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NONSOLVABLE FINITE GROUPS ALL OF WHOSE LOCAL SUBGROUPS ARE SOLVABLE, VI[1]

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This is the last paper in this series[2] and it contains the analysis of the remaining case, that is, $2 \in \pi_4$ and $e = 1$. As it happens, earlier work on this case was faulty, as I missed the group ${}^2F_4(2)$ and its simple subgroup of index 2. This lacuna is filled here, and the only change it necessitates in the earlier work is that the Main Theorem needs to be altered by added ${}^2F_4(2)'$ to the list of simple N-groups.[3]

16. The case $\mathfrak{T} \in \mathscr{M}^*$. All results in this paper are proved on the hypothesis that $2 \in \pi_4$ and $e = 1$. In this section, we also assume that if $\mathfrak{T}$ is a S_2-subgroup of $\mathfrak{G}$, then $\mathfrak{T} \in \mathscr{M}^*$. And we assume that $\mathfrak{G}$ is a minimal counterexample to the Main Theorem.

Set $\mathfrak{M} = \boldsymbol{M}(\mathfrak{T})$.

LEMMA 16.1. *If $\mathfrak{F} \lhd \mathfrak{M}$ and $\mathfrak{F}$ is an elementary abelian 2-group, then $\mathfrak{F} \notin \mathscr{M}^*$.*

Proof. Suppose false, so that $\mathfrak{M} = \boldsymbol{M}(\mathfrak{U})$ for every solvable subgroup of $\mathfrak{U}$ of $\mathfrak{G}$ which contains $\mathfrak{F}$. In particular, $\boldsymbol{C}(F) \subseteq \mathfrak{M}$ for all $F \in \mathfrak{F}^\#$, and also, of course $\boldsymbol{N}(\mathfrak{T}) \subseteq \mathfrak{M}$. By Lemma 13.2, there is a 2,3-subgroup $\mathfrak{H}$ of $\mathfrak{G}$ satisfying (a) through (e) of Lemma 13.2.

Let $\mathfrak{F}_0 = \mathfrak{F} \cap \mathfrak{H}$, $\mathfrak{F}_1 = \mathfrak{F} \cap \mathfrak{H}_1$, where $\mathfrak{H}_1 = \boldsymbol{O}_2(\mathfrak{H})$. Since $\mathfrak{F} \in \mathscr{M}^*$, we have $\mathfrak{F}_0 \subset \mathfrak{F}$. Since $e = 1$, $\mathfrak{H}_3$ is cyclic. Since $\boldsymbol{N}(\mathfrak{H}_2) \subseteq \mathfrak{M}$, it follows that $|\mathfrak{H}_2 : \mathfrak{H}_1| = 2$, whence $|\mathfrak{F}_0 : \mathfrak{F}_1| \leqq 2$.

If $\mathfrak{F}_0 = \mathfrak{F}_1$, then since $\mathfrak{F}_0 \subset \mathfrak{F}$, $\mathfrak{H}_1$ centralizes a subgroup $\mathfrak{E}/\mathfrak{F}_0$ of $\mathfrak{F}/\mathfrak{F}_0$ of order 2. Hence, $[\mathfrak{H}_1, \mathfrak{E}] \subseteq \mathfrak{F}_0 \subseteq \mathfrak{H}_1$, and so $\langle \mathfrak{H}_2, \mathfrak{E} \rangle$ is a 2-subgroup of $\boldsymbol{N}(\mathfrak{H}_1)$. Since Lemma 13.2(e) holds, we have $\mathfrak{E} \subseteq \mathfrak{H}_2 \cap \mathfrak{F} = \mathfrak{F}_0 = \mathfrak{F}_1$, against $|\mathfrak{E}/\mathfrak{F}_0| = 2$. Hence, $|\mathfrak{F}_0 : \mathfrak{F}_1| = 2$.

Choose $F \in \mathfrak{F}_0 - \mathfrak{F}_1$. Since $F \notin \boldsymbol{O}_2(H)$, we may assume that F normalizes $\mathfrak{H}_3$. Set $\mathfrak{K} = [\mathfrak{H}_1, \mathfrak{H}_3]$. Thus, $\mathfrak{H}_3$ has no fixed points on $\mathfrak{K}/\mathfrak{K}'$, and so

$$\mathfrak{K} = \langle [\mathfrak{K}, F], [\mathfrak{K}, F]^H \rangle ,$$

[1] An historical note is in order. In January, 1963, I announced at the meeting of the American Mathematical Society that with finitely many exceptions, the simple N-groups were $L_2(q)$ and $Sz(q)$. Had I been content to leave the explicit determination of the exceptions to someone else, I would have avoided the embarrassment of having missed ${}^2F_4(2)'$. Furthermore, several of the proofs would have been shortened considerably. But part of the fun and a great deal of the work involve pinning down the exceptions.

[2] The other papers are: Nonsolvable finite groups all of whose local subgroups are solvable, I-V: Bull. Amer. Math. Soc., 1968, Vol. 74, no. 3, pp. 383-437, Pacific J. Math., Vol. 33, no. 2, 1970, pp. 451-536, Vol. 39, no. 2, 1971, pp. 483-534, Vol. 48, No. 2, 1973, pp. 511-592, Vol. **50**, no. 1, (1974), 215-297.

[3] I have not taken the trouble to check Corollary 5 for the case $\mathfrak{G} = {}^2F_4(2)'$.

where H is a generator for $\mathfrak{H}_3$. Since $[\mathfrak{K}, F] \subseteq \mathfrak{F}$, it follows that $[\mathfrak{K}, F]$ is a normal elementary abelian subgroup of $\mathfrak{K}$ and so $\mathrm{cl}(\mathfrak{K}) \leqq 2$, $\mathfrak{K}' \subseteq [\mathfrak{K}, F] \cap [\mathfrak{K}, F]^H$. Since F centralizes $\mathfrak{K}'$, so does $\mathfrak{H}_3$. Since $C(F_1) \subseteq \mathfrak{M}$ for all $F_1 \in \mathfrak{F}^\#$, and since $\mathfrak{K}' \subseteq \mathfrak{F}$, while $\mathfrak{H}_3 \not\subseteq \mathfrak{M}$, we conclude that

$$\mathfrak{K} = [\mathfrak{K}, F] \times [\mathfrak{K}, F]^H \quad \text{is elementary abelian .}$$

Also, $C_{\mathfrak{K}}(F) = [\mathfrak{K}, F] = \mathfrak{F} \cap \mathfrak{K}$, $C_{\mathfrak{F}_1}(\mathfrak{H}_3) \cap \mathfrak{K} = 1$. Suppose $F_1 \in \mathfrak{F}_1$. Then $F_1 = UV$, where $U \in C_{\mathfrak{F}_1}(\mathfrak{H}_3)$, $V \in \mathfrak{K}$. Since $F_1 = F_1^F$, and since F normalizes $C_{\mathfrak{F}_1}(\mathfrak{H}_3)$ and $\mathfrak{K}$, it follows that $U = U^F$, $V = V^F \in C_{\mathfrak{K}}(F) = [\mathfrak{K}, F] \subseteq \mathfrak{F}$. Hence, $U = F_1 \cdot V^{-1} \in \mathfrak{F}$. Since $\mathfrak{H}_3 \not\subseteq \mathfrak{M}$, we get $U = 1$, and so $[\mathfrak{K}, F] = \mathfrak{F}_1$.

Set $\mathfrak{A} = [\mathfrak{K}, F]^H = \mathfrak{F}_1^H$, so that $\mathfrak{K} = \mathfrak{F}_1 \times \mathfrak{A}$. Furthermore, since $\mathfrak{H}_2$ is a S_2-subgroup of $N(\mathfrak{K})$, it follows that $N_{\mathfrak{F}}(\mathfrak{K}) = \mathfrak{F}_0$. Let $\mathfrak{F}^1/\mathfrak{F}_0$ be a subgroup of $\mathfrak{F}/\mathfrak{F}_0$ of order 2 which admits $\mathfrak{A}$. Thus, $[\mathfrak{K}, \mathfrak{F}^1] = [\mathfrak{A}, \mathfrak{F}^1] \subseteq \mathfrak{F}_0$, and $\mathfrak{F}^1 \not\subseteq N(\mathfrak{K})$, so that $[\mathfrak{A}, \mathfrak{F}^1] \not\subseteq \mathfrak{F}_1$. Let $\mathfrak{F}^1 = \mathfrak{F}_0 \times \langle U \rangle$, and choose A in $\mathfrak{A}$ such that $[A, U] = V \in \mathfrak{F}_0 - \mathfrak{F}_1$. Since $A^2 = 1$, we have $[A, V] = 1$, and so $V \in C_{\mathfrak{F}_0}(A)$. Since $A \in \mathfrak{F}_1$, and since $C_{\mathfrak{K}}(F) = \mathfrak{F}_1$, we get $V \in \mathfrak{F}_1$. This contradiction completes the proof.

Set $\mathfrak{Z} = \Omega_1(R_2(\mathfrak{M}))$, and let $\mathscr{J}$ be the set of involutions J of $\mathfrak{M}$ such that $C_{\mathfrak{M}}(J) \in \mathscr{M}^*$. Since $\mathfrak{T} \in \mathscr{M}^*$, we have

$$\Omega_1(Z(\mathfrak{T}))^\# \subseteq \mathscr{J} \ . \tag{16.1}$$

LEMMA 16.2. *One of the following holds*:

(a) $|\mathfrak{Z}| = 2$.

(b) $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$ *for every hyperplane* $\mathfrak{Z}_0$ *of* $\mathfrak{Z}$.

Proof. Suppose that (a) does not hold, so that $|\mathfrak{Z}| \geqq 4$. Let $\mathfrak{Z}^* = \Omega_1(Z(\mathfrak{T}))$, and suppose that $\mathfrak{Z}_0$ is a hyperplane of $\mathfrak{Z}$ with $C(\mathfrak{Z}_0) \not\subseteq \mathfrak{M}$. By (16.1), we have $\mathfrak{Z}_0 \cap \mathfrak{Z} = 1$, and so $\langle Z^* \rangle = \mathfrak{Z}^*$ is of order 2. Set $\mathfrak{C} = C(\mathfrak{Z})$ and let $\mathfrak{D}/\mathfrak{C}$ be a minimal normal subgroup of $\mathfrak{M}/\mathfrak{C}$. Since $\mathfrak{Z}$ is 2 reducible in $\mathfrak{M}$, $|\mathfrak{D}/\mathfrak{C}|$ is odd. Since $|\mathfrak{Z}^*| = 2$ and $|\mathfrak{Z}| > 2$, we have $|\mathfrak{D}/\mathfrak{C}| > 1$. Set $\mathfrak{Z}_1 = [\mathfrak{Z}, \mathfrak{D}]$, so that $1 \subset \mathfrak{Z}_1 \lhd \mathfrak{M}$, whence $Z^* \in \mathfrak{Z}_1$. Since $\mathfrak{Z}_1 = [\mathfrak{Z}_1, \mathfrak{D}]$, we have $Z^* \in \mathfrak{D}'$. By (16.1), we have $Z^{*M} \notin \mathfrak{Z}_0$ for all M in $\mathfrak{M}$.

Let $\mathfrak{Q}$ be a $S_{2'}$-subgroup of $\mathfrak{D}$, and let $\mathfrak{S} = \mathfrak{Z}_1\mathfrak{Q}$. Since $[\mathfrak{Z}_1, \mathfrak{D}] = \mathfrak{Z}_1$, so also $[\mathfrak{Q}, \mathfrak{Z}_1] = \mathfrak{Z}_1$. Let $\mathfrak{Z}^0 = \mathfrak{Z}_0 \cap \mathfrak{Z}_1$. Since $Z^* \in \mathfrak{Z}_1$, we get that $\mathfrak{Z}^0$ is a hyperplane of $\mathfrak{Z}_1$, and so $Z^{*Q} \in \mathfrak{Z}_1 - \mathfrak{Z}^0$ for all $Q \in \mathfrak{Q}$. This violates Lemma 5.38, and completes the proof.

LEMMA 16.3. *One of the following holds*:

(a) $|\mathfrak{Z}| = 2$.

(b) $C(\mathfrak{Z}_0) = C(\mathfrak{Z})$ *for every hyperplane* $\mathfrak{Z}_0$ *of* $\mathfrak{Z}$.

(c) $\mathfrak{M}$ *has a normal four-group* $\mathfrak{W}$ *such that* $A_{\mathfrak{M}}(\mathfrak{W}) = \mathrm{Aut}\,\mathfrak{W}$.

Proof. Suppose $|\mathfrak{Z}| > 2$ and $\mathfrak{Z}_0$ is a hyperplane of $\mathfrak{Z}$ such that $C(\mathfrak{Z}_0) \neq C(\mathfrak{Z})$. By Lemma 16.2, $C(\mathfrak{Z}_0) = C_{\mathfrak{M}}(\mathfrak{Z}_0)$. Since $\mathfrak{Z}_0 \subset \mathfrak{Z}$, we have

$C(\mathfrak{Z}_0) \supset C(\mathfrak{Z}) = \mathfrak{C}$. Set $\tilde{\mathfrak{C}} = C(\mathfrak{Z}_0)$. Since $\tilde{\mathfrak{C}}$ stabilizes the chain $\mathfrak{Z} \supset \mathfrak{Z}_0 \supset 1$, we see that $\tilde{\mathfrak{C}}/\mathfrak{C}$ is an elementary abelian 2-group. Choose $T \in \tilde{\mathfrak{C}} - \mathfrak{C}$, and let $\mathfrak{D}/\mathfrak{C} = O(\mathfrak{M}/\mathfrak{C})$, $\mathfrak{F}/\mathfrak{C} = F(\mathfrak{D}/\mathfrak{C})$. Since $O_2(\mathfrak{M}/\mathfrak{C}) = 1$, and $e = 1$, $\mathfrak{F}/\mathfrak{C}$ is a cyclic group and T does not centralize $\mathfrak{F}/\mathfrak{C}$. Let $\mathfrak{F}_1 = [\mathfrak{F}, \mathfrak{T}]\cdot\mathfrak{C}$, so that $\mathfrak{F}_1/\mathfrak{C} \neq 1$. Since T inverts $\mathfrak{F}_1/\mathfrak{C}$, and since $\mathfrak{F}_1/\mathfrak{C}$ acts faithfully on $\mathfrak{Z}$, it follows that $|\mathfrak{F}_1/\mathfrak{C}| = 3$, while $[\mathfrak{Z}, \mathfrak{F}_1] = \mathfrak{W}$ is a normal four-subgroup of $\mathfrak{M}$. The proof is complete.

LEMMA 16.4. $|\mathfrak{Z}| = 2$.

Proof. Suppose false. Define $\mathfrak{F}$ as follows: if Lemma 16.3(b) holds, take $\mathfrak{F} = \mathfrak{Z}$, and if Lemma 16.3(c) holds, but Lemma 16.3(b) does not hold, let $\mathfrak{F}$ be a normal four-subgroup of $\mathfrak{M}$ with $A_{\mathfrak{M}}(\mathfrak{F}) = \mathrm{Aut}\,(\mathfrak{F})$.

By Lemma 16.1, $\mathfrak{Z} \notin \mathscr{M}^*$. Set

$\mathscr{T} = \{\mathfrak{P} \mid \mathfrak{Z} \subseteq \mathfrak{P}, \mathfrak{P}$ is a 2-subgroup of $\mathfrak{M}$, $\mathfrak{P} \notin \mathscr{M}^*\}$. Choose $\mathfrak{T}_0$ in $\mathscr{T}$ with $|\mathfrak{T}_0|$ maximal. We assume without loss of generality that $\mathfrak{T}_0 \subseteq \mathfrak{T}$. This normalization is admissible, since $\mathfrak{Z} \lhd \mathfrak{M}$. Let $\mathfrak{S}$ be a solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{T}_0$ and is not contained in $\mathfrak{M}$, with $|\mathfrak{S}|$ minimal. Since $N_{\mathfrak{T}}(\mathfrak{T}_0) \in \mathscr{M}^*$, it follows that $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{S}$. By minimality of $|\mathfrak{S}|$, we have $\mathfrak{S} = \mathfrak{T}_0\mathfrak{Q}$, where $\mathfrak{Q}$ is a p-group and p is an odd prime. Since $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$ for every hyperplane $\mathfrak{Z}_0$ of $\mathfrak{Z}$, it follows that $\mathfrak{Z} \subseteq C(O(\mathfrak{S}))$, and so $\mathfrak{H} = O_2(\mathfrak{S}) \neq 1$. By maximality of $\mathfrak{T}_0$, it follows that $\mathfrak{T}_0$ is a S_2-subgroup of $N(\mathfrak{H})$. Since $O(N(\mathfrak{H})) = 1$ so also $O(\mathfrak{S}) = 1$. Since $e = 1$, $\mathfrak{Q}$ is cyclic.

Let $\mathfrak{V} = V(\mathrm{ccl}_{\mathfrak{S}}(\mathfrak{Z}); \mathfrak{T}_0)$. Thus, $\mathfrak{V} \ntriangleleft \mathfrak{S}$, since $N_{\mathfrak{T}}(\mathfrak{V}) \supset \mathfrak{T}_0$. Choose G in $\mathfrak{G}$ such that $\mathfrak{Z}^G = \mathfrak{X} \subseteq \mathfrak{T}_0$, $\mathfrak{X} \not\subseteq \mathfrak{H}$. Since $\mathfrak{Q}$ is a cyclic p-group, $\mathfrak{X} \cap \mathfrak{H} = \mathfrak{Y}$ is a hyperplane of $\mathfrak{X}$. On the other hand, $N_{\mathfrak{T}}(\mathfrak{H}) = \mathfrak{T}_0$, and so $Z(\mathfrak{T}) \subseteq \mathfrak{T}_0$, whence $Z(\mathfrak{T}) \subseteq Z(\mathfrak{H})$, as $C_{\mathfrak{S}}(\mathfrak{H}) = Z(\mathfrak{H})$. Set $\mathfrak{U} = \Omega_1(Z(\mathfrak{T}))^{\mathfrak{S}} = \Omega_1(Z(\mathfrak{T}))^{\mathfrak{Q}}$. Since $\mathfrak{Q} \not\subseteq \mathfrak{M}$, it follows from (16.1) that $\mathfrak{Q}$ does not centralize $\mathfrak{U}$, and so $C_{\mathfrak{S}}(\mathfrak{U}) = \mathfrak{H}\mathfrak{Q}_0$, where $\mathfrak{Q}_0 \subseteq D(\mathfrak{Q})$. Let $\mathfrak{X} = \mathfrak{Y} \times \langle X \rangle$. Since X inverts $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}$, we assume without loss of generality that X inverts $\mathfrak{Q}$. Thus, X does not centralize $\mathfrak{U}$. In particular, $C(\mathfrak{Y}) \supset C(\mathfrak{X})$, and so $\mathfrak{F}$ is a four-group.

