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INNER FUNCTIONS UNDER UNIFORM TOPOLOGY

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The structure of the space $\mathscr{F}$ of all inner functions in the unit disc $D = \{z: |z| < 1\}$ under the metric topology induced by the H^∞-norm is considered. It is proven that if two inner functions p and q belong to the same component of $\mathscr{F}$, then the variation of p/q on each open arc of ∂D (the boundary of D in the complex plane C) where they can be continued analytically is bounded by a constant $C = C(p, q)$, independent of the arc. This criterion is used to show that a component of $\mathscr{F}$ can contain nothing but Blaschke products with infinitely many zeroes, exactly one (up to a constant factor) singular inner function or infinitely many pairwise coprime singular inner functions.

The reader is assumed to be familiar with the basic theory of the space H^∞. We recall that the canonical form of an inner function is: $q = \lambda br = \lambda bds$, where $\lambda \in \partial D$ and

(1) $b(z) = \prod_k b_k(z) = \prod_k b(z, a_k) = \prod_k \overline{a_k}/|a_k| \cdot (a_k - z)/(1 - \overline{a_k}z)$,

$(\overline{a_k}/|a_k| = -1$, if $a_k = 0)$, $a_k \in D$ and $\sum_k(1 - |a_k|) < \infty$; $b(z)$ is the *Blaschke product* of q and the b_k's are the Blaschke factors;

$$(2) \qquad r(z) = e(z, \mu) = \exp\left\{\int_0^{2\pi} (z + e^{i\theta})/(z - e^{i\theta}) d\mu(\theta)\right\},$$

where μ is a nonnegative singular Borel measure on ∂D; $r(z)$ is the *singular part* of q. If $\mu = \nu + \nu_1$, where ν and ν_1 are the continuous and the purely atomic part, resp., of μ, then
(3) $s(z) = e(z, \nu)$ is the *continuous singular part* of q, and
(4) $d(z) = e(z, \nu_1) = \prod_j d_j(z) = \prod_j d(z; \theta_j, l_j)$, where $d(z; \theta, l) = \exp\{l(z + e^{i\theta})/(z - e^{i\theta})\}$, $0 \leqq \theta < 2\pi$ and $l > 0$; $d(z)$ is the *atomic singular part* of q. Clearly, $\sum_j l_j = \|\nu_1\| < \infty$; it will be assumed that $\theta_j \neq \theta_h$, whenever $j \neq h$.

The subsets of $\mathscr{F}$ containing all those functions defined by the conditions (1), (2), (3), and (4) (or the constant multiples of such functions) will be denoted by $\mathscr{F}_B$, $\mathscr{F}_S$, $\mathscr{F}_C$, and $\mathscr{F}_A$, resp.

It is well-known that the inner function $q(z)$ can be continued analytically across (a neighborhood of) the point $\lambda \in \partial D$ if and only if $\lambda \notin \mathrm{Sp}(q) = \mathrm{supp}(\mu) \cup \mathrm{closure}\{a_k\}$; furthermore, if $\mathrm{Sp}(q) \cap \partial D \neq \mathrm{Sp}(p) \cap \partial D (p, q \in \mathscr{F})$, then $\|q - p\|_\infty = 2$, and p and q belong to different components of $\mathscr{F}$.

Let $\mathscr{F}_\Gamma$ (Γ a closed subset of ∂D) be the set of all inner functions such that $\mathrm{Sp}(q) \cap \partial D = \Gamma$. Clearly, $\mathscr{F}_\phi$ is the family of all Blaschke products of finite order, and it is not difficult to see that $\mathscr{F}_\phi = \bigcup_{n=0}^\infty \mathscr{F}_n$ (disjoint union!), where $\mathscr{F}_n$ is the family of Blaschke products of order n and (for each n) $\mathscr{F}_n$ is an acrwise connected open and closed component of $\mathscr{F}$, topologically homeomorphic to $\partial D \times D^n$ (see [2; 4]).

