PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

ON A SPLITTING FIELD OF REPRESENTATIONS OF A FINITE GROUP

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The theorem of P. Fong about a splitting field of representations of a finite group G will be improved to the effect that the order of G mentioned in it will be replaced by the exponent of G. The proof depends on the Brauer-Witt theorem and properties of cyclotomic algebras.

Let Q denote the rational field. For a positive integer n, ζ_n is a primitive nth root of unity. Let χ be an irreducible character of a finite group G (an irreducible character means an absolutely irreducible one). Let K be a field of characteristic 0. Then $m_K(\chi)$ denotes the Schur index of χ over K. The simple component of the group algebra $K[G]$ corresponding to χ is denoted by $A(\chi, K)$. Its index is exactly $m_K(\chi)$. If L/K is normal, $\mathscr{G}(L/K)$ is the Galois group of L over K.

In this paper we will prove the following:

THEOREM. *Let G be a finite group of exponent $s = l^a n$, where l is a rational prime and $(l, n) = 1$. Let $k = Q(\zeta_n)$ if l is odd, let $k = Q(\zeta_n, \zeta_4)$ if $l = 2$. Then, $m_k(\chi) = 1$ for every irreducible character χ of G.*

REMARK. In Fong [2, Theorem 1], the above s denoted the order of G (instead of the exponent of G).

First we review

BRAUER-WITT THEOREM. *Let χ be an irreducible character of a finite group G of exponent s. Let q be a prime number. Let K be a field of characteristic 0 with $K(\chi) = K$. Let L be the subfield of $K(\zeta_s)$ over K such that $[K(\zeta_s): L]$ is a power of q and $[L: K] \not\equiv 0 \pmod q$. Then there is a subgroup F of G and an irreducible character ξ of F with the following properties: (1) there is a normal subgroup N of F and a linear character ψ of N such that $\xi = \psi^F$ and $L(\xi) = L$, (2) $F/N \cong \mathscr{G}(L(\psi)/L)$, (3) $m_L(\xi)$ is equal to the q-part of $m_K(\chi)$, (4) for every $f \in F$ there is a $\tau(f) \in \mathscr{G}(L(\psi)/L)$ such that $\psi(fnf^{-1}) = \tau(f)(\psi(n))$ for all $n \in N$, and (5) $A(\xi, L)$ is isomorphic to the crossed product $(\beta, L(\psi)/L)$ where, if S is a complete set of coset representatives of N in F $(1 \in S)$ with $ff' = n(f, f')f''$ for $f, f', f'' \in S$, $n(f, f') \in N$, then $\beta(\tau(f), \tau(f')) = \psi(n(f, f'))$.*

Proof. See, for instance, [1] and [4].

REMARK. The above crossed product is called a cyclotomic algebra (cf. [3]).

COROLLARY. *Let p be a prime number. Denote by Q_p the rational p-adic field. Suppose that $p \nmid s$ if $p \neq 2$, and that $4 \nmid s$ if $p = 2$, s being the exponent of G. Then $m_{Q_p}(\chi) = 1$ for every irreducible character χ of G.*

Proof. Set $K = Q_p(\chi)$. Then $m_K(\chi) = m_{Q_p}(\chi)$. Let q be any prime number. By the Brauer-Witt theorem, the q-part of $m_K(\chi)$ equals the index of some cyclotomic algebra of the form $(\beta, L(\psi)/L)$, where $Q_p \subset K \subset L \subset L(\psi) \subset Q_p(\zeta_s)$. It follows from the assumption that the extension $Q_p(\zeta_s)/Q_p$ is unramified, a fortiori, $L(\psi)/L$ is unramified. Because the values of the factor set β are roots of unity, it follows that $(\beta, L(\psi)/L) \sim L$. As q is an arbitrary prime, we conclude that $m_K(\chi) = 1$.

For the remainder of the paper we will use the same notation as in the theorem. Recall that $m_k(\chi)$ is the index of $A(\chi, k(\chi))$. Hence it suffices to prove $A(\chi, k(\chi)) \otimes_{k(\chi)} k(\chi)_{\mathfrak{p}} \sim k(\chi)_{\mathfrak{p}}$ for every prime $\mathfrak{p}$ of $k(\chi)$, where $k(\chi)_{\mathfrak{p}}$ is the completion of $k(\chi)$ with respect to $\mathfrak{p}$. For simplicity, set $K = k(\chi)_{\mathfrak{p}}$. Because $A(\chi, k(\chi)) \otimes_{k(\chi)} K$ is K-isomorphic to $A(\chi, K)$, we need to show $A(\chi, K) \sim K$, i.e., $m_K(\chi) = 1$. Note that $k(\chi)$ is a cyclotomic extension of the rational field Q. If M is a cyclotomic extension of Q containing $k(\chi)$, then $M^{\mathfrak{p}}$ represents the isomorphy type of the completion $M_{\mathfrak{P}}$, $\mathfrak{P}$ being any prime of M dividing $\mathfrak{p}$.