Let $\mathfrak{W} = V(\mathrm{ccl}_{\mathfrak{S}}(\mathfrak{F}); \mathfrak{T}_0)$. Since $N_{\mathfrak{T}}(\mathfrak{W}) \supset \mathfrak{T}_0$, it follows that $\mathfrak{W} \ntriangleleft \mathfrak{S}$, and $\mathfrak{W} \not\subseteq \mathfrak{H}$. Choose G_1 in $\mathfrak{G}$ such that $\mathfrak{F}^{G_1} = \mathfrak{L} \subseteq \mathfrak{T}_0$, $\mathfrak{L} \not\subseteq \mathfrak{H}$. Set $\mathfrak{L}_1 = \mathfrak{L} \cap \mathfrak{H}$, so that $|\mathfrak{L} : \mathfrak{L}_1| = 2$.

Now $\mathfrak{L}$ is a normal 4-group of $\mathfrak{M}^{G_1}$ and $A_{\mathfrak{M}^{G_1}}(\mathfrak{L}) = \mathrm{Aut}\,(\mathfrak{L})$, so every involution of $\mathfrak{L}$ is central in some S_2-subgroup of $\mathfrak{M}^{G_1}$. Hence, by (16.1), $C(L) \subseteq \mathfrak{M}^{G_1}$ for all $L \in \mathfrak{L}^\#$. In particular, $\mathfrak{U} \subseteq C(L_1) \subseteq \mathfrak{M}^{G_1}$, and so $[\mathfrak{U}, \mathfrak{L}] \subseteq \mathfrak{L}$. Since $\mathfrak{Q}$ does not centralize $\mathfrak{U}$, and since $\mathfrak{L} = \mathfrak{L}_1 \times \langle T \rangle$, where T inverts $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}$, T does not centralize $\mathfrak{U}$. Since $[\mathfrak{U}, \mathfrak{L}] \subseteq \mathfrak{L}$, we get that $[\mathfrak{U}, \mathfrak{L}] = \mathfrak{L}_1 \subseteq Z(\mathfrak{H})$, and so $\mathfrak{H} \subseteq \mathfrak{M}^{G_1}$, $|\mathfrak{Q}| = 3$. But now we get that $[\mathfrak{H}, \mathfrak{L}] = \mathfrak{L}_1$, whence $[\mathfrak{H}, \mathfrak{Q}] = \mathfrak{H}_1$ is a 4-group, $\mathfrak{H} = \mathfrak{H}_1 \times$

$\mathfrak{H}_2$, $\mathfrak{H}_2 = C_{\mathfrak{H}}(\mathfrak{Q})$. Furthermore, $\mathfrak{H}_1 \lhd \mathfrak{S}$, and so $\mathfrak{L}_1 \subseteq Z(\mathfrak{T}_0)$ whence $\mathfrak{T}_0 \subseteq \mathfrak{M} \cap \mathfrak{M}^{G_1}$. By maximality of $\mathfrak{T}_0$, we conclude that $\mathfrak{M} = \mathfrak{M}^{G_1}$, $\mathfrak{L} = \mathfrak{F}$.

Since $\mathfrak{T}_0 \subset \mathfrak{T}$, there is $U \in N_{\mathfrak{T}}(\mathfrak{T}_0) - \mathfrak{T}_0$ with $U^2 \in \mathfrak{T}_0$. Thus, U normalizes $C_{\mathfrak{T}_0}(\mathfrak{F}) = \mathfrak{F} \times \mathfrak{H}_2$, and so U normalizes $D(C_{\mathfrak{T}_0}(\mathfrak{F})) = D(\mathfrak{H}_2)$. Since $\mathfrak{Q} \subseteq N(D(\mathfrak{H}_2))$, and since $\langle \mathfrak{T}_0, U \rangle \in \mathscr{M}^*$, it follows that $\mathfrak{H}_2$ is elementary. Since $\mathfrak{H}_2 \cap \mathfrak{H}_2^U$ is normalized by $\langle \mathfrak{T}_0, U, \mathfrak{Q} \rangle$, we conclude that $\mathfrak{H}_2 \cap \mathfrak{H}_2^U = 1$, and so $|\mathfrak{H}_2| = 2^h$, where $h \leqq 2$. In this case, $\mathfrak{T}_0$ has precisely 2 elementary subgroups of order 2^{h+2}, namely, $\mathfrak{H}$ and $C_{\mathfrak{T}_0}(\mathfrak{F})$, whence U normalizes $\mathfrak{H}$. This is false, since $\mathfrak{T}_0$ is a S_2-subgroup of $N(\mathfrak{H})$. The proof is complete.

LEMMA 16.5. *If $\mathfrak{U} \in \mathscr{U}(\mathfrak{T})$, then $C(U) \subseteq \mathfrak{M}$ for all $U \in \mathfrak{U}^\#$.*

Proof. By Lemma 16.4, $\mathfrak{Z} = \Omega_1(Z(\mathfrak{T}))$ is of order 2, and $\mathfrak{Z} \subseteq Z(\mathfrak{M})$. We assume by way of contradiction that $C(U) = \mathfrak{C} \not\subseteq \mathfrak{M}$. Thus, $\mathfrak{U} = \mathfrak{Z} \times \langle U \rangle$. Let $\mathfrak{S}$ be an element of $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{C}$ and let $\mathfrak{G} \supseteq \mathfrak{R} \supset \mathfrak{S}$, such that $\mathfrak{S}$ is a maximal subgroup of $\mathfrak{R}$.

It is crucial to show that

$$(16.2) \qquad |\mathfrak{S} : \mathfrak{S} \cap \mathfrak{M}| = 3 .$$

In any case, since $\mathfrak{C} \subseteq \mathfrak{S}$, we have $\mathfrak{S} \not\subseteq \mathfrak{M}$, and so $|\mathfrak{S} : \mathfrak{S} \cap \mathfrak{M}| = d > 1$. Let $\mathfrak{T}_0 = C_{\mathfrak{T}}(U)$, so that $|\mathfrak{T} : \mathfrak{T}_0| = 2$. Since $\mathfrak{T} \subseteq N(\mathfrak{T}_0)$, we have $N(\mathfrak{T}_0) \subseteq \mathfrak{M}$. Since $\mathfrak{T} \in \mathscr{M}^*$, it follows that $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{S}$.

Since $\mathfrak{U}$ centralizes every element of $\mathsf{M}(\mathfrak{U}; 2')$, it follows that $\mathfrak{H} = O_2(\mathfrak{S}) \neq 1$. Since $\mathfrak{S} = N(\mathfrak{H})$, it follows that $O(\mathfrak{S}) = 1$. For each odd prime p, let $\mathfrak{S}_p$ be a S_p-subgroup of $\mathfrak{S}$ permutable with $\mathfrak{T}_0$, and let $\mathfrak{S}(p) = \mathfrak{T}_0 \cdot \mathfrak{S}_p$. Thus, $O(\mathfrak{S}(p)) = 1$ for all p. Since $J(\mathfrak{T}_0)$ and $Z(\mathfrak{T}_0)$ are normal in $\mathfrak{T}$, it follows that $\langle N(J(\mathfrak{T}_0)), N(Z(\mathfrak{T}_0)) \rangle \subseteq \mathfrak{M}$. Hence, $\mathfrak{S}(p) \subseteq \mathfrak{M}$ for all $p \geqq 5$, by Lemma 5.53. By Lemma 5.54, $\mho^1(\mathfrak{S}(3)) \subseteq \mathfrak{M}$. This is (16.2).

Next, set $\mathfrak{S}_0 = \mathfrak{S} \cap \mathfrak{M}$ and suppose that $\mathfrak{S}_0 \subset \mathfrak{M}_0 \subseteq \mathfrak{M}$, and that $\mathfrak{M}_0$ contains a S_2-subgroup of $\mathfrak{M}$. Let $\pi = \pi(\mathfrak{S}_0)$ and let $\widetilde{\mathfrak{M}}$ be a S_π-subgroup of $\mathfrak{M}_0$ which contains $\mathfrak{S}_0$. Let $\tilde{\mathfrak{H}} = O_2(\widetilde{\mathfrak{M}})$, $\tilde{\mathfrak{H}}_0 = \tilde{\mathfrak{H}} \cap \mathfrak{S}_0$. Since $\mathfrak{T}_0 \subseteq \mathfrak{S}_0$ and $\mathfrak{T}_0$ is of index 2 in a S_2-subgroup of $\widetilde{\mathfrak{M}}$, it follows that $|\tilde{\mathfrak{H}} : \tilde{\mathfrak{H}}_0| \leqq 2$.

We argue that $\mathfrak{M}_0 - \mathfrak{S}_0$ contains a 2-element T which normalizes $\mathfrak{S}_0$. This is clear if $|\tilde{\mathfrak{H}} : \tilde{\mathfrak{H}}_0| = 2$, since in this case, we may take $T \in \tilde{\mathfrak{H}} - \tilde{\mathfrak{H}}_0$. Suppose $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}_0$. Let $\tilde{\mathfrak{F}}/\tilde{\mathfrak{H}} = F(\widetilde{\mathfrak{M}}/\tilde{\mathfrak{H}})$, so that $\tilde{\mathfrak{F}}/\tilde{\mathfrak{H}}$ is a cyclic group of odd order, and $\widetilde{\mathfrak{M}}' \subseteq \tilde{\mathfrak{F}}$. Let $\mathfrak{K}_0$ be a $S_{2'}$-subgroup of $\mathfrak{S}_0$ and let $\mathfrak{K}$ be a $S_{2'}$-subgroup of $\widetilde{\mathfrak{M}}$ which contains $\mathfrak{K}_0$. Since $\mathfrak{K}$ is a Z-group, it follows that subgroups of $\mathfrak{K}$ are conjugate if and only if they have the same order. This implies that $N_{\widetilde{\mathfrak{M}}}(\mathfrak{K}_0)\tilde{\mathfrak{H}}$ contains a S_2-subgroup of $\widetilde{\mathfrak{M}}$, and since $\mathfrak{T}_0 \subseteq \tilde{\mathfrak{H}} . N_{\widetilde{\mathfrak{M}}}(\mathfrak{K}_0)$, it follows that $\mathfrak{S}_0$ is normalized by a S_2-subgroup of $\widetilde{\mathfrak{M}}$, so T exists.

Case 1. $\mathfrak{T}_0$ is not a S_2-subgroup of $\mathfrak{R}$.

Let $\mathfrak{T}_1$ be a S_2-subgroup of $\mathfrak{R}$ which contains $\mathfrak{T}_0$. Since $\mathfrak{T} \subseteqq N(\mathfrak{T}_0)$, we have $N(\mathfrak{T}_0) \subseteqq \mathfrak{M}$, and so $\mathfrak{T}_1 \subseteqq \mathfrak{M}$. Let $\mathfrak{M}_0 = \mathfrak{M} \cap \mathfrak{R}$, so that $\mathfrak{M}_0 \supseteqq \langle \mathfrak{T}_1, \mathfrak{S}_0 \rangle$. Choose $T \in N_{\mathfrak{M}_0}(\mathfrak{S}_0) - \mathfrak{S}_0$, T a 2-element. We may assume that $T^2 \in \mathfrak{S}_0$. Now $\mathfrak{S} \subset \mathfrak{R}$, and so $\mathfrak{R}$ is not solvable. Since $\mathfrak{G}$ is an N-group, so is $\mathfrak{R}$, and so 1 is the only solvable normal subgroup of $\mathfrak{R}$. In particular,

$$\bigcap_{R \in \mathfrak{R}} \mathfrak{S}^R = 1 ,$$

and so $\mathfrak{R}$ is represented faithfully as permutations of the cosets of $\mathfrak{S}$ in $\mathfrak{R}$. Since $\mathfrak{S}$ is a maximal subgroup of $\mathfrak{R}$, this permutation group is primitive. Since $\mathfrak{S} \cap \mathfrak{S}^T = \mathfrak{S}_0$, $\mathfrak{S}$ has an orbit of size 3. By the Main Theorem of Wong [Determination of a class of primitive permutation groups, Math. Zeitschr. 99, 235–246 (1967)], $\mathfrak{R}$ is isomorphic to one of the following groups:

A_5, S_5, PGL(2, 7), PSL(2, 11), PSL(2, 13), PSL(2, q)
(q a prime $\equiv \pm 1 \pmod{16}$), SL(3, 3), Aut (SL(3, 3)).

Since $2 \in \pi_4$, it follows that $\mathscr{SCN}_3(\mathfrak{T}_1) \neq \varnothing$. However, a S_2-subgroup of each of the above groups has no elementary normal subgroup of order 2^3.

Case 2. $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{R}$.

In this case, since $\mathfrak{T}_0$ is not a S_2-subgroup of $\mathfrak{G}$, we have $\mathfrak{R} \subset \mathfrak{G}$. Since $\mathfrak{G}$ is a minimal counterexample, $\mathfrak{R}$ contains a simple normal subgroup $\mathfrak{R}_0$ such that $C_{\mathfrak{R}}(\mathfrak{R}_0) = 1$, and such that $\mathfrak{R}_0$ is one of the groups listed in the (augmented) Main Theorem. Let $\mathfrak{R}_1 = \mathfrak{R}_0\mathfrak{T}_0$. If $\mathfrak{R}_0 \cong {}^2F_4(2)'$, then either $\mathfrak{R}_1 = \mathfrak{R}_0$ or $\mathfrak{R}_1 \cong {}^2F_4(2)$. Both possibilities are excluded since $\Omega_1(Z(\mathfrak{T}_0)) = \mathfrak{U}$, while S_2-subgroups of both ${}^2F_4(2)'$ and ${}^2F_4(2)$ have cyclic centers.

Suppose $\mathfrak{R}_0 \cong U_3(3) = \mathrm{PSU}(3, 3) (= \mathrm{SU}(3, 3))$. Since $\mathrm{GU}(3, 3) = \mathrm{SU}(3, 3) \times Z(\mathrm{GU}(3, 3))$ it follows that either $\mathfrak{R}_1 \cong U_3(3)$ or $\mathfrak{R}_1 \cong U_3(3)\langle S \rangle$, where S is induced by a field automorphism. In either case, a S_2-subgroup of $\mathfrak{R}_1$ contains no normal elementary subgroup of order 2^3, against $\mathfrak{T}_0 = C_{\mathfrak{T}}(U)$.

If $\mathfrak{R}_0 \cong M_{11}$, then $\mathfrak{R}_1 \cong M_{11} \cong \mathrm{Aut}\,(M_{11})$. This case is excluded since $\mathscr{SCN}_3(\mathfrak{T}_0) \neq \varnothing$.

If $\mathfrak{R}_0 \cong L_3(3)$, then $\mathfrak{R}_1$ is either isomorphic to $L_3(3)$ or to $L_3(3)\langle S \rangle$, where S is the transpose inverse map. This case is also excluded, since $\mathscr{SCN}_3(\mathfrak{T}_0) \neq \varnothing$.

If $\mathfrak{R}_0 \cong A_7$, then $\mathsf{M}\,(\mathfrak{T}_0; 3) \neq 1$, against $2 \in \pi_4$.

If $\mathfrak{R}_0 \cong Sz(q)$, then $\mathfrak{R}_0 = \mathfrak{R}_1$, since $\mathrm{Aut}\,(Sz(q))/I(Sz(q))$ has odd order. In this case, every 2-local subgroup of $\mathfrak{R}_1$ is 2-closed. This is false, since $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{S}$ and $\mathfrak{S} \nsubseteq \mathfrak{M}$, while $N(\mathfrak{T}_0) \subseteqq \mathfrak{M}$.

Suppose $\mathfrak{N}_0 \cong L_2(2^n)$. Since $\mathfrak{S}$ is not 2-closed, $\mathfrak{T}_0$ is non abelian. Since $\mathfrak{N}_1$ is an N-group, the only possibility is that $\mathfrak{N}_1 \subseteq \Sigma_5$. This violates $\mathscr{SCN}_3(\mathfrak{T}_0) \neq \emptyset$.

So $\mathfrak{N}_0 \cong L_2(q)$ for some odd q. Let $q = p^a$, p a prime. Then $\mathrm{Aut}\,(L_2(q))/I(L_2(q))$ is the direct product of a group of order 2 and a cyclic group of order a. Thus, $\mathfrak{N}_1/\mathfrak{N}_0$ is abelian, and is either cyclic or of type $(2, 2^b)$.

Let $\mathfrak{T}^0 = \mathfrak{T}_0 \cap \mathfrak{N}_0$. Thus, $\mathfrak{T}^0$ is a dihedral group. First, suppose $|\mathfrak{T}^0| \geqq 16$. In this case, $\mathscr{SCN}_2(\mathfrak{T}^0) = \emptyset$, and so it is straightforward to verify that $\mathscr{SCN}_3(\mathfrak{T}_0) = \emptyset$, the desired contradiction. Hence $|\mathfrak{T}^0| \leqq 8$. If $|\mathfrak{T}^0| = 4$, then $a = 0$, and so $b = 0$, and $\mathfrak{T}_0$ is non abelian of order 8, against $\mathscr{SCN}_3(\mathfrak{T}_0) \neq \emptyset$. Hence, $|\mathfrak{T}^0| = 8$.