During the "Sesquicentennial Seminar in Operator Theory" (Indiana University, Bloomington, Indiana, 1969–1970), R. G. Douglas asked for a characterization of the components of $\mathscr{F} \backslash \mathscr{F}_\phi$ containing a singular inner function; in particular, he asked whether or not every component of $\mathscr{F} \backslash \mathscr{F}_\phi$ contains a singular inner function. Here we partially answer the first question. A first counterexample to the second question was given by D. Sarason; his unpublished result follows from his paper [5] (some components of $\mathscr{F} \backslash \mathscr{F}_\phi$ contain nothing but Blaschke products whose zeroes converge nontangentially to $z = 1$). We shall give another counterexample by using a different approach.

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1. Components of $\mathscr{F}$ containing exactly one, or $\underline{c}$($\underline{c}$ = the power of the continuum) singular inner functions. We shall need a remarkable classical result due to O. Frostman.

THEOREM 1.1 ([3, § 59, p. 111]). *Let $q \in \mathscr{F}$ and let*

$$q_a(z) = [q(z) - a]/[1 - \bar{a}q(z)] , \quad a \in D . \tag{1}$$

Then, for all $a \in D$, except for a subset of logarithmic capacity zero, $q_a \in \mathscr{F}_B$.

For a precise definition of *logarithmic capacity* of a subset Λ of the complex plane, see [3; 6]. For our purposes, it is enough to recall that, if log cap $\Lambda = 0$, then Λ is a "very small" subset of $\boldsymbol{C}$; in fact, Λ has planar Lebesgue measure zero and, moreover, the projection of Λ on any line of $\boldsymbol{C}$ has linear measure zero. Thus, Theorem 1.1 implies, in particular, that $\mathscr{F}_B$ is dense in $\mathscr{F}$.

We are going to analyze the behavior of an inner function q on a closed arc $\Gamma = [e^{i\alpha}, e^{i\beta}]$ $(0 \leqq \alpha < 2\pi,\ \alpha < \beta < \alpha + 2\pi)$, contained in $\partial D \backslash \mathrm{Sp}\,(q)$. For an arbitrary continuously differentiable unimodular function $u(e^{i\theta})$ defined in a neighborhood (in ∂D) of Γ, we define the *variation* of u on Γ as

$$M(u;\, \alpha, \beta) = \arg u(e^{i\beta}) - \arg u(e^{i\alpha}) = \int_\alpha^\beta d \arg u(e^{i\theta}) ,$$

where $\arg u(e^{i\theta})$ denotes any continuously differentiable determination of the argument of $u(e^{i\theta})$.

LEMMA 1.2. *Let p and q be two inner functions and let Γ be as above. Then*

(i) $M(q; \alpha, \beta) \geqq 0$;

(ii) *If* $\Gamma \subset \partial D \backslash \mathrm{Sp}(p)$, $M(pq; \alpha, \beta) = M(p; \alpha, \beta) + M(q; \alpha, \beta) \geqq M(q; \alpha, \beta)$;

(iii) *If p belongs to the component of q, then* $\mathrm{Sp}(p) \cap \partial D = \mathrm{Sp}(q) \cap \partial D$ *and there exists a constant* $C = C(p, q)$ *such that* $|M(q; \alpha, \beta) - M(p; \alpha, \beta)| = |M(q/p; \alpha, \beta)| \leqq C$, *and C is independent of Γ.*

Proof. (i) This is trivial for finite Blaschke products. If $q(z) = \prod_{k=1}^{\infty} b(z, a_k)$ is an infinite Blaschke product such that $\Gamma \subset \partial D \backslash \mathrm{Sp}(q)$, then observe that $\{\prod_{k=1}^{n} b(z, a_k)\}_{n=1}^{\infty}$ converges to $q(z)$ uniformly on Γ, from which the result follows. Finally, if q is any inner function, then we can find (by *Frostman's theorem*) a sequence $\{a_n\} \subset D$ converging to zero, such that $q_n = q_{a_n}$ (defined by (1)) is a Blaschke product, for all n. Clearly, q_n belongs to the component of q and therefore (see the *introduction*) $\mathrm{Sp}(q_n) \cap \partial D = \mathrm{Sp}(q) \cap \partial D$. Since the result is true for all q_n, $n = 1, 2, \cdots$ and $|q_n(z) - q(z)| \to 0$, as $n \to \infty$, uniformly on Γ, the result is also true for q.

