation assigned to the knot (Fig. 2).

We shall call the directed arc between two successive crossing points of an oriented knot K a boundary arc. Thus to each pair of consecutive syllables of a word for K there corresponds a unique boundary arc on K. We now state a lemma which follows from the proof of Theorem 2 of [8].

BASIC LEMMA 2.1. Let K and L be two oriented knots possessing equivalent prime words. Then there is an isotopic deformation of $\boldsymbol{R}^3$, fixed on a neighborhood of double points of K, which transforms K into a knot K' such that

(a) The oriented projections $p(K')$ and $p(L)$ are plane equivalent under an autohomeomorphism of $\boldsymbol{R}^2$ which maps projected boundary arcs of K' onto the corresponding projected boundary arcs of L.

(b) K and L belong to the same knot type.

3. The prime factorization of a word. If W is a nonempty word which is not prime, then it can be written in the form ABC where AC, B are nonempty words such that $b \in B$ iff $b^{-1} \in B$. Clearly AC and B themselves are words for some knot. Moreover, W can be written in the above form where B is a prime word. If AC is not prime the process may be repeated and so on. In this manner every nonempty word W may be factored into a finite number of prime words, called the *prime factors* of W. Furthermore, it is easy to see that this factorization is unique except possibly for the order of the prime factors.

DEFINITION 3.1. Let $P(Q)$ denote the set of prime factors of the nonempty word $U(V)$. Then U and V have *equivalent prime factorizations* iff there is a bijection h between P and Q such that $W \in P$ implies $h(W)$ is equivalent to W.

LEMMA 3.2. *Let $K(K')$ be a knot with double points $d_1, \cdots, d_n(d'_1, \cdots, d'_n)$ and prime word $W(W')$. Suppose W is equivalent to W' and d_i corresponds to $d'_i (i = 1, \cdots, n)$ under this equivalence. Then $\varepsilon(d_1) = \varepsilon(d'_1)$ implies $\varepsilon(d_i) = \varepsilon(d'_i)$ for $i = 2, \cdots, n$.*

Proof. By Lemma 2.1 we may assume that there is an autohomeomorphism h of $\boldsymbol{R}^2$ mapping the oriented projection $p(K)$ onto the oriented projection $p(K')$ so that the projected boundary arcs of

K are mapped onto the corresponding projected boundary arcs of K'. It is well known (p. 158 of [3]) that we may color the unbounded region of $p(K)(p(K'))$ white and then alternately color the bounded regions of $p(K)(p(K'))$ black and white so that every projected boundary arc of $K(K')$ lies on the boundary of precisely one black region and one white region. We may suppose that the double points of K have been labelled so that there is a projected boundary arc d_1ad_2 joining d_1 to d_2. The arc d_1ad_2 lies on the boundary of precisely one black region, say D. Since K has a prime word it follows from Theorem 9 of [10] that Bd D is a simple closed curve, and by Theorem 7 of [10] the union of a finite number of projected boundary arcs form an arc d_2bd_1 such that Bd $D = d_1ad_2 \cup d_2bd_1$.

Since h preserves projected overpasses (underpasses) and their orientations, $\varepsilon(d_1) = \varepsilon(d_1')$ implies that the projected overpass at d_1 can be rotated through black regions by a clockwise rotation onto the projected underpass at d_1 iff the projected overpass at d_1' can be rotated through black regions by a clockwise rotation onto the projected underpass at d_1'. It then follows the preceding statement is true if $d_1(d_1')$ is replaced by $d_2(d_2')$, and therefore $\varepsilon(d_2) = \varepsilon(d_2')$. By successively applying this procedure $n - 1$ times we obtain $\varepsilon(d_i) = \varepsilon(d_i')$ for $i = 2, \cdots, n$.