Since $|\mathfrak{T}^0| = 8$, it follows that a is either odd or twice an odd number. Since $\mathfrak{T}_0$ contains an element of $\mathscr{SCN}_3(\mathfrak{T})$, it follows that $\mathsf{M}(\mathfrak{T}_0 ; 2') = \{1\}$. Thus, if Z_0 is the central involution of $\mathfrak{T}^0$, then $\boldsymbol{C}_{\mathfrak{N}_0}(Z_0)$ is core free, that is, if $\varepsilon \equiv q \pmod 4$, then $(q - \varepsilon)$ is a power of 2. Since $|\mathfrak{T}^0| = 8$, we get $q - \varepsilon = 8$, whence $q = 7$ or 9. If $q = 7$, we get $\mathscr{SCN}_3(\mathfrak{T}_0) = \emptyset$. So $q = 9$, and the only possibility is that $\mathfrak{N}_1 \cong \Sigma_6$. In this case, we see that $\mathfrak{S} \cong Z_2 \times \Sigma_4$. Thus, $\mathfrak{T}_0$ has precisely 2 elementary subgroups of order 8, one of which is an element of $\mathscr{SCN}(\mathfrak{T})$. Hence, $\mathfrak{T}$ normalizes both of these elementary subgroups, and so $\mathfrak{T}$ normalizes $\boldsymbol{O}_2(\mathfrak{S})$. This is false, since $\mathfrak{S} = \boldsymbol{N}(\boldsymbol{O}_2(\mathfrak{S}))$, and $\mathfrak{T} \not\subseteq \mathfrak{S}$. The proof is complete.

LEMMA 16.6. *If $\mathfrak{F}$ is a non cyclic normal elementary abelian 2-subgroup of $\mathfrak{M}$, then $\boldsymbol{C}(\mathfrak{F}_0) \subseteq \mathfrak{M}$ for every hyperplane $\mathfrak{F}_0$ of $\mathfrak{F}$.*

Proof. Since $\mathfrak{F}$ is non cyclic and $\boldsymbol{Z}(\mathfrak{T})$ is cyclic, $\mathfrak{F}$ contains an element $\mathfrak{U}$ of $\mathscr{U}(\mathfrak{T})$. Since $\mathfrak{F}_0 \cap \mathfrak{U} \neq 1$, this lemma is a consequence of Lemma 16.5.

By Theorems 13.5, 13.6, 13.7, $\mathfrak{M}$ contains a non cyclic normal elementary abelian 2-subgroup, so we can choose $\mathfrak{F}$ such that

(a) $\mathfrak{F} \lhd \mathfrak{M}$.

(b) $\mathfrak{F}$ is an elementary abelian 2-group.

(c) $\mathfrak{F}/\mathfrak{Z}$ is a chief factor of $\mathfrak{M}$.

LEMMA 16.7. *Suppose T is an involution of $\mathfrak{M}$. Then one of the following holds*:

(a) $[\mathfrak{F}, T] \subseteq \mathfrak{Z}$.

(b) *$\mathfrak{F}/\mathfrak{Z}$ is a free $F_2\langle T\rangle$-module.*

Proof. Suppose $[\mathfrak{F}, T] \not\subseteq \mathfrak{Z}$. Set $\bar{\mathfrak{F}} = \mathfrak{F}/\mathfrak{Z}$, and let $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{M}}(\bar{\mathfrak{F}})$, $\bar{\mathfrak{M}} = \mathfrak{M}/\mathfrak{C}$, $\bar{T} = \mathfrak{C}T$. Thus $\bar{T} \neq 1$, and $\boldsymbol{O}_2(\bar{\mathfrak{M}}) = 1$. Let $\bar{\bar{\mathfrak{F}}} = \boldsymbol{F}(\bar{\mathfrak{M}})$. Thus, $\bar{\bar{\mathfrak{F}}}$ is a cyclic group of odd order and $\boldsymbol{C}_{\bar{\mathfrak{M}}}(\bar{\bar{\mathfrak{F}}}) = \bar{\bar{\mathfrak{F}}}$. Hence, $\bar{T}$

inverts a subgroup $\mathfrak{P}$ of $\bar{\mathfrak{F}}$ of prime order. Since $\mathfrak{P}$ char $\bar{\mathfrak{F}}$, we have $\mathfrak{P} \lhd \bar{\mathfrak{M}}$, and since $\bar{\mathfrak{M}}$ acts faithfully and irreducibly on $\bar{\mathfrak{F}}$, we have $C_{\bar{\mathfrak{F}}}(\mathfrak{P}) = 1$. The lemma follows.

LEMMA 16.8. *Suppose $\mathfrak{X} = \langle X \rangle \times \langle Y \rangle$ is a four-group contained in $\mathfrak{F}$ and $\mathfrak{Z} \not\subseteq \mathfrak{X}$. Suppose $\mathfrak{S}$ is a S_2-subgroup of $C_{\mathfrak{M}}(X)$. Then there is S in $\mathfrak{S}$ such that $[Y, S]$ generates $\mathfrak{Z}$.*

Proof. Let $\mathfrak{C} = C(\mathfrak{F})$, $\mathfrak{D}/\mathfrak{C} = O_2(\mathfrak{M}/\mathfrak{C})$. Since $\mathfrak{F}$ is not 2-reducible in $\mathfrak{M}$, we get that $C_{\mathfrak{F}}(\mathfrak{D}) = \mathfrak{Z}$, $[\mathfrak{F}, \mathfrak{D}] = \mathfrak{Z}$, and so $\mathfrak{D}/\mathfrak{C}$ is isomorphic to the stability group of the chain $\mathfrak{F} \supset \mathfrak{Z} \supset 1$. Let $\mathfrak{S}_1$ be a S_2-subgroup of $C_{\mathfrak{D}}(X)$ and let $\mathfrak{S}_2$ be a S_2-subgroup of $C_{\mathfrak{M}}(X)$ which contains $\mathfrak{S}_1$. Since $[\mathfrak{D}, X] = \mathfrak{Z}$, $\mathfrak{S}_1$ is of index 2 in a S_2-subgroup of $\mathfrak{D}$. Also $\mathfrak{S}_2^M = \mathfrak{S}$ for some $M \in C_{\mathfrak{M}}(X)$, and so $\mathfrak{S}$ contains $\mathfrak{S}_1^M$. Since $\mathfrak{D} \lhd \mathfrak{M}$, $\mathfrak{S}_1^M$ is of index 2 in a S_2-subgroup of $\mathfrak{D}$, and so $\langle X, \mathfrak{Z} \rangle = C_{\mathfrak{F}}(\mathfrak{S}_1^M)$. So we can choose S in $\mathfrak{S}_1^M - C(Y)$, whence $[Y, S]$ generates $\mathfrak{Z}$, as required.

LEMMA 16.9. *If $|\mathfrak{F}| \leqq 2^4$, then $C(F) \subseteq \mathfrak{M}$ for all $F \in \mathfrak{F}^\#$.*

Proof. Since $|\mathfrak{F}| \leqq 2^4$ and $\mathfrak{F}/\mathfrak{Z}$ is a chief factor of M, while $F \sim FZ$ for all F in $\mathfrak{F} - \mathfrak{Z}$ (where $\mathfrak{Z} = \langle Z \rangle$), it follows that $\mathfrak{M}$ is transitive on $\mathfrak{F} - \mathfrak{Z}$, so this lemma is a consequence of Lemma 16.5.

LEMMA 16.10. *Suppose $\mathfrak{E}$ is a hyperplane of $\mathfrak{F}$ and T is an involution of $\mathfrak{M}$ with $C_{\mathfrak{F}}(T) = \mathfrak{E}$. Then one of the following holds:*
(a) $[\mathfrak{F}, T] = \mathfrak{Z}$.
(b) $|\mathfrak{F}| = 2^3$.

Proof. This lemma is a consequence of Lemma 16.7.

With these results at our disposal, we turn to the final configuration of this section. By Lemma 16.1, $\mathfrak{F} \notin \mathscr{M}^*$. Let

$$\mathscr{T} = \{\mathfrak{S} \mid \mathfrak{F} \subseteq \mathfrak{S} \subseteq \mathfrak{M}, \mathfrak{S} \notin \mathscr{M}^*, \mathfrak{S} \text{ is a 2-group}\} .$$

Choose $\mathfrak{T}_0$ in $\mathscr{T}$ with $|\mathfrak{T}_0|$ maximal. Since $\mathfrak{F} \lhd \mathfrak{M}$, we assume without loss of generality that $\mathfrak{T}_0 \subseteq \mathfrak{T}$. Thus, if $\mathfrak{T}_1$ is any 2-subgroup of $\mathfrak{G}$ which contains $\mathfrak{T}_0$ properly, then $\mathfrak{T}_1 \subseteq \mathfrak{M}$, and $\mathfrak{T}_1 \in \mathscr{M}^*$. Let

$$\mathscr{S} = \{\mathfrak{S} \mid \mathfrak{T}_0 \subseteq \mathfrak{S}, \mathfrak{S} \subseteq \mathfrak{G}, \mathfrak{S} \not\subseteq \mathfrak{M}, \mathfrak{S} \text{ solvable}\} ,$$

and choose $\mathfrak{S}$ in $\mathscr{S}$ of minimal order. Thus $\mathfrak{S} = \mathfrak{T}_0\mathfrak{P}$ where $\mathfrak{P}$ is a p-group for some odd prime p. Since $C(\mathfrak{F}_0) \subseteq \mathfrak{M}$ for all hyperplanes $\mathfrak{F}_0$ of $\mathfrak{F}$, it follows that $O(\mathfrak{S}) \subseteq \mathfrak{M}$. Hence, $\mathfrak{F}$ centralizes $O(\mathfrak{S})$, and so $O_2(\mathfrak{S}) = \mathfrak{H} \neq 1$. By maximality of $\mathfrak{T}_0$, it follows that $\mathfrak{T}_0$ is a S_2-subgroup of $N(\mathfrak{H})$, and so $O(\mathfrak{S}) = 1$. Since $e = 1$, $\mathfrak{P}$ is cyclic. By

Lemma 5.53, $p = 3$.

Set $\mathfrak{B} = \Omega_1(\mathbf{R}_2(\mathfrak{S}))$, so that $\mathfrak{Z} \subseteq \mathfrak{B}$. Since $\mathfrak{P} \not\subseteq \mathfrak{M}$, we have $[\mathfrak{B}, \mathfrak{P}] = \mathfrak{B}_0 \neq 1$. Set $\mathfrak{W} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}); \mathfrak{T}_0)$. Since $N_{\mathfrak{X}}(\mathfrak{W}) \supset \mathfrak{T}_0$, we have $\mathfrak{W} \not\triangleleft \mathfrak{S}$, and so $\mathfrak{W} \not\subseteq \mathfrak{H}$. Choose G in $\mathfrak{G}$ such that $\mathfrak{X} = \mathfrak{F}^G \subseteq \mathfrak{T}_0$, $\mathfrak{X} \not\subseteq \mathfrak{H}$. Let $\mathfrak{X}_1 = \mathfrak{X} \cap \mathfrak{H}$, so that $\mathfrak{X}_1$ is a hyperplane of $\mathfrak{X}$. Choose $F \in \mathfrak{X} - \mathfrak{X}_1$. Since F inverts some S_3-subgroup of $\mathfrak{S}$, we assume without loss of generality that F inverts $\mathfrak{P}$. Now $\mathfrak{B}_0$ is a free $F_2\langle F\rangle$-module, and $\mathfrak{B}_0 \subseteq C(\mathfrak{X}_1) \subseteq \mathfrak{M}^G$. Also, $\mathfrak{B}_0 \triangleleft \mathfrak{T}_0 = \mathfrak{H}\langle F\rangle$, and so $[\mathfrak{B}_0, F] \subseteq Z(\mathfrak{T}_0)$. We argue that $C([\mathfrak{B}_0, F]) \subseteq \mathfrak{M}^G$. This is clear if $\mathfrak{Z}^G \subseteq [\mathfrak{B}_0, F]$, and if $\mathfrak{Z}^G \not\subseteq [\mathfrak{B}_0, F]$, then Lemmas 16.10 and 16.9 imply that $C(V) \subseteq \mathfrak{M}^G$ for all $V \in \mathfrak{X}^\#$. So in any case, $C([\mathfrak{B}_0, F]) \subseteq \mathfrak{M}^G$. In particular, $\mathfrak{T}_0 \subseteq \mathfrak{M}^G$. By a previous remark, this forces $\mathfrak{M} = \mathfrak{M}^G$, $\mathfrak{X} = \mathfrak{F}$. So $\mathfrak{F} \triangleleft \mathfrak{T}_0$. Since F inverts $\mathfrak{P}$ and $\mho^1(\mathfrak{P}) \subseteq \mathfrak{M}$, we have $|\mathfrak{P}| = 3$.

Let $\mathfrak{H}_1 = [\mathfrak{H}, \mathfrak{P}]$, $\mathfrak{H}_2 = C_{\mathfrak{H}}(\mathfrak{P})$, and let P be a generator for $\mathfrak{P}$. Let $\mathfrak{E} = [\mathfrak{H}_1, F]$. Thus, $\mathfrak{E}$ is a normal elementary subgroup of $\mathfrak{H}_1$, and $\langle \mathfrak{E}, \mathfrak{E}^P\rangle = \mathfrak{H}_1$, the equality holding since $\mathfrak{H}_1/D(\mathfrak{H}_1)$ is a free $F_2\langle F\rangle$-module.

Since $\mathfrak{H}_1 = \mathfrak{E}^P \cdot \mathfrak{E}$, we have $\mathrm{cl}(\mathfrak{H}_1) \leqq 2$, $\mathfrak{H}_1' \subseteq \mathfrak{E} \cap \mathfrak{E}^P$. Since F centralizes $\mathfrak{H}_1'$, so does $\mathfrak{P}$, and so $\mathfrak{Z} \not\subseteq \mathfrak{H}_1'$. We argue that

$$|\mathfrak{H}_1'| \leqq 2 .$$

In any case $\mathfrak{H}_1' \triangleleft \mathfrak{S}$, so if $\mathfrak{H}_1' \neq 1$, we can choose $X \in \mathfrak{H}_1' \cap Z(\mathfrak{S})^\#$. Suppose $|\mathfrak{H}_1'| \geqq 4$, and $\langle X, Y\rangle$ is a four-group contained in $\mathfrak{H}_1'$. By maximality of $\mathfrak{T}_0$, it follows that $\mathfrak{T}_0$ is a S_2-subgroup of $C(X)$. By Lemma 16.8, $[Y, S]$ is a generator of $\mathfrak{Z}$ for some $S \in \mathfrak{T}_0$. This is false, since $\mathfrak{Z} \not\subseteq \mathfrak{H}_1'$, while $\mathfrak{H}_1' \triangleleft \mathfrak{S}$. So $|\mathfrak{H}_1'| \leqq 2$.

Case 1. $\mathfrak{H}_1$ is a four-group.

Here we get $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$. Choose $T \in N_{\mathfrak{X}}(\mathfrak{T}_0) - \mathfrak{T}_0$ with $T^2 \in \mathfrak{T}_0$. Then $\mathfrak{H}_2 \cap \mathfrak{H}_2^T = 1$, since $\langle \mathfrak{P}, \mathfrak{T}_0, T\rangle \subseteq N(\mathfrak{H}_2 \cap \mathfrak{H}_2^T)$, and $\langle \mathfrak{T}_0, T\rangle \in \mathscr{M}^*$. Since $\mathfrak{T}_0/\mathfrak{H}_2$ is a dihedral group of order 8, we conclude that $\mathfrak{H}_2$ is isomorphic to a subgroup of a dihedral group of order 8. First, suppose that $\mathfrak{H}_2$ is elementary abelian. Since $\mathfrak{H} \neq J(\mathfrak{T}_0)$, we conclude that F centralizes $\mathfrak{H}_2$. Thus, $C_{\mathfrak{T}_0}(\mathfrak{F})$ and $\mathfrak{H}$ are the only elementary subgroups of $\mathfrak{T}_0$ of index 2. Since T normalizes $C_{\mathfrak{T}_0}(\mathfrak{F})$, we get $T \in N(\mathfrak{H})$, which is false. Next, suppose $\mathfrak{H}_2$ is cyclic of order 4. If $\mathfrak{H}_2 \subseteq C(\mathfrak{F})$, then $\mathfrak{T}_0 = \mathfrak{H}_2 \times \langle \mathfrak{H}_1, F\rangle$, so that $D(\mathfrak{H}_2)$ char $\mathfrak{T}_0$. This is false, since $\mathfrak{H}_2 \cap \mathfrak{H}_2^T = 1$. If $[\mathfrak{H}_2, F] \neq 1$, we get that $D(\mathfrak{H}_2) \subseteq \mathfrak{F}$. In this case, since $|\mathfrak{F}| = 2^3$, Lemma 16.9 implies that $C(D(\mathfrak{H}_2)) \subseteq \mathfrak{M}$, against $\mathfrak{P} \subseteq C(D(\mathfrak{H}_2))$. So $\mathfrak{H}_2$ is a dihedral of order 8. Hence, $|\mathfrak{F}| \leqq 2^4$, and so by Lemma 16.9, $\mathfrak{F} \cap \mathfrak{H}_2 = 1$. In particular, $\mathfrak{F}$ centralizes $\mathfrak{H}_2$, and $\mathfrak{T}_0 = \mathfrak{H}_2 \times \langle \mathfrak{H}_1, F\rangle \cong D_8 \times D_8$. Since $C_{\mathfrak{T}_0}(F) = \mathfrak{H}_2 \times \langle [\mathfrak{H}_1, F]\rangle \times \langle F\rangle$, we get that T normalizes $C_{\mathfrak{T}_0}(F)' = \mathfrak{H}_2'$. This contradiction shows that this case does not arise.

Case 2. $\mathfrak{H}_1' = 1$.

Since $\mathfrak{P}$ acts faithfully on $\mathfrak{H}$, and since Case 1 does not hold, we have $|\mathfrak{H}_1| = 2^{2w}$, where $w \geqq 2$. Also $C_{\mathfrak{H}_1}(F) = [\mathfrak{H}_1, F] = \mathfrak{H}_1 \cap \mathfrak{F}$ is of order 2^w. A standard argument now shows that

$$\mathfrak{F}_1 \subseteqq C(\mathfrak{H}_1), \mathfrak{F}_1 = (\mathfrak{H}_2 \cap \mathfrak{F}_1) \times (\mathfrak{H}_1 \cap \mathfrak{F}_1) .$$

Since $w \geqq 2$, there is H in $\mathfrak{H}_1$ such that $[F, H] \notin \mathfrak{Z}$. Since $\mathfrak{F}_1 \subseteqq C(H)$, Lemma 16.10 implies that $|\mathfrak{F}| = 2^3$. Hence, $w = 2$. It follows readily that $J(\mathfrak{T}_0) \subseteqq \mathfrak{H}$, and so $J(\mathfrak{T}_0) \lhd \mathfrak{S}$, against $N(J(\mathfrak{T}_0)) \subseteqq \mathfrak{M}$.