(ii) This follows immediately from (i).

(iii) Let $\mathscr{C}_0(q) = \{q' \in \mathscr{F}: \|q' - q\|_\infty < 1\}$ and, by induction, define $\mathscr{C}_n(q) = \{q'' \in \mathscr{F}: \|q'' - q'\|_\infty < 1$, for some $q' \in \mathscr{C}_{n-1}(q)\}$, $n = 1, 2, \cdots$, and $\mathscr{C}(q) = \bigcup_{n=0}^{\infty} \mathscr{C}_n(q)$. Clearly, $\mathscr{C}(q)$ is open and closed in $\mathscr{F}$ and contains the component of q. Moreover, $\mathrm{Sp}(q') \cap \partial D = \mathrm{Sp}(q) \cap \partial D$, for all $q' \in \mathscr{C}(q)$.

Let $q' \in \mathscr{C}_0(q)$; then for every $e^{i\theta} \in \Gamma$, $|q(e^{i\theta}) - q'(e^{i\theta})| \leqq \|q - q'\|_\infty < 1$, and therefore, for a suitable continuously differentiable definition of the argument, we have $|\arg q(e^{i\theta})/q'(e^{i\theta})| < \pi/3$. Hence, $|M(q/q'; \alpha, \beta)| = |\arg q(e^{i\beta})/q'(e^{i\beta}) - \arg q(e^{i\alpha})/q'(e^{i\alpha})| \leqq |\arg q(e^{i\beta})/q'(e^{i\beta})| + |\arg q(e^{i\alpha})/q'(e^{i\alpha})| < 2\pi/3$.

By an elementary inductive argument, it follows that $|M(q/q''; \alpha, \beta)| < 2\pi(n + 1)/3$, for all $q'' \in \mathscr{C}_n(q)$, $n = 0, 1, 2, \cdots$, whence the result follows.

THEOREM 1.3. *Let $d(z) = d(z; 0, 1)$. Then, for each positive r, the component of d is isometric to the component of d^r; however, if $r \neq 1$, d and d^r belong to different components of $\mathscr{F}$. Thus, if $0 < r < 1$, then the subset $\{qd^r \in$ (component of d)$\}$ is isometric to the whole component of $d \cdot d^r$ is the only (up to a constant factor) singular inner function in its own component, for each $r > 0$.*

Proof. If $b(z)$ is a Blaschke factor, then the mapping $q(z) \to q(b(z))$ is an isometry from $\mathscr{F}$ onto itself. If the zero of $b(z)$ is the point $a = (1 - r)/(1 + r) \in D (r > 0)$, then $d(b(z)) = d^r(z)$, which proves the first part.

Observe that $\mathrm{Sp}(d) = \{1\}$ and $M(d^t; 1/n, \pi) \to +\infty$, as $n \to \infty$, for each positive t; hence, d^r and d^{r+t} cannot belong to the same component, as follows from Lemma 1.2. Since $\mathscr{F}_S \cap \mathscr{F}_{\{1\}} = \{\lambda d^t: \lambda \in \partial D, t > 0\}$, we conclude that every singular inner function in the component of d^r is a constant multiple of d^r.

The remaining statements are clear now.

COROLLARY 1.4. *Let $q \in \mathscr{F}_S$ and assume that $\{\theta: e^{i\theta} \in \mathrm{Sp}(q)\}$ is well-ordered (with the usual order of the interval $[0, 2\pi)$). Then the only singular inner functions in the component of q are the constant multiples of q.*

Proof. Clearly, $\mathrm{Sp}(q)$ is countable and therefore q has the form $q(z) = \lambda \prod_j d(z; \theta_j, l_j) = \lambda \prod_j d_j(z)$ (i.e., $q \in \mathscr{F}_A$). Moreover, if $p \in \mathscr{F}_S$ belongs to the component of q, then $p(z) = \lambda' \prod_h d(z; \theta'_h, l'_j)$, where $\lambda' \in \partial D$ and $\mathrm{clos}\{e^{i\theta'_h}\} = \mathrm{clos}\{e^{i\theta_j}\} = \mathrm{Sp}(q)$.