Since the mirror image of a knot K can be rotated to yield a knot L such that K and L have equivalent words and the characteristics of corresponding double points have opposite signs, we obtain

COROLLARY 3.3. *Let $K(K')$ be a knot with double points $d_1, \cdots, d_n(d_1', \cdots, d_n')$ and prime word $W(W')$. Suppose W is equivalent to W' and d_i corresponds to $d_i'(i = 1, \cdots, n)$ under this equivalence. Then either $\varepsilon(d_i') = \varepsilon(d_i)(i = 1, \cdots, n)$ or $\varepsilon(d_i') = -\varepsilon(d_i)(i = 1, \cdots, n)$.*

THEOREM 3.4. *Let $K(L)$ be an oriented knot with word $U(V)$ whose prime factors are $U_1, \cdots, U_m(V_1, \cdots, V_m)$. Suppose U and V have equivalent prime factorizations with U_i being equivalent to $V_i(i =, \cdots, m)$. Then, if the characteristic of a double point of U_i is equal to the characteristic of the corresponding double point of $V_i(i = 1, \cdots, m)$, the oriented knots K and L belong to the same oriented knot type.*

Proof. Now $K(L)$ can be represented as a composition of oriented knots $K_1, \cdots, K_m(L_1, \cdots, L_m)$ whose words are $U_1, \cdots, U_m(V_1, \cdots, V_m)$. Since the set of oriented knot types form a commutative semigroup (indeed, with unique factorization) [p. 140, 3], it suffices to prove the theorem in the case when $U = U_1$ and $V = V_1$.

By Lemma 2.1 we may assume that there is an autohomeomorphism h of R^2 mapping the oriented projection $p(K)$ onto the oriented projection $p(L)$ so that the projected boundary arcs of K are mapped onto the corresponding boundary arcs of L. Let $R_1(R_2)$ denote the unbounded region of $K(L)$. Without loss of generality we may assume that $\mathrm{Bd}\, R_i (i = 1, 2)$ lies on a circle C_i except in a neighborhood of double points. By Lemma 3.2 the characteristics of corresponding double points are equal. It follows that if $d_i d_k$ is a projected boundary arc of K, then $d_i d_k$ induces a clockwise orientation on C_1 iff $h(d_i d_k)$ induces a clockwise orientation on C_2. Hence we may suppose that h is the identity on $\boldsymbol{R}^2 \backslash \mathrm{Int}\, C$ where C denotes a circle in $\boldsymbol{R}^2$ whose interior contains $p(K)$ and $p(L)$. Let $f(x, y, z) = (h, (x, y), z)$. Then f is an orientation preserving autohomeomorphism of $\boldsymbol{R}^3$. Since $f(K)$ and L obviously belong to the same oriented knot type the result follows.

If in Theorem 3.4 it is only assumed that U and V have equivalent prime factorizations, then it follows from (b) of Lemma 2.1 that $K(L)$ can be represented as a composition of knots $K_1, \cdots, K_m(L_1, \cdots, L_m)$ where K_i and L_i belong to the same knot type $(i = 1, \cdots, m)$. Consequently we have

THEOREM 3.5. *Let K and L be two knots which may be given orientations so that their words have equivalent prime factorizations. If either K or L is a prime knot, then they belong to the same knot type.*

We note that it is possible for inequivalent nonprime knots to have words possessing equivalent prime factorizations. For instance, it is well known that the square and granny knots belong to different knot types ([2]). However, diagrams can be chosen for these two knots so as to yield identical words.

4. Diagram transformations. We remark that due to the work of Graeub ([5]) and Moise ([7]) the classical notion of equivalence for (oriented) knots (see [1]) agrees with the definitions given in §1. In [1] Alexander and Briggs have shown that two oriented knots belong to the same oriented knot type iff the diagram of one can be transformed into the diagram of the other by a finite sequence of the following types of deformations or their inverses:

(I) A boundary arc acquires a loop which creates a new double

FIGURE 3

point (Fig. 3).

(II) One boundary arc passes under another with the creation of two new double points (Fig. 4).

FIGURE 4

(III) If there is a region bounded by three double points and three projected boundary arcs where the endpoints of one of the boundary arcs are undercrossing points, then any one of the boundary arcs may be deformed past the crossing points determined by the other two boundary arcs (Fig. 5).

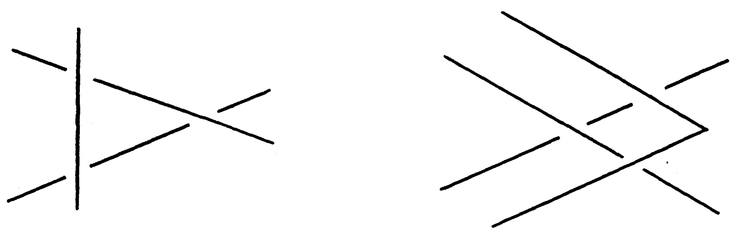


FIGURE 5

The above diagram transformations determine the following possible transformations between the associated words.