Case 3. $\mathfrak{H}_1' \neq 1$.

Let $\mathfrak{Y} = Z(\mathfrak{H}_1)$. Since $\mathfrak{P}$ has no fixed points on $\mathfrak{H}_1/\mathfrak{H}_1'$, it follows that $\mathfrak{Y} = \mathfrak{Y}_1 \times \mathfrak{H}_1'$, where $\mathfrak{Y}_1 = [\mathfrak{Z}, \mathfrak{P}]$. Hence, $\mathfrak{H}_1 = \mathfrak{Y}_1 \times \mathfrak{B}$, where $\mathfrak{B}$ is extra special. Since $\mathfrak{Z}$ centralizes $\mathfrak{H}_1$, we conclude that $Z \notin \mathfrak{B}$.

Case 3a. $\mathfrak{H}_1$ is extra special.

Since $\mathfrak{H}_1/D(\mathfrak{H}_1)$ is a free $F_2\langle F\rangle$-module, it follows that $\mathfrak{H}_1 \cap \mathfrak{F}$ contains a four-group $\mathfrak{A}$ with $\mathfrak{H}_1' = \langle X\rangle \subset \mathfrak{A}$. Since $\mathfrak{T}_0$ is a S_2-subgroup of $C(X)$, Lemma 16.8 implies that $\mathfrak{Z} \subseteqq \mathfrak{H}_1 = \mathfrak{B}$. This is false, and so this case does not occur.

Case 3b. $\mathfrak{H}_1$ is not extra special.

Here we can find a subgroup $\mathfrak{Y}^1$ of $\mathfrak{Y}$ of order 2^3 such that $\mathfrak{H}_1' \subset \mathfrak{Y}^1 \lhd \mathfrak{S}$. Since $\mathfrak{H} = \mathfrak{H}_1\mathfrak{H}_2$, we get $\mathfrak{Y}^1 \subseteqq Z(\mathfrak{H})$, and so $[\mathfrak{Y}^1, \mathfrak{P}] = \mathfrak{Y}^* \lhd \mathfrak{S}$, $\mathfrak{Y}^*$ a four-group. Choose $Y^* \in \mathfrak{Y}^* - C_{\mathfrak{Y}^*}(F)$. Thus, $C_{\mathfrak{F}}(Y^*) = \mathfrak{F}_1$ is a hyperplane of F. If $[Y^*, F]$ is not a generator for $\mathfrak{Z}$, then Lemma 16.10 implies that $|\mathfrak{F}| = 2^3$. Since $\langle [Y^*, F]\rangle \times \mathfrak{H}_1' \subseteqq \mathfrak{F}_1$, we get that $\mathfrak{F}_1 = \langle [Y^*, F]\rangle \times \mathfrak{H}_1'$. This is false, since $\mathfrak{H}_1/\mathfrak{H}_1'$ is a free $F_2\langle F\rangle$-module, and since $[\mathfrak{H}_1, F] \subseteqq \mathfrak{F}_1$. So $[Y^*, F]$ is a generator for $\mathfrak{Z}$.

Let $\mathfrak{F}_2 = \mathfrak{F} \cap \mathfrak{H}_1$, and suppose that $|\mathfrak{B}| = 2^{2n+1}$. Then $n \geqq 2$, and so $|\mathfrak{F}| > 2^3$. Furthermore, $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \mathfrak{H}_1'$ and if $H \in \mathfrak{H}_1$ and $[H, F] \subseteqq \mathfrak{H}_1'$, then $[H, F] = 1$. This is so since $[\mathfrak{H}_1, F]$ covers $C_{\mathfrak{H}_1/\mathfrak{H}_1'}(F)$, and $[\mathfrak{H}_1, F] \subseteqq \mathfrak{F}$, so that $[\mathfrak{H}_1, F, F] = 1$. Now suppose $F_1 \in \mathfrak{F}_1$. Then $F_1 = UV$, $U \in \mathfrak{H}_2$, $V \in \mathfrak{H}_1$. Since F normalizes both $\mathfrak{H}_1$ and $\mathfrak{H}_2$, we get $UV = U^F \cdot V^F$, $U^{-F} \cdot U = V^F \cdot V^{-1} \in \mathfrak{H}_1 \cap \mathfrak{H}_2$. Thus, $U^{-F} \cdot U = V^F \cdot V^{-1} = 1$, and so $V \in C_{\mathfrak{H}_1}(F) = \mathfrak{F} \cap \mathfrak{H}_1$, whence $U \in \mathfrak{F}$. So once again we have $\mathfrak{F}_1 = (\mathfrak{F} \cap \mathfrak{H}_1) \times (\mathfrak{F} \cap \mathfrak{H}_2)$. Since $\mathfrak{F} \cap \mathfrak{H}_2$ centralizes $\mathfrak{P}$ and $\mathfrak{F} \cap \mathfrak{H}_1$, we conclude that $\mathfrak{F} \cap \mathfrak{H}_2$ centralizes $\mathfrak{H}_1$.

Let $|\mathfrak{Y}| = 2^{1+2r}$. If $r \geqq 2$, there is $W \in \mathfrak{Y}$ such that $[W, F] \notin \mathfrak{Z}$. By Lemma 16.10, we get $|\mathfrak{F}| = 2^3$. This is false, as we have seen. So $r = 1$.

Choose $H \in (\mathfrak{F} \cap \mathfrak{H}_1)^P - \mathfrak{H}_1'$, where P is a generator for $\mathfrak{P}$. Then $[\mathfrak{F}_1, H] \subseteq \mathfrak{H}_1'$, and $[\mathfrak{F}_1 \cap \mathfrak{H}_1, H] = \mathfrak{H}_1'$. Since $\mathfrak{H}_1' \neq \mathfrak{Z}$, we conclude that $H \notin \boldsymbol{C}_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{Z})$. By Lemma 16.7, $\mathfrak{F}/\mathfrak{Z}$ is a free $F_2\langle H\rangle$-module. Since $\mathfrak{F}_1$ is a hyperplane of $\mathfrak{F}$ and $\boldsymbol{C}_{\mathfrak{F}_1}(H)$ is a hyperplane of $\mathfrak{F}_1$, we have $|\mathfrak{F}: \boldsymbol{C}_{\mathfrak{F}}(H)| = 2^2$. Hence, $|\mathfrak{F}| = 2^5$. Since $\mathfrak{F}/\mathfrak{Z}$ is a chief factor of $\mathfrak{M}$, we get that $5 \mid |\boldsymbol{A}_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{Z})|$. If 3.5 divides $\boldsymbol{A}_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{Z})$, then $\mathfrak{M}$ is transitive on $\mathfrak{F} - \mathfrak{Z}$, and so $\boldsymbol{C}(X) \subseteq \mathfrak{M}$ for all $X \in \mathfrak{F}^\#$. This is false, since $\boldsymbol{C}(\mathfrak{H}_1') \not\subseteq \mathfrak{M}$. So a $S_{2'}$-subgroup of $\boldsymbol{A}_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{Z})$ is of order 5, and $\boldsymbol{A}_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{Z})$ is isomorphic to a subgroup of a Frobenius group of order 20. In particular, $\boldsymbol{A}_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{Z})$ contains no four-group.

On the other hand, $n \geqq 2$, and $|\mathfrak{B} \cap \mathfrak{F}| = 2^3$. Let $\tilde{\mathfrak{F}}$ be a complement to $\mathfrak{H}_1'$ in $\mathfrak{B} \cap \mathfrak{F}$, so that $\tilde{\mathfrak{F}}^P$ is a four-group normalizing $\mathfrak{F}$ and acting faithfully on $\mathfrak{F}/\mathfrak{Z}$. This contradiction shows that

$$\mathfrak{T} \notin \mathscr{M}^* ,$$

the aim of this section.

17. Some properties of $\mathscr{M}(\mathfrak{T})$[4].

From now on, $\mathfrak{T}$ denotes a S_2-subgroup of $\mathfrak{G}$. For each solvable subgroup $\mathfrak{S}$ of $\mathfrak{G}$, $\mathscr{M}(\mathfrak{S})$ is the set of elements of $\mathscr{M}\mathscr{S}(\mathfrak{G})$ which contain $\mathfrak{S}$. Thus, $|\mathscr{M}(\mathfrak{T})| \geqq 2$.

Set

$$\mathfrak{Z}_0 = \boldsymbol{Z}(\mathfrak{T}) , \quad \mathfrak{Z}_1 = \boldsymbol{Z}(\boldsymbol{J}(\mathfrak{T})) , \quad \mathfrak{Z}_2 = \boldsymbol{Z}(\boldsymbol{J}_1(\mathfrak{T})) .$$

If $\mathfrak{L}$ is a solvable subgroup of $\mathfrak{G}$, denote by $f(\mathfrak{L})$ the number of integers i such that

$$0 \leqq i \leqq 2 , \qquad \mathfrak{Z}_i \lhd \mathfrak{L} .$$

As it turns out, $f(\mathfrak{L})$ is an important invariant.

HYPOTHESIS 17.1. There are $\mathfrak{M}, \mathfrak{N} \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{M} \neq \mathfrak{N}$ such that $\mathfrak{M} \cap \mathfrak{N} \supset \mathfrak{T}$.

Lemmas 17.1 through 17.11 are proved under Hypothesis 17.1.

LEMMA 17.1. *Let p be the largest prime in*

$$\bigcup_{\mathfrak{X} \in \mathscr{M}(\mathfrak{T})} \pi(\mathfrak{X}) .$$

Then $p \geqq 7$.

[4] I am indebted to I. M. Isaacs for making available to me some notes which he took, based on lectures of mine given several years ago.

Proof. Suppose false. Let $\mathfrak{D}$ be a $S_{2'}$-subgroup of $\mathfrak{M} \cap \mathfrak{N}$. Then $\mathfrak{D} \neq 1$ since $\mathfrak{M} \cap \mathfrak{N} \supset \mathfrak{T}$. Let $\mathfrak{D}_1$, $\mathfrak{D}_2$ be $S_{2'}$-subgroups of $\mathfrak{M}$, $\mathfrak{N}$ respectively, such that $\mathfrak{D} \subseteqq \mathfrak{D}_1 \cap \mathfrak{D}_2$.

Since $p \leqq 5$, and $\mathfrak{D}_1$, $\mathfrak{D}_2$ are Z-groups, they are both cyclic and so $\mathfrak{E} = \langle \mathfrak{D}_1, \mathfrak{D}_2 \rangle \subseteqq C(\mathfrak{D})$, so that $\mathfrak{E}$ is solvable. Since $\mathfrak{M} = \mathfrak{T}\mathfrak{D}_1$, and $\mathfrak{N} = \mathfrak{T}\mathfrak{D}_2$, it follows that $\mathfrak{T}\mathfrak{E} = \mathfrak{L}$ is a group. Now

$$\bigcap_{L \in \mathfrak{L}} \mathfrak{M}^L = \bigcap_{E \in \mathfrak{E}} \mathfrak{M}^E \supseteqq \mathfrak{D} ,$$

and so $\mathfrak{L}$ has a non identity solvable normal subgroup. As $\mathfrak{G}$ is an N-group, $\mathfrak{L}$ itself is solvable, and so $\mathfrak{M} = \mathfrak{L}$, $\mathfrak{N} = \mathfrak{L}$. The proof is complete.

LEMMA 17.2. *$\mathscr{M}(\mathfrak{T})$ has a unique element of order divisible by p (where p is as in Lemma 17.1).*

Proof. Choose $\mathfrak{X} \in \mathscr{M}(\mathfrak{T})$ with $p \mid |\mathfrak{X}|$, and let $\mathfrak{X}_p$ be a S_p-subgroup of $\mathfrak{X}$. Let $\mathfrak{S}$ be a $S_{2'}$-subgroup of $\mathfrak{X}$ containing $\mathfrak{X}_p$, so that $\mathfrak{S}$ is a Z-group and $\mathfrak{X}_p \lhd \mathfrak{S}$. Let $\mathfrak{L} = \mathfrak{T} \cdot \mathfrak{X}_p$. By Lemma 5.53, $f(\mathfrak{L}) \geqq 2$.

Suppose $\mathfrak{X}^* \in \mathscr{M}(\mathfrak{T})$ and $p \mid |\mathfrak{X}^*|$. Let $\mathfrak{X}_p^*$ be a S_p-subgroup of $\mathfrak{X}^*$, let $\mathfrak{S}^*$ be a $S_{2'}$-subgroup of $\mathfrak{X}^*$ which contains $\mathfrak{X}_p^*$, and set $\mathfrak{L}^* = \mathfrak{T} \cdot \mathfrak{X}_p^*$. Since $f(\mathfrak{L}^*) \geqq 2$, there is $i \in \{0, 1, 2\}$ such that $\mathfrak{Z}_i \lhd \langle \mathfrak{L}, \mathfrak{L}^* \rangle$. Set $\mathfrak{K} = \langle \mathfrak{L}, \mathfrak{L}^* \rangle$. Since $\mathfrak{K}$ is solvable and p is the largest prime in $\pi(\mathfrak{K})$, while a $S_{2'}$-subgroup of $\mathfrak{K}$ is a Z-group, it follows that $\mathfrak{T}\mathfrak{P} = \mathfrak{T} \cdot \mathfrak{P}^*$, where $\mathfrak{P} = \Omega_1(\mathfrak{X}_p)$, $\mathfrak{P}^* = \Omega_1(\mathfrak{X}_p^*)$. Thus, there is T in $\mathfrak{T}$ such that $\mathfrak{P}^T = \mathfrak{P}^*$. Hence, $\mathfrak{F} = \langle \mathfrak{S}^T, \mathfrak{S}^* \rangle \subseteqq N(\mathfrak{P}^*)$, and so $\mathfrak{F}$ is solvable. Since $\mathfrak{X} = \mathfrak{T}\mathfrak{S}$, so also $\mathfrak{X} = \mathfrak{T} \cdot \mathfrak{S}^T$, and so $\mathfrak{R} = \mathfrak{T}\mathfrak{F}$ is a solvable group. Hence, $\mathfrak{X} = \mathfrak{R}$, $\mathfrak{X}^* = \mathfrak{R}$. The proof is complete.

Next, let

$$\mathscr{P} = \{(\mathfrak{S}_0, \mathfrak{S}_1) \mid \mathfrak{S}_i \in \mathscr{M}(\mathfrak{T}),\ \mathfrak{S}_0 \neq \mathfrak{S}_1,\ \mathfrak{S}_0 \cap \mathfrak{S}_1 \supset \mathfrak{T}\} .$$

We may assume that notation is chosen so that

$$\max_{q \in \pi(\mathfrak{M} \cap \mathfrak{N})} \{q\} \geqq \max_{q \in \pi(\mathfrak{S}_0 \cap \mathfrak{S}_1)} \{q\} ,$$

for all $(\mathfrak{S}_0, \mathfrak{S}_1) \in \mathscr{P}$.

Let $\mathfrak{D}$ be a $S_{2'}$-subgroup of $\mathfrak{M} \cap \mathfrak{N}$ and let $\mathfrak{E}$, $\mathfrak{F}$ be $S_{2'}$-subgroups of $\mathfrak{M}$, $\mathfrak{N}$ respectively with $\mathfrak{D} \subseteqq \mathfrak{E} \cap \mathfrak{F}$.

LEMMA 17.3. *If $1 \subset \mathfrak{D}_1 \lhd \mathfrak{D}$, then either $\mathfrak{D}_1 \ntriangleleft \mathfrak{E}$ or $\mathfrak{D}_1 \ntriangleleft \mathfrak{F}$.*

Proof. If $\mathfrak{D}_1 \lhd \langle \mathfrak{E}, \mathfrak{F} \rangle$, then $\langle \mathfrak{E}, \mathfrak{F} \rangle$ is solvable, and $\mathfrak{L} = \mathfrak{T}\langle \mathfrak{E}, \mathfrak{F} \rangle$ is also solvable, whence $\mathfrak{M} = \mathfrak{L}$, $\mathfrak{N} = \mathfrak{L}$. The proof is complete.

Let $\mathfrak{X}$ be the unique element of $\mathscr{M}(\mathfrak{T})$ such that $p \mid |\mathfrak{X}|$.

LEMMA 17.4. *Either* $\mathfrak{E}' \neq 1$ *or* $\mathfrak{F}' \neq 1$.

Proof. This lemma is an immediate consequence of Lemma 17.3.

Choose notation so that $\mathfrak{E}' \neq 1$.

LEMMA 17.5. $\mathfrak{M} = \mathfrak{X}$.

Proof. Let r, s be primes such that a $S_{r,s}$-subgroup of $\mathfrak{E}$ is non abelian, with $r > s$. Let $\mathfrak{K}$ be a $S_{\{2,r,s\}}$-subgroup of $\mathfrak{M}$, $\mathfrak{K} \supset \mathfrak{T}$, and choose a Sylow system $\mathfrak{T}$, K_r, $\mathfrak{K}_s$ for $\mathfrak{K}$. Then $\mathfrak{K}_r = (\mathfrak{K}_r\mathfrak{K}_s)'$, since $\mathfrak{K}_r$ and $\mathfrak{K}_s$ are cyclic. Also, $r \equiv 1 \pmod{s}$, and so $r \geqq 7$. If $\mathfrak{U}/\mathfrak{V}$ is a chief factor of $\mathfrak{K}$ of order r, then $\mathfrak{K}'$ centralizes $\mathfrak{U}/\mathfrak{V}$. Since $\mathfrak{K}_s$ does not centralize $\mathfrak{K}_r$, it follows that $\mathfrak{K}_s \not\subseteq \mathfrak{K}'$, and so $\mathfrak{T}\mathfrak{K}_r \lhd \mathfrak{K}$, so that $\mathfrak{T} \lhd \mathfrak{T}\mathfrak{K}_s$. Hence, $f(\mathfrak{T}\mathfrak{K}_s) = 3$ and so $f(\mathfrak{T}\mathfrak{K}_r\mathfrak{K}_s) \geqq 2$. By Lemma 17.2, we conclude that $\mathfrak{K} \subseteq \mathfrak{X}$.