Let $q(z) = \lambda q_1(z) q'_1(z)$, where $q_1(z)$ is the product of all d_j's corresponding to the isolated points of $\mathrm{Sp}(q)$. It is not difficult to see, by using Lemma 1.2 and the arguments of the proof of Theorem 1.3, that $p(z) = \lambda' q_1(z) p'_1(z)$, where $\mathrm{Sp}(p'_1) = \mathrm{Sp}(q'_1) \subset \mathrm{Sp}(q)'$ (here Γ' denotes the *derived* set of the set Γ).

Now, let $q(z) = \lambda q_1(z) q_2(z) q'_2(z)$, where $q_2(z)$ is the product of the d_j's corresponding to the isolated points of $\mathrm{Sp}(q'_1)$. Using the above arguments and the fact that, if $e^{i\theta} \in \mathrm{Sp}(q'_1)$, then the arc $(e^{i\theta}, e^{i(\theta+\varepsilon)}]$ (for some $\varepsilon = \varepsilon(\theta) > 0$) does not intersect $\mathrm{Sp}(q)$ (here we are using the "well-order property"!), we see that $p(z) = \lambda' q_1(z) q_2(z) p'_2(z)$.

The result follows by a transfinite inductive argument.

It is clear that, if $q \in \mathscr{F}$ is nonconstant, then $q(D)$ is an open subset of D; in general, $q(D) \neq D$ (namely, if q is singular, then $0 \notin q(D)$). In [6, Theorem 10], W. Seidel proved that if q is a nonconstant inner function and $D \backslash q(D)$ contains more than one point, then $\mathrm{Sp}(q) \cap \partial D$ is a nonempty perfect subset of ∂D. The set $D \backslash q(D)$ has been completely determined by O. Frostman ([3, § 61, p. 113]; see also [6, Theorem 13]): If $q \in \mathscr{F}$ is nonconstant, then $D \backslash q(D)$ is a closed subset of D of logarithmic capacity zero; conversely, if Λ is a closed subset of D and $\log \mathrm{cap}\, \Lambda = 0$, then the *uniformizer* of the Riemann surface $D \backslash \Lambda$ is an inner function q such that $D \backslash q(D) = \Lambda$. Moreover, if Λ is compact, then $\mathrm{Sp}(q) \cap \partial D$ has linear measure zero (see [6, p. 218]). The uniformizer of $D \backslash \{a\} (a \in D)$ can be taken equal to $d_{(-a)}(z; \theta, l) = q(z)$ (defined by (1); θ and $l > 0$ can be arbitrarily chosen); in this case $\mathrm{Sp}(q) = \{e^{i\theta}\}$ consist of a single point and we know (by Theorem 1.3) that the component of q contains exactly one (up to a constant factor) singular inner function. On the other hand, if Λ is a compact subset of logarithmic capacity zero of D, and Λ contains more than one point, the uniformizer of $D\backslash\Lambda$, $p(z)$ is an inner function such that $\mathrm{Sp}(p)$ is a nonempty perfect subset of ∂D of linear measure zero; since, for every $a \in \Lambda$, $p_a \in \mathscr{F}_S$, we conclude that the component of p contains (at least!) two coprime singular inner functions. In fact, if $q = rs$ and $q_a = rt$, where $q, r, s, t \in \mathscr{F}$ and $a \neq 0$, then $q - a = rs - a = (1 - \bar{a}q)rt$; hence $t(1 - \bar{a}q) = s - a/r \in H^\infty$. It follows that r and a/r belong to H^∞, but this is impossible unless r is a constant. In other words, q and q_a ard coprime (this example and Theorem 1.5 below are due to D. Sarason).