(I) $A \rightarrow Ax^e x^{-e} (e = \pm 1)$

(II) $AB \rightarrow Ax^e y^e B x^{-e} y^e$
$AB \rightarrow Ax^e y^e B y^{-e} x^{-e}$ $\quad (e = \pm 1)$

(III) $Ax_1y_1Bx_1^{-1}z_1Cy_1^{-1}z_1^{-1} \longrightarrow Ay_2x_2Bz_2x_2^{-1}Cz_2^{-1}y_2^{-1}$
$Ay_1x_1Bx_1^{-1}z_1Cy_1^{-1}z_1^{-1} \longrightarrow Ax_2y_2Bz_2x_2^{-1}Cz_2^{-1}y_2^{-1}$
$Ax_1y_1Bx_1^{-1}z_1Cz_1^{-1}y_1^{-1} \longrightarrow Ay_2x_2Bz_2x_2^{-1}Cy_2^{-1}z_2^{-1}$
$Ay_1x_1Bx_1^{-1}z_1Cz_1^{-1}y_1^{-1} \longrightarrow Ax_2y_2Bz_2x_2^{-1}Cy_2^{-1}z_2^{-1}$.

The words in (III) may occur with the order of the pairs $x_1y_1(y_1x_1)$, $x_1^{-1}z_1$, $y_1^{-1}z_1^{-1}(z_1^{-1}y_1^{-1})$ being permuted provided that the analogous rearrangement is made for the pairs $y_2x_2(x_2y_2)$, $z_2x_2^{-1}$, $z_2^{-1}y_2^{-1}(y_2^{-1}z_2^{-1})$.

We now list three additional types of transformations of words.

(0) Any transformation between two equivalent words.

(IV) A transformation which results from a reversal of the orientation of the knot,

$$\text{i.e., } x_1^{e_1} \cdots x_{2n}^{e_{2n}} \longrightarrow x_{2n}^{e_{2n}} \cdots x_1^{e_1} .$$

(V) A transformation which results from a reflection through the xy-plane,

$$\text{i.e., } x_1^{e_1} \cdots x_{2n}^{e_{2n}} \longrightarrow x_1^{-e_1} \cdots x_{2n}^{-e_{2n}} .$$

DEFINITION 4.1. A sequence $W_1, W_2, \cdots, W_n$ of words of knots is called a *fundamental sequence of words* between U and V iff $W_1 = U$, $W_n = V$ and W_i is obtained from $W_{i-1} (i = 2, \cdots, n)$ by a transformation of one of the types (0)–(V).

DEFINITION 4.2. A finite sequence of words and characteristics of double points of knots is said to be *fundamental* if each word and its successor are related by a transformation of one of the types (0)–(III) and under this transformation corresponding double points have the same characteristic.

THEOREM 4.3. *Let K, L be two prime knots with words U, V respectively. Then K and L belong to the same knot type iff there is a fundamental sequence of words between U and V.*

Proof. (*Necessity*) Suppose K and L are prime knots which belong to the same knot type. We regard K, L as oriented knots with the orientations being determined by U, V respectively. Let σ denote reversal of knot orientation and let $\rho(x, y, z) = (x, y, -z)$. Then the oriented knot K must belong to the same oriented knot type as one of the oriented knots $L, \rho L, \sigma L, \sigma\rho L$. Hence the necessity of the theorem follows from [1].

(*Sufficiency*) Let $W_1, W_2, \cdots, W_n$ be a fundamental sequence of words between U and V. Let $K_2, \cdots, K_{n-1}$ be a sequence of knots with words $W_2, \cdots, W_{n-1}$ respectively. If W_2 is obtained from W_1 by a type (0) transformation it is easy to show that W_1 and W_2 have equivalent prime factorizations. Hence K is equivalent to K_2 by Theorem 3.5. If W_2 is not obtained from W_1 by a type (0) transformation, then it is evident that there exist equivalent knots L_1, L_2 with words W_1, W_2 respectively. By Theorem 3.5 K is equivalent to L_1 and L_2 is equivalent to K_2. Thus K is equivalent to K_2. Using an inductive argument we obtain the desired result that K and L belong to the same knot type.