Let $\mathfrak{S}$ be a $S_{2'}$-subgroup of $\mathfrak{X}$, $\mathfrak{S} \supseteq \mathfrak{K}_r\mathfrak{K}_s$. Then $\mathfrak{K}_r \lhd \mathfrak{S}$, and so $\langle \mathfrak{S}, \mathfrak{E} \rangle \subseteq N(\mathfrak{K}_r)$, whence $\mathfrak{L} = \mathfrak{T}\langle \mathfrak{S}, \mathfrak{E} \rangle$ is solvable, and so $\mathfrak{L} = \mathfrak{X}$, $\mathfrak{L} = \mathfrak{M}$. The proof is complete.

Note that we now know that $\mathfrak{F}' = 1$, since $\mathfrak{N} \neq \mathfrak{M}$.

LEMMA 17.6. $\mathfrak{E}$ *is a Frobenius group with complement* $\mathfrak{D}$ *and kernel* $\mathfrak{E}'$.

Proof. Suppose $1 \subset \mathfrak{D}_0 \subseteq \mathfrak{D}$. Since $\mathfrak{D}$ is cyclic and is permutable with $\mathfrak{T}$, so is $\mathfrak{D}_0$. Hence, $N_{\mathfrak{E}}(\mathfrak{D}_0)$ is also permutable with $\mathfrak{T}$, and so $\mathfrak{T}\langle N_{\mathfrak{E}}(\mathfrak{D}_0), \mathfrak{F}\rangle$ is a solvable group, whence $\mathfrak{T}\langle N_{\mathfrak{E}}(\mathfrak{D}_0), \mathfrak{F}\rangle = \mathfrak{N}$. Thus, $\mathfrak{M} \cap \mathfrak{N} = \mathfrak{T}\mathfrak{D} \supseteq \mathfrak{T}\cdot N_{\mathfrak{E}}(\mathfrak{D}_0) \supseteq \mathfrak{M} \cap \mathfrak{N}$, and so $\mathfrak{D} = N_{\mathfrak{E}}(\mathfrak{D}_0)$. The proof is complete.

From Lemma 17.6, we conclude the $\mathfrak{D} \cap \mathfrak{M}' = 1$, and so $\mathfrak{D}$ has a normal complement in $\mathfrak{M}$, namely, $\mathfrak{T}\cdot\mathfrak{E}'$. Hence, $\mathfrak{T} \lhd \mathfrak{T}\mathfrak{D}$.

LEMMA 17.7. *If* $1 \subset \mathfrak{D}_0 \subseteq \mathfrak{D}$, *then* $C_{\mathfrak{T}}(\mathfrak{D}_0)$ *contains no four-group.*

Proof. Suppose false. We may assume that $\mathfrak{D}_0$ is of prime order r, and that $\mathfrak{W}$ is a four-group in $C_{\mathfrak{T}}(\mathfrak{D}_0)$. Thus, S_r-subgroups of $\mathfrak{G}$ are cyclic and $\mathfrak{D}_0 \subseteq N(\mathfrak{D}_0)'$.

Choose $X \in N(\mathfrak{D}_0)$; then $\mathfrak{T}, \mathfrak{T}^X \in \mathscr{U}^*(\mathfrak{D}_0; 2)$. Since $C_{\mathfrak{T}}(\mathfrak{D}_0) \neq 1$ and $C_{\mathfrak{T}^X}(\mathfrak{D}_0) \neq 1$, it follows that $\mathfrak{T} = \mathfrak{T}^{XC}$ for some C in $C(\mathfrak{D}_0)$. Hence $N(\mathfrak{T})$ covers $N(\mathfrak{D}_0)/C(\mathfrak{D}_0)$, and in particular, $\mathfrak{D}_0 \subseteq N(\mathfrak{T})'$. Then $N(\mathfrak{T})/\mathfrak{T}$ is non cyclic. Hence, $N(\mathfrak{T}) \subseteq \mathfrak{M}$, since otherwise $(\mathfrak{M}, \mathfrak{M}_1) \in \mathscr{P}$ for some $\mathfrak{M}_1 \in \mathscr{M}(\mathfrak{T})$ such that $N(\mathfrak{T}) \subseteq \mathfrak{M}_1$. This violates $\mathfrak{F}' = 1$. So $N(\mathfrak{T}) \subseteq \mathfrak{M}$, whence $\mathfrak{D}_0 \subseteq N(\mathfrak{T})' \subseteq \mathfrak{M}'$. This is false, since $\mathfrak{D}$ has a normal complement in $\mathfrak{M}$, and $\mathfrak{D}' = 1$. The proof is complete.

LEMMA 17.8. $|\mathfrak{D}| = r$ *is a prime.*

Proof. $\mathfrak{E}$ is faithfully represented on $\mathfrak{B} = O_2(\mathfrak{M})/D(O_2(\mathfrak{M}))$. Hence, $\mathfrak{B}$ contains a free $F_2\mathfrak{D}$-submodule $\mathfrak{B}_0$. If $|\mathfrak{D}|$ is not a prime, then $|C_{\mathfrak{B}_0}(\mathfrak{D}_0)| \geqq 2^3$ for each subgroup $\mathfrak{D}_0$ of D of prime order. Hence $O_2(\mathfrak{M}) \cap C(\mathfrak{D}_0)$ has a section which is not generated by two elements, and so $O_2(\mathfrak{M}) \cap C(\mathfrak{D}_0)$ has more than one involution, so contains a four-group, against Lemma 17.7. The proof is complete.

Let $\mathfrak{X}_p$ be the unique subgroup of $\mathfrak{E}$ of order p and let $\mathfrak{L} = \mathfrak{T}\mathfrak{D}\cdot\mathfrak{X}_p$. Thus, $\mathfrak{L}$ is a subgroup of $\mathfrak{M} = \mathfrak{X}$, and $\mathfrak{D}\cdot\mathfrak{X}_p$ is a Frobenius group with kernel $\mathfrak{X}_p$.

LEMMA 17.9. *One of the following holds*:

(a) $\mathfrak{T} \lhd \mathfrak{L}$.

(b) $\mathfrak{L}$ *has a unique central involution.*

Proof. Let $\mathfrak{H} = O_2(\mathfrak{L})$. Suppose (a) does not hold. Then $\mathfrak{H} \subset \mathfrak{T}$. Let $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{H}$. Now $O_{2,2'}(\mathfrak{L}) \cap \mathfrak{D}\mathfrak{X}_p \neq 1$, and so $\mathfrak{X}_p \subseteqq O_{2,2'}(\mathfrak{L})$. Since $O_2(\bar{\mathfrak{L}}) = 1$, it follows that $F(\bar{\mathfrak{L}}) = \mathfrak{H}\cdot\mathfrak{X}_p/\mathfrak{H} = \bar{\mathfrak{X}}_p$. Thus, $\bar{\mathfrak{L}}$ is a Frobenius group with kernel $\bar{\mathfrak{X}}_p$ and complements of order $2^a\cdot r$, where $a \geqq 1$. Since $C_{\mathfrak{T}}(\mathfrak{D})$ contains no four-group, this implies that $\mathfrak{X}_p$ centralizes every characteristic abelian subgroup of $\mathfrak{H}$. Let $\mathfrak{H}_1 = [\mathfrak{H}, \mathfrak{X}_p]$, $\mathfrak{B} = \mathfrak{H}_1/D(\mathfrak{H}_1)$. Then since $C_{\mathfrak{T}}(\mathfrak{D})$ contains no four-group, and since $a \geqq 1$, it follows that $|\mathfrak{B}| = 2^{2r}$, and $C_{\mathfrak{B}}(\mathfrak{D})$ is a four-group. Since $\mathfrak{H}_1$ is special and $C_{\mathfrak{H}_1}(\mathfrak{D})$ contains no four-group, it follows that $C_{\mathfrak{H}_1}(\mathfrak{D})$ is a quaternion group of order 8. Since $\mathfrak{B}$ is a chief factor of $\mathfrak{L}$, we have $[\mathfrak{H}, \mathfrak{H}_1] \subseteqq \mathfrak{H}_1'$, and so $[\mathfrak{H}, \mathfrak{H}_1, \mathfrak{H}_1] = 1$. Hence, $\mathfrak{H}_1' \subseteqq Z(\mathfrak{H})$, and so $C_{\mathfrak{H}_1'}(\mathfrak{D}) = \langle Z \rangle$, where Z is a central involution of $\mathfrak{L}$. Since $C_{\mathfrak{T}}(\mathfrak{D})$ contains no four-group, Z is unique.

LEMMA 17.10. $\mathfrak{T} \lhd \mathfrak{L}$.

Proof. Suppose false. Now $\mathfrak{N} = \mathfrak{T}\mathfrak{F}$, $\mathfrak{F}$ cyclic, $\mathfrak{F} \supset \mathfrak{D}$. Let $\mathfrak{A} = Z(\mathfrak{T})^{\mathfrak{N}}$ be the normal closure of $Z(\mathfrak{T})$ in $\mathfrak{N}$. Since $Z(\mathfrak{T}) \subseteqq Z(O_2(\mathfrak{N}))$, $\mathfrak{A}$ is abelian. Also $Z \in Z(\mathfrak{T}) \subseteqq \mathfrak{A}$, where Z is the unique central involution of $\mathfrak{L}$. Since $p \mid |C_{\mathfrak{M}}(Z)|$, we have $C(Z) \subseteqq \mathfrak{M}$. But $O_2(\mathfrak{N}) \cap C(\mathfrak{D})$ contains no four-group and so Z is the only involution in $C_{\mathfrak{A}}(\mathfrak{D})$. Since $\mathfrak{F}$ normalizes $\mathfrak{A}$ and centralizes $\mathfrak{D}$, we get $\mathfrak{F} \subseteqq C(Z) \subseteqq \mathfrak{M}$, $\mathfrak{N} \subseteqq \mathfrak{M}$. The proof is complete.

Since $\mathfrak{T} \lhd \mathfrak{L}$, we have $f(\mathfrak{T}\mathfrak{X}_p) = 3$. Hence, $N(\mathfrak{Z}_i) \subseteqq \mathfrak{M}$, $i = 0, 1, 2$, by Lemma 17.2.

LEMMA 17.11. $r > 3$ *and* $|\mathfrak{N} : \mathfrak{N} \cap \mathfrak{M}| = 3$.

Proof. The assertion $|\mathfrak{N}: \mathfrak{N} \cap \mathfrak{M}| = 3$ is a consequence of Lemmas 5.53 and 5.54, together with $N(\mathfrak{Z}_i) \subseteq \mathfrak{M}$, $i = 0, 1, 2$. If $r = 3$, then $\mathfrak{F}$ is a cyclic group of order 3^2, and $\mathfrak{D} = \Omega_1(\mathfrak{F})$. Since $\mathfrak{T} \lhd \mathfrak{T}\mathfrak{D}$, we get $\mathfrak{T} \lhd \mathfrak{N}$, so $\mathfrak{N} \subseteq N(\mathfrak{T}) \subseteq \mathfrak{M}$, which is false. The proof is complete.

Since $\mathfrak{N} = \mathfrak{T}\cdot\mathfrak{F}$, and $|\mathfrak{F}: \mathfrak{D}| = 3$, we have $\mathfrak{F} = \mathfrak{D} \times \mathfrak{A}$, where $|\mathfrak{A}| = 3$, $|\mathfrak{D}| = r > 3$. Let $\mathfrak{L}_1 = \mathfrak{T}\mathfrak{A}$, $\mathfrak{L}_2 = O_2(\mathfrak{L}_1)$. Since $\mathfrak{L}_1 \not\subseteq \mathfrak{M}$, while $N(\mathfrak{T}) \subseteq \mathfrak{M}$, we have $\mathfrak{T} \ntriangleleft \mathfrak{L}_1$, and so $\mathfrak{L}_1/\mathfrak{L}_2 \cong \Sigma_3$.

Let $\mathfrak{K} = O_2(\mathfrak{N})$, $\mathfrak{Z} = Z(\mathfrak{K})$. Then $\mathfrak{Z} \supseteq Z(\mathfrak{T})$, and since $f(\mathfrak{L}) = 3$, $\mathfrak{A}$ does not centralize $Z(\mathfrak{T})$. Hence, $\mathfrak{W} = [\Omega_1(\mathfrak{Z}), \mathfrak{A}] \neq 1$, and $\mathfrak{W}$ admits $\mathfrak{D}$.

Let $d = \max\{m(\mathfrak{B}) \mid \mathfrak{B} \subseteq \mathfrak{T}, \mathfrak{B}' = 1\}$. Since $J(\mathfrak{T}) \ntriangleleft \mathfrak{L}_1$, there is $\mathfrak{B} \subseteq \mathfrak{T}$, $\mathfrak{B}' = 1$, $m(\mathfrak{B}) = d$, such that $\mathfrak{B} \not\subseteq \mathfrak{L}_2$. Let $\mathfrak{B}_2 = \mathfrak{B} \cap \mathfrak{L}_2$, so that $|\mathfrak{B}: \mathfrak{B}_2| = 2$. Let $\mathfrak{C} = \mathfrak{B} \cap \mathfrak{K}$. Thus, $\mathfrak{B}/\mathfrak{C}$ acts faithfully on $\mathfrak{F}\mathfrak{K}/\mathfrak{K}$, and $\mathfrak{B}/\mathfrak{C}$ does not centralize $\mathfrak{A}\mathfrak{K}/\mathfrak{K}$. Since $\mathfrak{W}$ contains a four-group, it follows that $[\mathfrak{W}, \mathfrak{D}] = \mathfrak{W}_1 \neq 1$, and $\mathfrak{W}_1 \lhd \mathfrak{N}$. Since $m(\mathfrak{B}/\mathfrak{C}) \leqq 2$, and since $m(\langle \mathfrak{C}, \mathfrak{W}_1 \rangle) = m(\mathfrak{C}) + m(\mathfrak{W}_1/\mathfrak{W}_1 \cap \mathfrak{C})$, it follows that $\mathfrak{W}_1 \cap \mathfrak{C}$ is of index at most 4 in $\mathfrak{W}_1$. Since $\mathfrak{F}\mathfrak{W}_1$ is a Frobenius group with kernel $\mathfrak{W}_1$, it follows that $|\mathfrak{W}_1| = 2^4$, $r = 5$. But this forces $m(\mathfrak{B}/\mathfrak{C}) = 2$, and so $\mathfrak{B}/\mathfrak{C}$ has an involution which inverts $\mathfrak{K}\mathfrak{F}/\mathfrak{K}$. This is false, since the elements of GL(4, 2) of order 15 are not real. So we have shown that $\mathscr{P} = \emptyset$, that is

$$\mathfrak{M} \cap \mathfrak{N} = \mathfrak{T} \text{ if } \mathfrak{M}, \mathfrak{N} \in \mathscr{M}(\mathfrak{T}),\ \mathfrak{M} \neq \mathfrak{N}\,. \tag{17.1}$$

HYPOTHESIS 17.2. $\mathfrak{T} \subset N(\mathfrak{T})$.

Suppose Hypothesis 17.2 is satisfied. Let $\mathfrak{M} = M(N(\mathfrak{T}))$, and choose $\mathfrak{N} \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{N} \neq \mathfrak{M}$. By (17.1), Lemmas 5.53 and 5.54, we have $|\mathfrak{N}| = 3|\mathfrak{T}|$.

Since $\mathfrak{T} \subset N(\mathfrak{T})$, $\mathfrak{N}$ has an orbit of size 3 on the cosets of $\mathfrak{N}$ in $\mathfrak{G}$. By the Main Theorem of Wong already used, we conclude that $\mathfrak{N}$ is not a maximal subgroup of $\mathfrak{G}$.

Let $\mathfrak{N} \subset \mathfrak{R} \subset \mathfrak{G}$, with $\mathfrak{N}$ a maximal subgroup of $\mathfrak{R}$. Since $\mathfrak{G}$ is a minimal counterexample, $\mathfrak{R}$ satisfies the conclusion of the (augmented) Main Theorem of this paper. Let $\mathfrak{R}_0$ be the simple normal subgroup of $\mathfrak{R}$, and let $\mathfrak{R}_1 = \mathfrak{R}_0\mathfrak{T}$.

It is straightforward to verify that if $\mathfrak{T}^*$ is a S_2-subgroup of ${}^2F_4(2)$ or of its simple subgroup of index 2, then Aut$(\mathfrak{T}^*)$ is a 2-group. Since Aut$(\mathfrak{T})$ is not a 2-group, we have $\mathfrak{R}_0 \ncong {}^2F_4(2)'$. Similarly, we see that $\mathfrak{R}_0$ is none of $U_3(3)$, $L_3(3)$, A_7, M_{11}. Since $\mathfrak{N}$ is not 2-closed, $\mathfrak{R}_0 \ncong Sz(q)$, and since Aut$(\mathfrak{T})$ is not a 2-group, while $\mathscr{SCN}_3(\mathfrak{T}) \neq \emptyset$, $\mathfrak{R}_0$ is not isomorphic to $L_2(q)$ for any odd q. Since $\mathfrak{N}$ is not 2-closed, $\mathfrak{R}_0$ is not isomorphic to $L_2(2^n)$ for any n. This contradiction shows that

$$\mathfrak{T} = N(\mathfrak{T})\,. \tag{17.2}$$

18. An exceptional case.

HYPOTHESIS 18.1. $|\mathfrak{M}| = 3|\mathfrak{T}|$ for all $\mathfrak{M} \in \mathscr{M}(\mathfrak{T})$.

All the results in this section are proved under Hypothesis 18.1.

LEMMA 18.1. *If $\mathfrak{X}$ is a 2-local subgroup of $\mathfrak{G}$, then $|\mathfrak{X}|_{2'} = 1$ or 3.*

Proof. This lemma is an easy consequence of Lemmas 5.53, 5.54, and Hypothesis 18.1.

HYPOTHESIS 18.2. There is $\mathfrak{M} = \mathfrak{T}\mathfrak{P} \in \mathscr{M}(\mathfrak{T})$ such that $\mathfrak{P} = C_{\mathfrak{M}}(\mathfrak{P})$ is of order 3.

Lemmas 18.2 through 18.5 are proved under Hypothesis 18.2.