Corollary 1.4 may be considered an improvement of the above mentioned result of W. Seidel for a very particular class of singular inner functions (in fact, Seidel's result can be reformulated as: If $\mathrm{Sp}(q) \cap \partial D$ is not perfect, then $D\backslash q(D)$ contains, at most, one point) and we guess that the "well-order" condition could be replaced by the weaker condition "$\mathrm{Sp}(q) \cap \partial D$ is countable"; however, Lemma 1.2 is not sufficient to prove this stronger conjecture. On the other hand, an analysis of the function p_a of the above example shows that no "reasonable" condition weaker than "countable" can work to get the same result.

Let $p(z)$ be the uniformizer of $D\backslash\Lambda$, where Λ is any nonempty perfect subset of D of logarithmic capacity zero (e.g., take as Λ a suitable "Cantor type" subset of the real interval $[0, 1/2]$; see [3; 6]). Then, for each $a \in \Lambda$, p_a is a singular inner function. Since $c(\Lambda) = \underline{c}$, the power of the continuum, it follows from the previous observations that

THEOREM 1.5 (D. Sarason). *There exists a compact of $\mathscr{F}$ containing $\underline{c}$ pairwise coprime singular inner functions.*

The perfect set Λ can be replaced by a finite subset or by a sequence of points in D converging to ∂D. This suggests that, for each $n = 0, 1, 2, \cdots, \aleph_0$, there exists a component of $\mathscr{F}$ containing exactly n coprime singular inner functions or, at least, exactly n "essentially different" (i.e., p/q is not a constant) singular inner functions, but we have been unable to prove it.

From Theorem 1.1, we obtain

COROLLARY 1.6.
(i) *$\mathscr{F}_S$ is a closed nowhere dense subset of $\mathscr{F}$.*
(ii) *$\mathscr{F}_B$ is a dense, but not open subset of $\mathscr{F}$.*

Proof. By the observations following Theorem 1.1, we only have to prove that $\mathscr{F}_S$ is closed and $\mathscr{F}_B$ is not open. The first fact is immediate, because $\mathscr{F}_S$ is clearly closed with respect to the compact-open topology restricted to $\mathscr{F}$ and the norm-topology in H^∞ is stronger than the compact-open topology.

To see that $\mathscr{F}_B$ is not open, write $d(z; 0, 1) = \prod_{j=1}^\infty d_j(z)$, where $d_j(z)$ is the 2^j-root of $d(z; 0, 1)$, and set $b(z) = \prod_{j=1}^\infty p_j(z)$, $p_j(z) = [d_j(z) - 1/2j]/[1 - (1/2j)d_j(z)]$. It follows from Theorem 1.3 that $p_j \in \mathscr{F}_B$, for all j, and $b = \prod_j p_j \in \mathscr{F}_B$. Given any $\varepsilon > 0$, choose n so that $2/n < \varepsilon$. Then $[\prod_{j\neq n} p_j]d_n \notin \mathscr{F}_B$ and $\| b - [\prod_{j\neq n} p_j]d_n \|_\infty = \| (d_n)_{(1/2n)} - d_n \|_\infty < 2/n < \varepsilon$.

Therefore, $b(z)$ does not belong to the interior of $\mathscr{F}_B$.

We close this section with two *conjectures*:

(1) The component, in $\mathscr{F}_S$, of a singular inner function p is the set of the constant multiples of p (i.e., $\mathscr{F}_S$ is "essentially" a totally disconnected space).

(2) $\mathscr{F}_C$ and $\mathscr{F}_A$ are closed in $\mathscr{F}$.

2. Components contained in $\mathscr{F}_B$.

THEOREM 2.1. *Given any sequence $0 \leqq r_1 \leqq r_2 \leqq \cdots \leqq r_k \leqq \cdots < 1$ of radii such that $\sum_{k=1}^\infty (1 - r_k) < \infty$, it is possible to choose $\underline{c}$ sequences $\{\theta_k(t): 0 \leqq t < \infty\}$ of arguments such that, for each $t \in [0, \infty)$, the component of*

$$b_t(z) = \prod_{k=1}^{\infty} b(z, r_k \exp\{i\theta_k(t)\})$$

in $\mathscr{F}$ is contained in $\mathscr{F}_B$. Moreover, if $t \neq t'$, then b_t and $b_{t'}$ belong to different components.