By using an argument similar to that used above and evoking Theorem 3.4 instead of Theorem 3.5 we obtain

THEOREM 4.4. *Two oriented knots K and L belong to the same oriented knot type iff there is a fundamental sequence relating a* *word and the characteristics of the double points of K to a word and the characteristics of the double points of L.*

REMARKS 4.5. L. B. Treybig has conjectured that if U and V are words of length $\leqq m$ which may be connected by a fundamental sequence of words, then there is an integer $N(m)$ and a fundamental sequence of words between U and V such that each word in the sequence has length $\leqq N(m)$. In [13] Treybig obtains some partial results in this direction. The proof of such a conjecture together with Theorem 4.4 would provide an algorithm for determining oriented knot types.

5. The group of a prime word. By Corollary 3.3 there are precisely two sets of characteristics associated with a prime word, one set being the negatives of the other. Given a prime word, a set of characteristics associated with its alphabet may be obtained by simply constructing *any* diagram associated with the word.

Let $W = b_1^{e_1} b_2^{e_2} \cdots b_{2n}^{e_{2n}}$ be a prime word. A segment (consider W to be cyclically ordered) of the form $b_i \cdots b_{i+k}$ where $e_{i-1} = e_{i+k+1} = -1$ is called an over segment, and a segment of the form $b_j^{-1} \cdots b_{j+m}^{-1}$ where $e_{j-1} = e_{j+m+1} = 1$ is called and under segment. The group of W is presented as follows: The alphabet of W is the set of generators. For each over segment $b_i \cdots b_{i+k}$ we associate the relation $b_i = \cdots = b_{i+k}$. For each under segment $b_j^{-1} \cdots b_{j+m}^{-1}$ we associate the relation $b_{j-1} b_j^{\varepsilon(bj)} \cdots b_{j+m}^{\varepsilon(bj+m)} b_{j+m+1}^{-1} b_{j+m}^{-\varepsilon(bj+m)} \cdots b_j^{-\varepsilon(bj)} = 1$. The above generators and relations present a group which we shall call the group of W.

THEOREM 5.1. *A group is a knot group iff it has a presentation which presents the group of a prime word.*

Proof. To prove the theorem it suffices to show that every oriented knot type has a representative diagram which possesses a prime word, and that the group of a prime word W is the group of any knot whose diagram yields W.

Let K be an oriented knot with word W, and let $W_1, \cdots, W_m$ denote the prime factors of W. Then K can be represented as a composition of oriented knots $K_1, \cdots, K_m$ possessing prime words $W_1, \cdots, W_m$ respectively. If K is a prime knot it follows that only one of $K_1, \cdots, K_m$ belongs to a nontrivial knot type. Hence it can be assumed that every prime knot possesses a prime word. Thus given an oriented knot K, we may assume that K is a composition of oriented prime knots $K_1, \cdots, K_m$ with prime words $W_1, \cdots, W_m$ respectively, and that the word of K is $W_1 W_2 \cdots W_m$. An isotopic deformation of R^3 can be chosen so as to deform an arc in K_1 so that it passes under $K_2, \cdots, K_m$ in such a way that K is transformed into a new knot K' which possesses a prime word.

Now suppose K is an oriented knot with prime word W. We may assume that the characteristics associated with W are the characteristics of K. (For otherwise, the characteristics would be associated with the diagram of a knot $\alpha\rho K$ where α is a rotation and $\rho(x, y, z) = (x, y, -z)$. Since K and $\alpha\rho K$ are equivalent, it would suffice to show that the presentation for the group of W presents the group of $\alpha\rho K$.) By using the Tietze operations we can eliminate all the generators in the presentation except for the first letter in each over segment. It is then easy to check that the resulting presentation is precisely the over presentation of the group of K given by Fox and Torres (p. 212, [4]).

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Received October 10, 1972. This paper represents a portion of the author's Ph. D. dissertation, written at Tulane University under the direction of Professor L. B. Treybig, and was supported partly by the National Research Council of Canada (grant A8205).

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