Set $\mathfrak{H} = O_2(\mathfrak{M})$, so that $\mathfrak{M}/\mathfrak{H} \cong \Sigma_3$.

LEMMA 18.2. $\mathfrak{H}\mathfrak{P} \in \mathscr{M}^*(\mathfrak{G})$.

Proof. If $\mathfrak{H}\mathfrak{P} \subseteq \mathfrak{S}$, and $\mathfrak{S}$ is a solvable subgroup of $\mathfrak{G}$, then by Lemma 18.1, together with $\mathfrak{M} = N(\mathfrak{H})$, we conclude that $\mathfrak{H} = O_2(\mathfrak{S})$, whence $\mathfrak{S} \subseteq \mathfrak{M}$. The proof is complete.

Set $\mathfrak{Z} = Z(\mathfrak{H})$.

LEMMA 18.3. $|\mathfrak{Z}| \leqq 2^4$.

Proof. If false, then since $\mathfrak{Z}\mathfrak{P}$ is a Frobenius group, we have $|\mathfrak{Z}| \geqq 2^6$. If $\mathfrak{Y}$ is of index at most 4 in $\mathfrak{Z}$, and if $\mathfrak{P} = \langle P \rangle$, then $\mathfrak{Y} \cap \mathfrak{Y}^P \neq 1$, and $\mathfrak{Y} \cap \mathfrak{Y}^P$ admits $\mathfrak{P}$. Hence, $C(\mathfrak{Y}) \subseteq C(\mathfrak{Y} \cap \mathfrak{Y}^P) \subseteq N(\mathfrak{Y} \cap \mathfrak{Y}^P)$, and since $\mathfrak{H}\mathfrak{P} \subseteq N(\mathfrak{Y} \cap \mathfrak{Y}^P)$, we conclude that $C(\mathfrak{Y}) \subseteq \mathfrak{M}$.

Set

$$\mathfrak{A}_0 = Z(\mathfrak{T}),\ \mathfrak{A}_1 = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{Z}); \mathfrak{T}),\ \mathfrak{A}_2 = \langle V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{Z}_1); \mathfrak{T}) \mid |\mathfrak{Z} : \mathfrak{Z}_1| = 2 \rangle .$$

We argue that if $\mathfrak{X} \in \mathscr{M}(\mathfrak{T})$, and σ is a permutation of $\{0, 1, 2\}$, then $\mathfrak{X} \subseteq (\mathfrak{X} \cap N(\mathfrak{A}_{\sigma(0)}))(\mathfrak{X} \cap N(\mathfrak{A}_{\sigma(1)}))$. Namely, if $\mathfrak{Y}$ is a subgroup of $\mathfrak{Z}$ of index at most 4, then $C(\mathfrak{Y}) = \mathfrak{H} = C(\mathfrak{Z})$. The desired factorizations are straightforward consequences of these equalities. But then $N(\mathfrak{A}_0) \cdot N(\mathfrak{A}_1)$ is the only member of $\mathscr{M}(\mathfrak{T})$. This contradiction completes the proof.

LEMMA 18.4. $|\mathfrak{Z}| = 4$.

Proof. Suppose false, so that $|\mathfrak{Z}| = 2^4$. If $\mathfrak{Y}$ is of index 2 in $\mathfrak{Z}$, then $C(\mathfrak{Y}) \subseteq C(\mathfrak{Y} \cap \mathfrak{Y}^P) \subseteq N(\mathfrak{Y} \cap \mathfrak{Y}^P)$, and since $\mathfrak{H}\mathfrak{P} \subseteq N(\mathfrak{Y} \cap \mathfrak{Y}^P)$, we conclude that $C(\mathfrak{Y}) = \mathfrak{H}$. Set $\mathfrak{A} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{Z}); \mathfrak{T})$. Since $C(\mathfrak{Y}) = C(\mathfrak{Z})$ for every subgroup $\mathfrak{Y}$ of index 2 in $\mathfrak{Z}$, it follows that $\mathfrak{A} \subseteq \mathfrak{H}$, whence $\mathfrak{A} \lhd \mathfrak{M} = \mathfrak{T}\mathfrak{P}$.

Choose $\mathfrak{X} \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{X} \neq \mathfrak{M}$, and let $\mathfrak{K} = O_2(\mathfrak{X})$, so that $\mathfrak{X}/\mathfrak{K} \cong \Sigma_3$. Since $\mathfrak{M} = N(\mathfrak{A})$, there is G in $\mathfrak{G}$ such that $\mathfrak{Z}^G \subseteq \mathfrak{T}$, $\mathfrak{Z}^G \not\subseteq \mathfrak{K}$. Set $\mathfrak{Z}^* = \mathfrak{Z}^G$, $\mathfrak{Z}_1^* = \mathfrak{K} \cap \mathfrak{Z}^*$. Thus, $\mathfrak{Z}_1^*$ is of index 2 in $\mathfrak{Z}^*$. Let $\mathfrak{X} = \mathfrak{T}\mathfrak{Q}$, where $\langle Q \rangle = \mathfrak{Q}$ is of order 3. We see that $\mathfrak{M} \cap \mathfrak{Z}^{*Q} = \mathfrak{Z}_1^{*Q}$ is of order 8. If $\mathfrak{Z}_1^{*Q} \subseteq \mathfrak{H}$, we get $\mathfrak{Z} \subseteq C(\mathfrak{Z}_1^{*Q}) = C(\mathfrak{Z}^{*Q})$, and so $\mathfrak{Z}^{*Q} \subseteq C(\mathfrak{Z}) = \mathfrak{H}$, which is false. So $\mathfrak{Z}_1^{*Q} \not\subseteq \mathfrak{H}$, and $\mathfrak{Z}_1^{*Q} \cap \mathfrak{H} = \mathfrak{Z}_2^{*Q}$, where $\mathfrak{Z}_2^{*Q}$ has index 4 in $\mathfrak{Z}^{*Q}$. Set $\mathfrak{C} = C(\mathfrak{Z}_2^{*Q})$. We argue that $\mathfrak{M}^{GQ} \cap \mathfrak{C}$ contains a full S_2-subgroup of $\mathfrak{C}$. In fact, if $\mathfrak{U}$ is any non identity subgroup of $\mathfrak{Z}$ then since $\mathfrak{H} \subseteq C(\mathfrak{U})$, $|\mathfrak{T}: \mathfrak{H}| = 2$, and $N(\mathfrak{H}) = \mathfrak{M}$, it follows that $\mathfrak{M} \cap C(\mathfrak{U})$ contains a full S_2-subgroup of $C(\mathfrak{U})$. So $\mathfrak{M}^{GQ} \cap \mathfrak{C}$ contains a full S_2-subgroup of $\mathfrak{C}$.

Set $\mathfrak{T}_0 = \mathfrak{M}^{CQ} \cap \mathfrak{C}$, so that $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{C}$, and $\mathfrak{Z} \subseteq \mathfrak{T}_0$. Since $|\mathfrak{T}_0 : O_2(\mathfrak{C})| \leqq 2$, it follows that $\mathfrak{Z} \cap O_2(\mathfrak{C}) = \mathfrak{Z}_0$ is of index at most 2 in $\mathfrak{Z}$.

Next, $\mathfrak{Z}_1^{*Q} \subseteq \mathfrak{M}$ and $\mathfrak{Z} \lhd \mathfrak{M}$, so $[\mathfrak{Z}_1^{*Q}, \mathfrak{Z}_0] \subseteq \mathfrak{Z}$. Also, $O_2(\mathfrak{C})$ normalizes $\mathfrak{Z}^{*Q}$, so $[\mathfrak{Z}_1^{*Q}, \mathfrak{Z}_0] \subseteq \mathfrak{Z}^{*Q}$. Hence,

$$[\mathfrak{Z}_1^{*Q}, \mathfrak{Z}_0] \subseteq \mathfrak{Z} \cap \mathfrak{Z}^{*Q} .$$

Now, $\mathfrak{Z}_1^{*Q} \not\subseteq \mathfrak{H} = C(\mathfrak{Z}) = C(\mathfrak{Z}_0)$, so $[\mathfrak{Z}_1^{*Q}, \mathfrak{Z}_0] \neq 1$. Choose an involution U in $[\mathfrak{Z}_1^{*Q}, \mathfrak{Z}_0]$.

Let $\mathfrak{D} = C(U) \supseteq \langle \mathfrak{Z}, \mathfrak{Z}^{*Q} \rangle$. Thus, $\mathfrak{D} \cap \mathfrak{M}^{GQ}$ contains a S_2-subgroup of $\mathfrak{D}$. On the other hand, $\mathfrak{H} \subseteq \mathfrak{D}$, since $U \in \mathfrak{Z}$. Thus, $\mathfrak{H}_0 = \mathfrak{H} \cap \mathfrak{M}^{GQ} \lhd O_2(\mathfrak{D}) \cap \mathfrak{H}$. Since $O_2(\mathfrak{D})$ has index at most 2 in every S_2-subgroup of $\mathfrak{D}$, we get $|\mathfrak{H}: \mathfrak{H}_0| \leqq 2$. Also, $[\mathfrak{Z}_1^{*Q}, \mathfrak{H}_0] \subseteq \mathfrak{Z}^{*Q} \cap \mathfrak{H} = \mathfrak{Z}_2^{*Q}$, a group of order 4.

Choose Z in $\mathfrak{Z}_1^{*Q} - \mathfrak{H}$. Let $\mathfrak{H}_1 = \mathfrak{H}_0 \cap \mathfrak{H}_0^P$. Thus, $\mathfrak{H}_1$ has index at most 4 in $\mathfrak{H}$ and $\mathfrak{H}_1$ admits $\mathfrak{P}$. Now $[\mathfrak{H}_1, \mathfrak{Z}] \subseteq [\mathfrak{H}_0, \mathfrak{Z}_1^{*Q}] \subseteq \mathfrak{Z}_2^{*Q} \subseteq \mathfrak{H} \cap \mathfrak{M}^{GQ} = \mathfrak{H}_0$, that is, Z normalizes $\mathfrak{H}_1$. We claim that $|\mathfrak{H}_1| \leqq 2^4$. Since $|[\mathfrak{H}_1, Z]| \leqq 2^2$, $C_{\mathfrak{H}_1}(Z)$ has index at most 2^2 in $\mathfrak{H}_1$. Since $\mathfrak{H}_1\mathfrak{P}$ is a Frobenius group, and since Z inverts some element of $\mathfrak{M}/\mathfrak{H}_1$ of order 3, our assertion follows. Since $|\mathfrak{H}_1| \leqq 2^4$, we get $|\mathfrak{H}| \leqq 2^6$, and so $|\mathfrak{H}| = 2^4$ or 2^6. There are no groups of order 2^6 which has a fixed point free automorphism of order 3 and whose center is of index 4. Hence, $|\mathfrak{H}| = 2^4$, and so $\mathfrak{H} = \mathfrak{Z}$ is abelian. Since $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{T}) \neq \emptyset$, it follows that $\mathfrak{H}$ is elementary abelian. So $\mathfrak{T}' = Z(\mathfrak{T})$ is a four-group, and $\mathfrak{T}' = D(\mathfrak{T})$. Hence, $C(Z(\mathfrak{T}))/Z(\mathfrak{T})$ has abelian S_2-subgroups, and so $C(Z(\mathfrak{T})) = \mathfrak{T}$. On the other hand, $\mathfrak{Z}_2^{*Q} \subseteq \mathfrak{H} = \mathfrak{Z}$, and $\mathfrak{T} = \mathfrak{H} \cdot \mathfrak{Z}_1^{*Q}$, whence $\mathfrak{Z}_2^{*Q} = Z(\mathfrak{T})$. This implies that $\mathfrak{Z}^{*Q} = \mathfrak{Z}$, the unique elementary subgroup of $C(Z(\mathfrak{T}))$ of order 2^4. This contradiction completes the proof.

Set $\mathfrak{Z}_0 = \boldsymbol{Z}(\mathfrak{T})$, so that $|\mathfrak{Z}_0| = 2$. Since $\mathfrak{H}\mathfrak{P}$ is a Frobenius group, it follows that if J is any involution of $\mathfrak{H} - \mathfrak{Z}$, then $[\mathfrak{H}, J] = \mathfrak{Z}$. This then implies that $\mathfrak{Z}$ is the only normal four-subgroup of $\mathfrak{T}$, an important fact.

LEMMA 18.5. *Let* $\mathfrak{N} = \boldsymbol{C}(\mathfrak{Z}_0)$. *Then* $\mathscr{M}(\mathfrak{T}) = \{\mathfrak{M}, \mathfrak{N}\}$.

Proof. Choose $\mathfrak{X} \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{X} \neq \mathfrak{M}$, and let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{X}$. Since $|\mathfrak{X}| = 3|\mathfrak{T}|$, we have $|\mathfrak{U}| \leqq 2^2$. If $\mathfrak{U}$ is a four-group, we have $\mathfrak{U} = \mathfrak{Z}$, $\mathfrak{X} = \mathfrak{M}$. This is false, and so $\mathfrak{U} = \mathfrak{Z}_0$, $\mathfrak{X} = \mathfrak{N}$. The proof is complete.

Since $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{T}) \neq \emptyset$, it follows that $\mathfrak{H} \supset \mathfrak{Z}$, and so $\mathfrak{H}' = \mathfrak{Z}$. Let $\mathfrak{N} = \mathfrak{T}\mathfrak{Q}$, $\langle Q \rangle = \mathfrak{Q}$, $Q^3 = 1$. Since $\mathfrak{Z}$ centralizes $\mathfrak{T}/\mathfrak{Z}_0$, we get $\mathfrak{Z} \subseteqq \boldsymbol{O}_2(\mathfrak{N}) = \mathfrak{K}$. Also $\mathfrak{Z}^Q \neq \mathfrak{Z}$. Since $\mathfrak{Z} \subseteqq \mathfrak{K}$, so also $\mathfrak{Z}^Q \subseteqq \mathfrak{K}$. Let $\mathfrak{E} = \langle \mathfrak{Z}, \mathfrak{Z}^Q \rangle$. Since $\mathfrak{Z} \cap \mathfrak{Z}^Q = \mathfrak{Z}_0$, we have $|\mathfrak{E}| = 2^3$. If $\mathfrak{E}' \neq 1$, let $\mathfrak{H}_0 = \mathfrak{H} \cap \mathfrak{K}$ so that $|\mathfrak{H} : \mathfrak{H}_0| = 2$, and $[\mathfrak{Z}^Q, \mathfrak{H}_0] \subseteqq [\mathfrak{Z}^Q, \mathfrak{K}] = \mathfrak{Z}_0^Q = \mathfrak{Z}_0$, and so $\mathfrak{Z}^Q$ centralizes $\mathfrak{H}_0/\mathfrak{Z}_0$, whence centralizes $\mathfrak{H}_0/\mathfrak{Z}$. Since $\mathfrak{Z}^Q \not\subseteqq \mathfrak{H}$, it follows that if $Y \in \mathfrak{Z}^Q - \mathfrak{H}$, then Y centralizes a hyperplane of $\mathfrak{H}/\mathfrak{Z}$. This forces $|\mathfrak{H}/\mathfrak{Z}| = 2^2$, against $\mathfrak{H}' = \mathfrak{Z}$. So $\mathfrak{E}' = 1$.

Set $\mathfrak{C} = \boldsymbol{C}(\mathfrak{E})$. Then $\mathfrak{C} \subseteqq \boldsymbol{C}(\mathfrak{Z}) = \mathfrak{H}$, and $\mathfrak{C} \subseteqq \boldsymbol{C}(\mathfrak{Z}^Q) = \mathfrak{H}^Q$. Since $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{Z}^Q)$ admits $\mathfrak{P}$, and $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{H}^Q}(\mathfrak{Z})$ admits $\mathfrak{P}^Q$, and since $\mathfrak{C} \subseteqq \mathfrak{H}$, we get $N(\mathfrak{C}) \supseteqq \langle \mathfrak{H}, \mathfrak{P}, \mathfrak{P}^Q \rangle$. This forces $\mathfrak{P}^Q \subseteqq \mathfrak{M}$. But $\mathfrak{H}^Q \subseteqq N(\mathfrak{C})$, and so $\mathfrak{H}\mathfrak{P} = (\mathfrak{P}\mathfrak{H})^Q$, which gives $Q \in \mathfrak{M}$, which is false. This contradiction gives us

(18.1) Every element of $\mathscr{M}(\mathfrak{T})$ contains an element of order 6.

LEMMA 18.6. *S_3-subgroups of $\mathfrak{G}$ are not cyclic.*

Proof. Suppose false. Choose $\mathfrak{X}_1 \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{X}_i = \mathfrak{T}\mathfrak{P}_i$, $|\mathfrak{P}_i| = 3$, $i = 1, 2$. Let $\mathfrak{H}_i = \boldsymbol{O}_2(\mathfrak{X}_i)$. Then $\mathfrak{H}_i$ is a maximal element of $N(\mathfrak{P}_i; 2)$. Since $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are conjugate, the transitivity theorem (or rather its proof) implies that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjugate. Since $\mathfrak{T}$ is self normalizing in $\mathfrak{G}$, this gives $\mathfrak{H}_1 = \mathfrak{H}_2$, which is false if we take $\mathfrak{X}_1 \neq \mathfrak{X}_2$ (as we may). The proof is complete.

LEMMA 18.7. *If* $\mathfrak{X} = \mathfrak{T}\mathfrak{P} \in \mathscr{M}(\mathfrak{T})$, $|\mathfrak{P}| = 3$, *then* $\boldsymbol{C}_{\mathfrak{X}}(\mathfrak{P})$ *does not contain a four-group.*

Proof. This lemma is a consequence of the preceding lemma.

LEMMA 18.8. $\boldsymbol{C}(\mathfrak{V})$ *is a 2-group for every four-subgroup* $\mathfrak{V}$ *of* $\mathfrak{G}$.

Proof. This lemma is also a consequence of Lemma 18.6.

We introduce the following notation: $\mathscr{M}(\mathfrak{T}) = \{\mathfrak{M}_1, \cdots, \mathfrak{M}_n\}$,

$\mathfrak{H}_i = O_2(\mathfrak{M}_i)$, $\mathfrak{Z}_i = \Omega_1(Z(\mathfrak{H}_i))$, $\mathfrak{P}_i$ of order 3 in $\mathfrak{M}_i$, $\mathfrak{P}_i = \langle P_i \rangle$.