Proof. First of all observe that $\prod_{k=1}^\infty b(z, r_k)$ converges uniformly on each of the subsets $A_n = \{z: |z| \leqq 1, |1 - z| \geqq 2^{-n}\}$. Therefore, we can choose $k_0 = 0 < k_1 < k_2 < \cdots$ in such a way that

$$(2)\qquad \begin{aligned} &\left| M\Big(\prod_{k=m_1}^{m_2} b(e^{i\theta}, r_k); 2^{-n}, 2\pi - 2^{-n}\Big)\right| < 2^{-n},\ n = 1, 2, \cdots, \quad \text{and} \\ &2\pi(m_2 - m_1 + 1) - 2^{-n} < \left| M\Big(\prod_{k=m_1}^{m_2} b(e^{i\theta}, r_k); -2^{-n}, 2^{-n}\Big)\right| \\ &\qquad\qquad < 2\pi(m_2 - m_1 + 1),\ n = 1, 2, \cdots, \end{aligned}$$

whenever $k_n \leqq m_1 \leqq m_2 < \infty$.

Define $b(z) = b_0(z) = \prod_{h=1}^{\infty} b'_h(z)$, where

$$b'_h(z) = \prod\{b(z, r_k \exp[i2^{-h}]): k_{h-1} < k \leqq k_h\},\ h = 1, 2\cdots .$$

Then the first inequality of (2) implies that $M(b_0(e^{i\theta}); \pi, 2\pi - \varepsilon)$ is uniformly bounded with respect to $\varepsilon, 0 < \varepsilon < \pi$. Since $\mathrm{Sp}(b_0) \cap \partial D = \{1\}$, it is easy to see from Lemma 1.2 and the properties of the functions $d(z; 0, l)$, $l > 0$, that the component of b_0 in $\mathscr{F}$ only contains Blaschke products. It is also clear that this property of b_0 only depends on the fact that the sequence $\{k_h\}_{h=0}^{\infty}$ tends to infinity rapidly enough. Thus, if $k_h(t)$ denotes the integer part of $k_h h^t$, $0 \leqq t < \infty$, and b_t is defined by $b_t(z) = \prod_{h=1}^{\infty} b'_{t,h}(z)$, where $b'_{t,h}(z) = \prod\{b(z, r_k \exp[i2^{-h}]): k_{h-1}(t) < k \leqq k_h(t)\}$, then the component of b_t is also contained in $\mathscr{F}_B$ and $\mathrm{Sp}(b_t) \cap \partial D = \{1\}$.

Finally observe that, if $0 \leqq t < t' < \infty$, then the second inequality of (2) implies that $M(b_t/b_{t'}; 2^{-n}, \pi) \cong 2\pi\{k_{n-1}(t) - k_{n-1}(t') + (1/2)[k_n(t) - k_{n-1}(t)] - (1/2)[k_n(t') - k_{n-1}(t')]\} = \pi\{k_{n-1}(t) - k_{n-1}(t') + k_n(t) - k_n(t')\} = -\pi\{k_n(n^{t'} - n^t) + k_{n-1}((n-1)^{t'} - (n-1)^t) + F(n; t, t')\}$, where $F(n; t, t') = 0(k_n)$; hence, $M(b_t/b_{t'}; 2^{-n}, \pi)$ tends to $-\infty$, as n tends to ∞. It follows from Lemma 1.2 that b_t cannot belong to the component of $b_{t'}$.