(i) Suppose that $\mathfrak{p}$ is an infinite prime. Denote by R (resp. C) the field of real numbers (resp. complex numbers). If $k(\chi)$ is not real, then $\mathfrak{p}$ is a complex prime, and so $m_K(\chi) = 1$. Suppose that $k(\chi)$ is real. Then $K = k(\chi)_{\mathfrak{p}} = R$, $l \neq 2$, and $n = 1$ or 2, i.e., $k = Q(\zeta_n) = Q$ and χ is real valued. Therefore, 4 does not divide s, the exponent of G. If $s = 1$ or 2, then G is abelian, and so $m_k(\chi) = 1$. Hence we assume that $s > 2$, so that the field $Q(\zeta_s)$ is imaginary and $R = K \subset Q(\zeta_s)^{\mathfrak{p}} = C$. Note that $m_K(\chi) = 1$ or 2. By the Brauer-Witt theorem there are subgroups F and N of G and a linear character ψ of N such that $F \triangleright N$ and $R(\psi^F) = R(\chi) = R$ and that $m_R(\chi)$ is equal to the index of a cyclotomic algebra of the form $(\beta, R(\psi)/R)$. Recall that $\mathscr{G}(R(\psi)/R) \cong F/N$. If $R(\psi) = R$, then $(\beta, R(\psi)/R) \sim R$. If $R(\psi) = C$, then $[F: N] = 2$. Set $F = N \cup Nf$. We have

$$(\beta, R(\psi)/R) = (\psi(f^2), C/R, \rho), \qquad (\rho(\sqrt{-1}) = -\sqrt{-1})$$

where the right side denotes a cyclic algebra over R and $\psi(f^2)$ is a root of unity contained in R so that $\psi(f^2) = \pm 1$. If $\psi(f^2) = -1$, then the order of f would be divisible by 4, which is a contradiction. Consequently, $\psi(f^2) = 1$ and so $(\psi(f^2), C/R, \rho) \sim R$, yielding that $m_K(\chi) = 1$.

(ii) Suppose that $\mathfrak{p}$ does not divide $s = l^a n$. Then the corollary implies that $m_K(\chi) = 1$.

(iii) Suppose that $\mathfrak{p} \mid l$ and $l = 2$. Then $\zeta_4 \in k$, and so $\zeta_4 \in K$. It follows from [3, Satz 12] that $m_K(\chi) = 1$.

(iv) Suppose that $\mathfrak{p} \mid l$ and $l \neq 2$. Let q be a prime number. Let L be the subfield of $M = Q(\zeta_{l^a}, \zeta_n)^{\mathfrak{p}}$ over $K = k(\chi)_{\mathfrak{p}} = Q(\zeta_n, \chi)_{\mathfrak{p}}$ such that $q \nmid [L: K]$ and $[M: L]$ is a power of q. By the Brauer-Witt theorem there exist subgroups F and N of G and a linear character ψ of N such that $G \supset F \triangleright N$, $\mathscr{G}(L(\psi)/L) \cong F/N$, $[F: N]$ is a power of q, and the q-part of $m_K(\chi)$ is equal to the index of a cyclotomic algebra of the form $(\beta, L(\psi)/L)$. Since $l \neq 2$ and $\mathscr{G}(M/K)$ is canonically isomorphic to a subgroup of $\mathscr{G}(Q(\zeta_{l^a})/Q)$, it follows that M/K is cyclic, and so $L(\psi)/L$ is cyclic. Let $q^c = [F: N] = [L(\psi): L]$, $\langle \sigma \rangle = \mathscr{G}(L(\psi)/L)$ and $F = \bigcup_{i=0}^{q^c-1} N f^i$. Then we have

$$(\beta, L(\psi)/L) = (\psi(f^{q^c}), L(\psi)/L, \sigma) , \qquad \psi(f^{q^c}) \in L .$$

As ψ is a linear character, $\psi(f^{q^c})$ is a primitive tth root of unity for some integer t. Let $t = q^d h$, $(q, h) = 1$. Then we can write $\psi(f^{q^c}) = \zeta_{q^d}\zeta_h$, which implies that the order of f is divisible by q^{c+d}. Consequently, q^{c+d} divides n, and so a primitive q^{c+d}th root of unity $\zeta_{q^{c+d}}$ belongs to L. We may assume that $\zeta_{q^{c+d}}^{q^c} = \zeta_{q^d}$. Let r be an integer satisfying $rq^c \equiv 1 \pmod{h}$. Since both $\zeta_{q^{c+d}}$ and ζ_h belong to L, it follows that

$$N_{L(\psi)/L}(\zeta_{q^{c+d}}\zeta_h^r) = \zeta_{q^{c+d}}^{q^c}\zeta_h^{rq^c} = \zeta_{q^d}\zeta_h ,$$

which yields that $(\psi(f^{q^c}), L(\psi)/L, \sigma) \sim L$. Therefore, the q-part of $m_K(\chi)$ is equal to 1. As q is an arbitrary prime, it follows that $m_K(\chi) = 1$.

(v) Suppose that $\mathfrak{p} \mid n$ and $\mathfrak{p} \nmid 2$. Then k contains a primitive pth root of unity ζ_p, p being the rational prime divided by $\mathfrak{p}$. It follows from [3, Satz 12] that $m_K(\chi) = 1$.

(vi) Suppose that $\mathfrak{p} \mid n$ and $\mathfrak{p} \mid 2$. Then $k = Q(\zeta_n)$. If $4 \mid n$ then $\zeta_4 \in K$ and so $m_K(\chi) = 1$. If $4 \nmid n$, then $4 \nmid s$. It follows from the corollary that $m_K(\chi) = 1$.

The theorem is completely proved.

REFERENCES

1. W. Feit, *Characters of Finite Groups*, Benjamin, New York, 1967.

2. P. Fong, *A note on splitting fields of representations of finite groups*, Illinois J. Math., **7** (1963), 515-520.
3. E. Witt, *Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einem Zahlkörper*, J. Reine Angew. Math., **190** (1952), 231-245.
4. T. Yamada, *Characterization of the simple components of the group algebras over the p-adic number field*, J. Math. Soc. Japan, **23** (1971), 295-310.

Received February 21, 1973. This research was done while the author was a Visiting Associate Professor of Queen's University for 1971/72.

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