LEMMA 18.9. *Suppose $Z(\mathfrak{T})$ is not cyclic. Then for $i = 1, \cdots, n$, $\mathfrak{Z}_i$ contains a hyperplane $\mathfrak{Y}_i$ with $C(\mathfrak{Y}_i) \supset C(\mathfrak{Z}_i)$.*

Proof. Suppose false for i. Let $\mathfrak{W}$ be the weak closure of $\mathfrak{Z}_i$ in $\mathfrak{T}$ with respect to $\mathfrak{G}$. We will show that $\mathfrak{W}$ centralizes $\mathfrak{Z}_j$ for every j. Choose X in $\mathfrak{G}$ with $\mathfrak{Z}_i^X \subseteqq \mathfrak{T}$. Then $\mathfrak{Z}_i^X \cap \mathfrak{H}_j$ is of index at most 2 in $\mathfrak{Z}_i^X$, and $\mathfrak{Z}_i^X \cap \mathfrak{H}_j$ centralizes $\mathfrak{Z}_j$. Hence, $\mathfrak{Z}_i^X$ also centralizes $\mathfrak{Z}_j$, since this lemma is assumed false for i. Thus, $\mathfrak{W}$ centralizes $\mathfrak{Z}_j$ for all j, and so $\mathfrak{W} \subseteqq \mathfrak{H}_j$, whence $\mathfrak{W} \lhd \mathfrak{M}_j$, all j. This is false, since $n \geqq 2$. The proof is complete.

LEMMA 18.10. *If $Z(\mathfrak{T})$ is not cyclic, then $|\mathfrak{Z}_i| = 2^3$ for all i.*

Proof. $Z(\mathfrak{T}) \subseteqq Z(\mathfrak{H}_i)$ and so $\mathfrak{Z}_i \supseteqq \Omega_1(Z(\mathfrak{T}))$. By Lemma 18.8, $C(\mathfrak{Z}_i)$ is a 2-group, and so $\mathfrak{H}_i = C(\mathfrak{Z}_i)$. And since $Z(\mathfrak{T})$ is non cyclic, it follows that $|\mathfrak{Z}_i| \geqq 2^3$.

Suppose $|\mathfrak{Z}_i| \geqq 2^4$. Set $\mathfrak{U}_i = [\mathfrak{Z}_i, \mathfrak{P}_i]$. Since $|C_{\mathfrak{Z}_i}(\mathfrak{P}_i)| \leqq 2$, we have $|\mathfrak{U}| \geqq 2^3$, and so $|\mathfrak{U}_i| \geqq 2^4$. Let $\mathfrak{Y}_i$ be a hyperplane of $\mathfrak{Z}_i$. Set $\mathfrak{X} = (\mathfrak{U}_i \cap \mathfrak{Y}) \cap (\mathfrak{U}_i \cap \mathfrak{Y}_i)^{P_i}$. Then $\mathfrak{X}$ admits $\mathfrak{P}_i$ and $\mathfrak{X} \neq 1$. Since $\mathfrak{X}$ contains a four-group, it follows that $C(\mathfrak{X}) = \mathfrak{H}_i$. This then implies that $C(\mathfrak{Y}_i) = \mathfrak{H}_i$, against Lemma 18.9. The proof is complete.

We continue to treat the case where $Z(\mathfrak{T})$ is not cyclic. We have $\mathfrak{Z}_i = [\mathfrak{Z}_i, \mathfrak{P}_i] \times \mathfrak{F}_i$, where $\mathfrak{F}_i = C_{\mathfrak{Z}_i}(\mathfrak{P}_i)$ is of order 2, and $[\mathfrak{Z}_i, \mathfrak{P}_i]$ is a four-group.

LEMMA 18.11. *Suppose $Z(\mathfrak{T})$ is not cyclic and Z is an involution in $\mathfrak{Z}_i$. Let $\mathfrak{T}_0$ be a S_2-subgroup of $C(Z)$. Then the following hold:*
(a) *$\mathfrak{T}_0 \cap \mathfrak{M}_i$ has index at most 2 in $\mathfrak{T}_0$.*
(b) *$C(\mathfrak{Y}) \subseteqq \mathfrak{M}_i$ for all hyperplanes $\mathfrak{Y}$ of $\mathfrak{Z}_i$.*

Proof. Let $\mathfrak{C} = C(Z)$. $\mathfrak{M}_i$ has 3 S_2-subgroups, each with a distinct centralizer in $\mathfrak{Z}_i$, so each involution of $\mathfrak{Z}_i$ centralizes one of these S_2-subgroups of $\mathfrak{M}_i$. Thus, $|\mathfrak{C}| = d\,|\mathfrak{T}|$, where $d = 1$ or 3. Thus, $|\mathfrak{C} : O_2(\mathfrak{C})|_2 \leqq 2$, so if $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{C}$, then $|\mathfrak{T}_0 : \mathfrak{T}_0 \cap \mathfrak{M}_i| \leqq |T_0, M_i, T^*| \leqq 2$, where $\mathfrak{T}^*$ is a suitable S_2-subgroup of $\mathfrak{M}_i \cap \mathfrak{C}$. This is (a). As for (b), observe that $C(\mathfrak{Y})$ is a 2-group containing $\mathfrak{H}_i$. Thus, $|C(\mathfrak{Y}) : \mathfrak{H}_i| \leqq 2$, and so $C(\mathfrak{Y}) \subseteqq N(\mathfrak{H}_i) = \mathfrak{M}_i$.

LEMMA 18.12. *$Z(\mathfrak{T})$ is cyclic.*

Proof. Suppose false. Set $\mathfrak{Z} = \mathfrak{Z}_1$, and let $\mathfrak{W}$ be the weak closure of $\mathfrak{Z}$ in $\mathfrak{T}$ with respect to $\mathfrak{G}$. Now $\mathfrak{Z}$ is elementary of order 8 and $C(\mathfrak{Y}) \subseteq \mathfrak{M} = \mathfrak{M}_1$ for all hyperplanes $\mathfrak{Y}$ of $\mathfrak{Z}$. Choose i such that $\mathfrak{W} \ntriangleleft \mathfrak{M}_i$, and then choose G in $\mathfrak{G}$ such that $\mathfrak{Z}^G \subseteq \mathfrak{T}$, $\mathfrak{Z}^G \nsubseteq \mathfrak{H}_i$. Set $\mathfrak{Z}^* = \mathfrak{Z}^G$, and let $\mathfrak{Y}^* = \mathfrak{Z}^* \cap \mathfrak{H}_i$, so that $\mathfrak{Y}^*$ is a hyperplane of $\mathfrak{Z}^*$. So $\mathfrak{Z}_i \subseteq C(\mathfrak{Y}^*) \subseteq \mathfrak{M}^* = \mathfrak{M}^G$. Since $\mathfrak{Z}^* \nsubseteq \mathfrak{H}_i = C(\mathfrak{Z}_i)$, we get that $\mathfrak{X}^* = [\mathfrak{Z}_i, \mathfrak{Z}^*] \neq 1$. Also, $\mathfrak{X}^* \subseteq \mathfrak{Z} \cap \mathfrak{Z}^*$. So $C(\mathfrak{X}^*) \supseteq \langle \mathfrak{H}^*, \mathfrak{H}_i \rangle$, where $\mathfrak{H}^* = O_2(\mathfrak{M}^G)$. Since $|\mathfrak{Z}_i| = 8$, we get that $|\mathfrak{X}^*| = 2$. By Lemma 18.11, we conclude that $|\mathfrak{H}_i : \mathfrak{H}_i \cap \mathfrak{M}^*| \leqq 2$.

Choose $Z^* \in \mathfrak{Z}^* - \mathfrak{Y}^*$. Then Z^* is an involution of $\mathfrak{M}_i - \mathfrak{H}_i$, and so we may assume that Z^* inverts $\mathfrak{P}_i$. Since $|\mathfrak{H}_i : \mathfrak{H}_i \cap \mathfrak{M}^*| \leqq 2$, it follows that $|\mathfrak{H}_i : C_{\mathfrak{H}_i}(Z^*)| \mid 2^2$. On the other hand, $\mathfrak{Z}_i = \mathfrak{U}_i \times \mathfrak{F}_i$, where $\mathfrak{U}_i = [\mathfrak{Z}_i, \mathfrak{P}_i]$, and $\mathfrak{F}_i = C_{\mathfrak{Z}_i}(\mathfrak{P}_i)$, and Z^* does not centralize $\mathfrak{U}_i$. Hence, $|\mathfrak{H}_i : \mathfrak{U}_i \cdot C_{\mathfrak{H}_i}(Z^*)| \leqq 2$.

Write $\bar{\mathfrak{H}}_i = \mathfrak{H}_i/\mathfrak{U}_i$. The dihedral group $\langle Z^*, \mathfrak{P}_i \rangle$ acts on $\bar{\mathfrak{H}}_i$, and Z^* centralizes a subgroup of $\bar{\mathfrak{H}}_i$ of index 2.

Case 1. $\mathfrak{P}_i$ centralizes $\bar{\mathfrak{H}}_i$.

In this case, we have $\mathfrak{H}_i = C_{\mathfrak{H}_i}(\mathfrak{P}_i) \times \mathfrak{U}_i$. By Lemma 18.8, $C_{\mathfrak{H}_i}(\mathfrak{P}_i)$ contains no four-group. Hence, every involution of $\mathfrak{H}_i$ is central in $\mathfrak{H}_i$, and in particular, $\mathfrak{Y}^* \subseteq Z(\mathfrak{H}_i)$. Since $\mathfrak{H}_i \subseteq \mathfrak{M}^*$, it follows that Z^* centralizes a subgroup of $\mathfrak{H}_i$ of index 2. Hence, Z^* centralizes $C_{\mathfrak{H}_i}(\mathfrak{P}_i)$. Let $\mathfrak{E} = \langle \mathfrak{Z}^*, C_{\mathfrak{H}_i}(\mathfrak{P}_i) \rangle$. Then $\mathfrak{E} \subseteq N(\mathfrak{P}_i) = \mathfrak{N}$, say. Enlarge $\mathfrak{E}\mathfrak{P}_i$ to a $S_{2,3}$-subgroup of $\mathfrak{N}$, say $\mathfrak{L}$. Since a S_3-subgroup of $\mathfrak{E}$ is not cyclic, a S_3-subgroup of $\mathfrak{L}$ is not cyclic, and so $O_2(\mathfrak{L}) = 1$.

Now $\mathfrak{F}_i \times \langle Z^* \rangle$ normalizes $O_3(\mathfrak{L})$, and since $|C(J)|_{2'} \leqq 3$ for every involution J of $\mathfrak{G}$, we get that $|O_3(\mathfrak{L})| \leqq 3^3$. If $C_{\mathfrak{H}_i}(\mathfrak{P}_i)$ contains a cyclic subgroup $\mathfrak{A}$ of order 4, then since $\mathfrak{A} \times \langle Z^* \rangle$ acts faithfully on $O_3(\mathfrak{L})$, we get $|C(J)|_3 \geqq 3^2$ for some involution J of $\mathfrak{G}$. As this is false, we conclude that $C_{\mathfrak{H}_i}(\mathfrak{P}_i) = \mathfrak{F}_i$ is of order 2, and $|\mathfrak{T}| = 2^4$. Thus, $\mathfrak{T} \cong Z_2 \times D_8$, and so $\mathfrak{T} \nsubseteq \mathfrak{G}'$, by a standard transfer argument. This contradiction shows that this case does not arise.

Case 2. $\mathfrak{P}_i$ does not centralize $\bar{\mathfrak{H}}_i$.

Let $[\bar{\mathfrak{H}}_i, \mathfrak{P}_i] = \mathfrak{R}_i/\mathfrak{U}_i$. Since Z^* centralizes a subgroup of $\bar{\mathfrak{H}}_i$ of index 2, $\mathfrak{R}_i/\mathfrak{U}_i$ is a four-group so $|\mathfrak{R}_i| = 2^4$. Since $\mathfrak{P}_i\mathfrak{R}_i$ is a Frobenius group, $\mathfrak{R}_i$ is abelian. Also, setting $\mathfrak{S}_i = C_{\mathfrak{H}_i}(\mathfrak{P}_i)$, we have $\mathfrak{H}_i = \mathfrak{S}_i\mathfrak{R}_i$, $\mathfrak{S}_i \cap \mathfrak{R}_i = 1$. Suppose $[\mathfrak{S}_i, \mathfrak{R}_i] = 1$, so that $\mathfrak{H}_i = \mathfrak{S}_i \times \mathfrak{R}_i$. In this case, since $\mathfrak{S}_i$ contains no four-group, we conclude that every involution of $\mathfrak{H}_i$ is in $Z(\mathfrak{H}_i)$, and so $\mathfrak{Y}^* \subseteq Z(\mathfrak{H}_i)$. But then Z^* centralizes a subgroup of index 2 in $\mathfrak{H}_i$, as $\mathfrak{H}_i \subseteq C(\mathfrak{Y}^*) \subseteq \mathfrak{M}^*$. This is false, since $C_{\mathfrak{R}_i}(Z^*)$ has index 4 in $\mathfrak{R}_i$. So, $[\mathfrak{S}_i, \mathfrak{R}_i] \neq 1$.

Since $|\mathfrak{H}_i : C_{\mathfrak{H}_i}(Z^*)| \leqq 2^2$, we conclude that Z^* centralizes $\mathfrak{S}_i$. Thus,

$|\mathfrak{S}_i| = 2$, since otherwise $\mathfrak{S}_i$ has a cyclic subgroup $\mathfrak{A}$ of order 4, and since $\mathfrak{A} \times \langle Z^* \rangle$ acts faithfully on some 3-subgroup of $\mathfrak{G}$, we would get $|C(J)|_3 \geqq 3^2$ for some involution J or $\mathfrak{G}$. So $\mathfrak{S}_i$ is of order 2. Thus, $\mathfrak{S}_i = \mathfrak{F}_i \subseteqq Z(\mathfrak{T})$, against $[\mathfrak{R}_i, \mathfrak{S}_i] \neq 1$. The proof is complete.

Let Z be the central involution of $\mathfrak{T}$, and set $\mathfrak{M} = C(Z)$.

LEMMA 18.13. $\mathfrak{M} \in \mathscr{M}(\mathfrak{T})$.

Proof. The only other possibility is that $\mathfrak{M} = \mathfrak{T}$. Let $\mathfrak{Z}_1 = \Omega_1(Z(\mathfrak{H}_1)) \supseteqq \mathfrak{Z}$. Since $\mathfrak{H}_1$ has index 2 in $\mathfrak{T}$, it follows that $\mathfrak{Z}_1$ is a four-group. Choose $U \in \mathfrak{Z}_1$, $U \neq Z$. Then U and Z are fused in $\mathfrak{M}_1$, since $\mathfrak{M} = \mathfrak{T}$. Let $\mathfrak{W}$ be the weak closure of $\mathfrak{Z}_1$ in $\mathfrak{T}$ with respect to $\mathfrak{G}$. Now $\mathfrak{W}$ is not normal in every $\mathfrak{M}_j \in \mathscr{M}(\mathfrak{T})$, so choose $\mathfrak{M}_j$ such that $\mathfrak{W} \ntriangleleft \mathfrak{M}_j$. So there is G in $\mathfrak{G}$ such that $\mathfrak{Z}_1^G \nsubseteq \mathfrak{H}_j$, $\mathfrak{Z}_1^G \subseteqq \mathfrak{T}$. Since $\mathfrak{Z}_j$ is a four-group with $C(\mathfrak{Z}_j) = \mathfrak{H}_j$, we have $[\mathfrak{Z}_j, \mathfrak{Z}_1^G] \neq 1$. Since $\mathfrak{Z}_1^G$ normalizes $\mathfrak{Z}_j$, $\tilde{\mathfrak{U}} = [\mathfrak{Z}_j, \mathfrak{Z}_1^G] = \langle \tilde{U} \rangle \subset \mathfrak{Z}_j$. Now $\mathfrak{Z}_1^G$ does not act faithfully on $\mathfrak{Z}_j$ and so $\mathfrak{Z}_j \subseteqq \mathfrak{M}^G$. Hence, $\tilde{U} \in \mathfrak{Z}_1^G$. But now $\mathfrak{H}_j \subseteqq C(\tilde{U}) \subseteqq \mathfrak{M}_1^G$, and so $\mathfrak{H}_j$ normalizes $\mathfrak{Z}_1^G$. This implies that $\mathfrak{H}_j = \mathfrak{Z}_j \times C_{\mathfrak{H}_j}(\mathfrak{P}_j)$, which contradicts $\mathfrak{Z}_j = \Omega_1(Z(\mathfrak{H}_j))$. The proof is complete.

Choose $\mathfrak{N} \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{N} \neq \mathfrak{M}$, and let $\mathfrak{U}$ be the minimal normal subgroup of $\mathfrak{N}$, so that $\mathfrak{U} \supseteqq \langle Z \rangle$. Since $Z \notin Z(\mathfrak{N})$, it follows that $\mathfrak{U}$ is a 4-group. Hence, $\mathfrak{U} = \Omega_1(Z(O_2(\mathfrak{N})))$, and $A_{\mathfrak{G}}(\mathfrak{U}) = \mathrm{Aut}(\mathfrak{U})$. Since $[\mathfrak{T}, \mathfrak{U}] = \langle Z \rangle$, it follows that $\mathfrak{U} \subseteqq O_2(\mathfrak{M})$. Let $\mathfrak{W} = \mathfrak{U}^{\mathfrak{M}} = \mathfrak{U}^{\mathfrak{T}\mathfrak{P}} = \mathfrak{U}^{\mathfrak{P}}$ (where $\mathfrak{P} = \langle P \rangle$ is of order 3).

LEMMA 18.14. *$\mathfrak{W}$ is elementary of order* 8.

Proof. Suppose false. Then $\mathfrak{U} = \mathfrak{U}_0 \times \langle Z \rangle$, $\mathfrak{U}_0 = \langle U \rangle$, so $\mathfrak{W} = \langle Z, U, U^P, U^{P^2} \rangle$. Since $\mathfrak{U} \triangleleft \mathfrak{T}$, so also $\mathfrak{U} \triangleleft \mathfrak{H} = O_2(\mathfrak{M})$, and so $\mathfrak{U}^{P^i} \triangleleft \mathfrak{M}$, whence $|\mathfrak{W}| \leqq 2^4$. Let $\mathfrak{W}_1 = [\mathfrak{W}, \mathfrak{P}]$. Since $\mathfrak{P} \nsubseteq \mathfrak{N} = N(\mathfrak{U})$, it follows that $\mathfrak{W}_1 \neq 1$.