THEOREM 2.2. *If $b(z) = \prod_{k=1}^{\infty} b(z, a_k)$ has the properties:*

(1) *The component of b in $\mathscr{F}$ is contained in $\mathscr{F}_B$,*

(2) *$a_k \neq 0$ for all k and $\pi > \arg a_1 > \arg a_2 > \cdots > \arg a_k > \arg a_{k+1} > \cdots > 0$, and $\lim \arg a_k = 0$, and*

(3) *$M(b; \pi, 2\pi - \varepsilon)$ is uniformly bounded with respect to ε, $0 < \varepsilon < \pi$; then there exists $\underline{c}$ subproducts $b_\omega (0 \leqq \omega \leqq \pi/4)$ of b enjoying the properties* (1), (2), *and* (3); *moreover, if $\omega \neq \omega'$, then b_ω and $b_{\omega'}$ belong to different component of $\mathscr{F}$.*

It is completely apparent that, if $b(z)$ is a Blaschke product satisfying any of the properties (1), (2), or (3), then the same result is true for every subproduct of b with infinitely many zeroes. Therefore, we only have to show that the subproducts of b can be chosen in such a way that they belong to different components. The proof of the next lemma is a minor modification of the proof given in [1].

LEMMA 2.3. *Let N be the set of natural numbers. There exist $\underline{c}$ subsets $\{A_\omega: 0 \leqq \omega \leqq \pi/4\}$ of N such that, if $\omega \neq \omega'$, then*

(i) *$A_\omega \cap A_{\omega'}$ is finite and*

(ii) *given any $N \in N$, A_ω contains a finite sequence of N consecutive numbers $n_N, n_N + 1, \cdots, n_N + N - 1$ which are not in $A_{\omega'}$.*

Proof. Enumerate the points with integral coordinates in $\{(x, y): 0 \leqq x \leqq y\}$ as follows:

$1 \to (0, 0)$
$2 \to (0, 1)$, $3 \to (1, 1)$
$4 \to (0, 2)$, $5 \to (1, 2)$, $6 \to (2, 2)$
$7 \to (0, 3)$, $8 \to (1, 3)$, $9 \to (2, 3)$, $10 \to (3, 3)$. etc.

Let $P_0 = \{(x, y): y > x^2\}$ and let P_ω, $0 \leqq \omega \leqq \pi/4$, be the result of rotating P_0 in $-\omega$ about the origin. Now it is enough to take A_ω equal to the subset of numbers in the above lattice lying inside P_ω.

Proof of Theorem 2.2. Properties (2) and (3) show that $b(z)$ can be factored as $b(z) = b'(z) \cdot b''(z)$, where $b'(z)$ is a finite or infinite subproduct and $b''(z) = \prod_{k=1}^\infty b(z, a_k'')$ satisfies (1), (2) (with a_k replaced by a_k''), and (3), and moreover,

$$(3) \qquad | M(b''; \arg a_k'', \pi) - 2(k - 1/2)\pi | < 2\pi .$$

Set $b_\omega(z) = b'(z) \cdot \prod \{b(z, a_k''): k \in A_\omega\}$, $0 \leqq \omega \leqq \pi/4$, where A_ω is the subset of N defined in Lemma 2.3. Now observe that the definition of A_ω and (3) imply that, if $0 \leqq \omega < \omega' \leqq \pi/4$, then

$$\limsup \{M(b_\omega/b_{\omega'}; \arg a_k'', \pi): k \to \infty\} = +\infty$$

or $\liminf \{M(b_\omega/b_{\omega'}; \arg a_k'', \pi): k \to \infty\} = -\infty$. Thus, the result follows from Lemma 1.2.

REMARK. It is completely apparent that the same argument can be used to prove, e.g., that $\mathscr{F}$ has $\underline{c}$ components containing nothing but Blaschke products whose sequences of zeroes converges nontangentially to $z = 1$. To see this, set $b(z) = \prod_{k=1}^\infty b(z, r_k)$, where $\{r_k\}_{k=1}^\infty$ is a sequence of radii converging "very rapidly" to 1 so that, for a suitable sequence of arguments $\{\theta_n\}_{n=1}^\infty$, decreasing to zero, $M(d(e^{i\theta}; 0, 1/n)/b(e^{i\theta}); \theta_n, \pi\} > n$, $n = 1, 2, \cdots$. Clearly, the same result is true for any of the subproducts b_ω of b (defined as in the proof of Theorem 2.2), and the result follows from Lemma 1.2.

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