Since $\mathfrak{M}$ does not normalize $\mathfrak{U}$, we have $|\mathfrak{W}| = 8$ or 16. If $|\mathfrak{W}| = 8$, then since $\mathfrak{W}$ is generated by involutions, and since a dihedral group of order 8 has no automorphism of order 3, it follows that $\mathfrak{W}$ is elementary. So we conclude that $|\mathfrak{W}| = 16$.

If $\mathfrak{W}$ is elementary abelian, then since $\mathfrak{Z} \subset \mathfrak{W}$ and $\mathfrak{P}$ centralizes $\mathfrak{Z}$, it follows that $C_{\mathfrak{W}}(\mathfrak{P})$ is a four-group. This violates Lemma 18.8. Hence, $\mathfrak{W}$ is not elementary, and since $\mathfrak{W}$ is generated by involutions, we conclude that $\mathfrak{W}' \neq 1$. Since $\mathfrak{W}/\mathfrak{Z}$ is elementary, it follows that $Z(\mathfrak{W}) \supset \mathfrak{Z}$. Since $\mathfrak{W}' \neq 1$, $|\mathfrak{W}: Z(\mathfrak{W})| \geqq 2^2$, and so $Z(\mathfrak{W})$ is of order 4. Since $C_{\mathfrak{W}}(\mathfrak{P})$ contains no four-group, it follows that $Z(\mathfrak{W})$ is cyclic. Thus, $\mathfrak{W}$ is the central product of $Z(\mathfrak{W})$ and $\mathfrak{W}_1$, and $\mathfrak{W}_1$ is a quaternion group. Since $\mathfrak{W}_1 Z(\mathfrak{W}) = \mathfrak{W}$ has just one quaternion subgroup, we conclude that $\mathfrak{W}_1 \lhd \mathfrak{M}$.

Set $\mathfrak{N} = N(\mathfrak{U}) = \mathfrak{T}\mathfrak{Q}$, $\mathfrak{Q} = \langle Q\rangle$, $Q^3 = 1$, $\mathfrak{K} = O_2(\mathfrak{N}) = C(\mathfrak{U})$. Since $C_{\mathfrak{W}}(\mathfrak{W}_1) = Z(\mathfrak{W})$ is cyclic, we have $\mathfrak{W}_1 \not\subseteq \mathfrak{K}$. Let $\mathfrak{K}_1 = [\mathfrak{K}, \mathfrak{Q}]$. Then $\mathfrak{U} = [\mathfrak{U}, \mathfrak{Q}] \subseteq \mathfrak{K}_1$.

If $\mathfrak{K}_1 = \mathfrak{U}$, then $\mathfrak{K} = \mathfrak{U} \times C_{\mathfrak{K}}(\mathfrak{Q})$, and $C_{\mathfrak{K}}(\mathfrak{Q}) \lhd \mathfrak{N}$. This is false, since $\mathfrak{U}$ is the only minimal normal subgroup of $\mathfrak{N}$. So $\mathfrak{K}_1 \supset \mathfrak{U}$.

Suppose $\mathfrak{K}_1$ is non abelian. Then $\mathfrak{K}_1' \lhd \mathfrak{N}$, and so $\mathfrak{U} \subseteq \mathfrak{K}_1'$. Hence, $\mathfrak{C} = [\mathfrak{K}_1, \mathfrak{W}_1]\mathfrak{K}_1'/\mathfrak{K}_1'$ has order precisely 2. Choose $W \in \mathfrak{W}_1 - \mathfrak{K}$. Then W centralizes a hyperplane of $\mathfrak{K}_1/\mathfrak{K}_1'$, and so $\mathfrak{K}_1' = 2$. This is impossible since $\mathfrak{K}_1\mathfrak{Q}$ is a Frobenius group. We conclude that $\mathfrak{K}_1$ is abelian. Since $[\mathfrak{K}_1, \mathfrak{W}_1] \subseteq \mathfrak{W}_1$, it follows that $[\mathfrak{K}_1, \mathfrak{W}_1]$ is cyclic of order 4, and so $\mathfrak{K}_1$ is the direct product of two cyclic groups of order 4.

Let $\mathfrak{K}_2 = C_{\mathfrak{K}}(\mathfrak{Q})$, so that $\mathfrak{K} = \mathfrak{K}_1 \cdot \mathfrak{K}_2$. Since $\mathfrak{U}$ is the only minimal normal subgroup of $\mathfrak{N}$, $\mathfrak{K}_2$ acts faithfully on $\mathfrak{K}_1$. Furthermore, by Lemma 18.8, $\mathfrak{K}_2$ contains no four-group. Since $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{T}) \neq \emptyset$, it follows that $\mathfrak{K}_2 \neq 1$. Since $\mathfrak{K}_2$ stabilizes the chain $\mathfrak{K}_1 \supset \mathfrak{U} \supset 1$, it follows that $\mathfrak{K}_2$ is elementary abelian, and so $\mathfrak{K}_2 = \langle K\rangle$ is of order 2, which gives $|\mathfrak{T}| = 2^6$. The isomorphism type of $\mathfrak{T}$ is uniquely determined by the preceding data, and we see that $\mathfrak{H}$ is the central product of 2 quaternion groups. Furthermore, $\mathfrak{T}$ has an element of order 8 which is fused in $\mathfrak{G}$ to all of its odd powers. Since $\langle Z\rangle = \mathfrak{Z}$ char $C_{\mathfrak{T}}(K)$, it follows that K is not fused to Z in $\mathfrak{G}$. By a theorem of Brauer and Fong [11] we have $\mathfrak{G} \cong M_{12}$. Since M_{12} is not an N-group, the proof of this lemma is complete.

We use the following notation: $\mathfrak{M} = \mathfrak{T}\mathfrak{P}$, $\mathfrak{P} = \langle P\rangle$, $\mathfrak{N} = \mathfrak{T}\mathfrak{Q}$, $\mathfrak{Q} = \langle Q\rangle$, $\mathfrak{Z} = \langle Z\rangle$, $\mathfrak{U} = \mathfrak{Z}^{\mathfrak{N}}$, $\mathfrak{W} = \mathfrak{U}^{\mathfrak{M}}$, $P^3 = 1$, $Q^3 = 1$. And we set $\mathfrak{X} = \mathfrak{W}^{\mathfrak{N}}$.

LEMMA 18.15. *$\mathfrak{X}$ is abelian.*

Proof. We have $\mathfrak{W} = \mathfrak{U} \times \langle X\rangle$, and $\mathfrak{W}^{\mathfrak{N}} = \mathfrak{W}^{\mathfrak{T}\mathfrak{Q}} = \langle \mathfrak{U}, X, X^Q, X^{Q^2}\rangle$. Also, $\mathfrak{W} \subseteq C(\mathfrak{U}) = \mathfrak{K} = O_2(\mathfrak{N})$, so $\mathfrak{W} \lhd \mathfrak{K}$, and $\mathfrak{W}^{Q^i} \lhd \mathfrak{K}$. Hence, $|\mathfrak{X}| \leqq 2^5$. Since $\mathfrak{W} \ntriangleleft \mathfrak{N}$, $2^4 \leqq |\mathfrak{X}|$.

Suppose $\mathfrak{X}' \neq 1$. Since $\mathfrak{U}$ is a minimal normal subgroup of $\mathfrak{N}$, we have $\mathfrak{X}' = \mathfrak{U}$. If $|\mathfrak{X}| = 2^4$, then $\mathfrak{X}$ is of maximal class, so does not have an automorphism of order 3. Hence, $|\mathfrak{X}| = 2^5$. Since $\mathfrak{X}/\mathfrak{U}$ is elementary, and since $C(\mathfrak{Q})$ contains no four-group, it follows that $\mathfrak{X}/\mathfrak{U} = \mathfrak{X}_0/\mathfrak{U} \times \mathfrak{X}_1/\mathfrak{U}$, where $\mathfrak{X}_0/\mathfrak{U} = C_{\mathfrak{X}/\mathfrak{U}}(\mathfrak{Q})$ is of order 2, and $\mathfrak{X}_1/\mathfrak{U} = [\mathfrak{X}, \mathfrak{Q}]\mathfrak{U}/\mathfrak{U}$ is a four-group. Since $\mathfrak{X}_1\mathfrak{Q}$ is a Frobenius group, $\mathfrak{X}_1$ is abelian. Also, $\mathfrak{X}_0$ is elementary of order 8, and $\mathfrak{X}_0 \neq \mathfrak{W}$, since $\mathfrak{Q}$ does not normalize $\mathfrak{W}$. Let $\mathfrak{X}^0 = C_{\mathfrak{X}}(Q)$, so that $\mathfrak{X}^0$ is of order 2. Since $\mathfrak{X}' \neq 1$, it follows that $C_{\mathfrak{X}_1}(\mathfrak{X}^0) = \mathfrak{U}$, and so $Z(\mathfrak{X}) = \mathfrak{U}$. Since $\mathfrak{X}_1$ is the unique abelian subgroup of index 2 in $\mathfrak{X}$, we conclude that $\mathfrak{X}_1 \lhd \mathfrak{N}$, and so $\mathfrak{W} \not\subseteq \mathfrak{X}_1$. Choose $W \in \mathfrak{W} - \mathfrak{X}_1$. Then $W = X^0X_1$, where X^0 generates $\mathfrak{X}^0$, and $X_1 \in \mathfrak{X}_1$.

Since $W^2 = 1$, X^0 inverts X_1. Since $\mathfrak{X}_0 \neq \mathfrak{W}$, we have $X_1 \notin \mathfrak{U}$. Hence X^0 does not centralize X_1, and so X_1 is not an involution. Thus, $\mathfrak{X}_1$ is the direct product of two cyclic groups of order 4, and X^0 inverts $\mathfrak{X}_1$.

Since X^0 inverts $\mathfrak{X}_1$, it follows that $\mathfrak{X}$ has precisely 4 elementary abelian subgroups of order 2^3, one of which is $\mathfrak{W}$. Thus, $\mathfrak{N}$ permutes these 4 subgroups, and the orbit which contains $\mathfrak{W}$ has cardinal 3. This implies that $\mathfrak{X}_0 \lhd \mathfrak{N}$. Hence, $[\mathfrak{T}, \mathfrak{X}_0] \subseteq \mathfrak{U}$. This then implies that $[\mathfrak{T}, \mathfrak{X}] \subset \mathfrak{X}_1$, and $[\mathfrak{T}, \mathfrak{X}] \supseteq \mathfrak{U}$. Since $\mathfrak{X} - \mathfrak{X}_1$ is a set of involutions, we can choose an involution I in $\mathfrak{X} - \mathfrak{X}_1$ such that $[I, \mathfrak{W}] \not\subseteq \mathfrak{Z}$. Since $\mathfrak{W}/\mathfrak{Z} \subseteq Z(\mathfrak{H}/\mathfrak{Z})$, we have $I \notin \mathfrak{H}$. Since $[I, \mathfrak{H}]\mathfrak{U} \subset \mathfrak{X}_1$, it follows that $|[I, \mathfrak{H}]\mathfrak{W}/\mathfrak{W}| \leqq 2$. If I centralizes $\mathfrak{H}/\mathfrak{W}$, then $[\mathfrak{H}, \mathfrak{P}] \subseteq \mathfrak{W}$, which gives $[\mathfrak{P}, \mathfrak{H}] = [\mathfrak{W}, \mathfrak{P}] \lhd \mathfrak{H}$. This is false, since $Z(\mathfrak{T})$ is cyclic, and $Z(\mathfrak{T}) \cap [\mathfrak{W}, \mathfrak{P}] = 1$. We conclude that $[\mathfrak{H}/\mathfrak{W}, I]$ is of order 2. Set $\mathfrak{H}_1 = [\mathfrak{H}, \mathfrak{P}]$. Then $\mathfrak{H}_1 \supset \mathfrak{W}$ and $|\mathfrak{H}_1| \leqq 2^5$. Since $\mathfrak{H}_1/\mathfrak{W}$ admits $\mathfrak{P}$, we have $|\mathfrak{H}_1| = 2^5$. If $\mathfrak{W} \subseteq Z(\mathfrak{H}_1)$, then S_3-subgroups of $\mathfrak{M}/C(\mathfrak{W})$ are normal, and so $[\mathfrak{W}, \mathfrak{P}] \lhd \mathfrak{M}$. This is false, as we have already seen, and so $\mathfrak{W} \not\subseteq Z(\mathfrak{H}_1)$. Thus, as $\mathfrak{P}$ acts without fixed points on $\mathfrak{H}_1/\mathfrak{Z}$, we get that $\mathfrak{H}_1' = \mathfrak{Z}$. If $Z(\mathfrak{H}_1) \supset \mathfrak{Z}$, then $\mathfrak{H}_1 = \mathfrak{W} \cdot Z(\mathfrak{H}_1)$ is abelian. This is false, and so $Z(\mathfrak{H}_1) = \mathfrak{Z}$, whence $\mathfrak{H}_1$ is extra special. Since $\mathfrak{W}$ exists, $\mathfrak{H}_1$ is the central product of two quaternion groups.

Since $[I, \mathfrak{H}] \subset \mathfrak{X}_1$, it follows that $[I, \mathfrak{H}]$ contains no elementary subgroup of order 8, and so I fixes both the quaternion subgroups of $\mathfrak{H}_1$. We assume without loss of generality that I inverts $\mathfrak{P}$. This then implies that $[\mathfrak{H}_1, I]$ is abelian of type (2, 4), and so $[\mathfrak{H}_1, I] = [\mathfrak{T}, \mathfrak{X}]$. Let $\mathfrak{H}_2 = C_{\mathfrak{H}}(\mathfrak{P})$, so that $\mathfrak{H} = \mathfrak{H}_1\mathfrak{H}_2$, and $\mathfrak{H}_2$ admits I, while $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \mathfrak{Z}$. As we saw in a previous argument, $N(\mathfrak{P})$ contains no non cyclic abelian subgroup of order 8, and since $[\mathfrak{H}_2, I] \subseteq [\mathfrak{T}, \mathfrak{X}] \subseteq \mathfrak{H}_1$, it follows that $[\mathfrak{H}_2, I] \subseteq \mathfrak{Z}$. Since $\mathfrak{H}_2$ is either cyclic or generalized quaternion, it follows that $\langle \mathfrak{H}_2, I \rangle$ contains a non cyclic abelian subgroup of order 8 unless $\mathfrak{H}_2$ is cyclic of order at most 4. So $\mathfrak{H}_2$ is cyclic, and $|\mathfrak{H}_2| \leqq 4$.

Suppose $|\mathfrak{H}_2| = 4$. Then $|\mathfrak{H}| = 2^6$, $|\mathfrak{T}| = 2^7$, $|\mathfrak{K}| = 2^6$. Thus, $\mathfrak{Q}$ acts on $\mathfrak{K}/\mathfrak{X}_1$, a group of order 4, and $\mathfrak{Q}$ centralizes $\mathfrak{X}/\mathfrak{X}_1$, whence $\mathfrak{Q}$ centralizes $\mathfrak{K}/\mathfrak{X}_1$, whence $[\mathfrak{K}, \mathfrak{Q}] = \mathfrak{X}_1$. Let $\mathfrak{X}_2 = C_{\mathfrak{K}}(\mathfrak{Q})$, so that $|\mathfrak{X}_2| = 4$, $\mathfrak{X}_1 \cap \mathfrak{X}_2 = 1$, whence $\mathfrak{X}_2$ is cyclic of order 4. But $\mathfrak{X}_2$ stablizes the chain $\mathfrak{X}_1 \supset \mathfrak{U} \supset 1$, and so $\mathfrak{X}^0$ is forced to centralize $\mathfrak{X}_1$. This is false, since $\mathfrak{X}' \neq 1$. So $|\mathfrak{H}_2| = 2$. Since $\mathfrak{H}_2 \supseteq \mathfrak{Z}$, we conclude that $\mathfrak{H}_2 = \mathfrak{Z}$, and so $\mathfrak{H}_1 = \mathfrak{H} = \mathfrak{Q}_1 \circ \mathfrak{Q}_2$, where $\mathfrak{Q}_i$ is a quaternion group of order 8, $i = 1, 2$. Also, $\mathfrak{T} = \mathfrak{H}\langle I \rangle$. Since $|\mathfrak{T}| = 2^6$, it follows that $\mathfrak{X} = \mathfrak{K}$, and since X^0 inverts $\mathfrak{X}_1$, the isomorphism type of $\mathfrak{T}$ is uniquely determined. It is straightforward to check that $\mathfrak{G}$ has more than 1 class of involutions, and so the theorem of Brauer-Fong [11] implies that $\mathfrak{G} \cong M_{12}$, which is false. The proof is complete.

Since $\mathfrak{X}' = 1$, we see that $\mathfrak{X}$ is elementary of order 2^4 or 2^5.

For each conjugate $\mathfrak{W}^*$ of $\mathfrak{W}$, define $Z^*(\mathfrak{W}^*)$ to be the unique central involution in $N(\mathfrak{W}^*)$. Thus, $Z^*(\mathfrak{W}) = Z$.

Now $\mathfrak{M}$ acts on $\mathscr{I}$, the set of involutions of $\mathfrak{W}$, and on $\mathscr{H}$, the set of hyperplanes of $\mathfrak{W}$. In its action on $\mathscr{I}$, $\mathfrak{M}$ has two orbits, of sizes 1 and 6; and in its action on $\mathscr{H}$, $\mathfrak{M}$ has two orbits, of sizes 3 and 4, and $\mathfrak{U}$ is in an orbit of size 3. For each conjugate $\mathfrak{W}^*$ of $\mathfrak{W}$, let $\mathscr{U}(\mathfrak{W}^*)$ be the orbit of size 3 of $N(\mathfrak{W}^*)$ on the hyperplanes of $\mathfrak{W}^*$.

LEMMA 18.16. *If $\mathfrak{A}$ is a hyperplane of $\mathfrak{W}$, and $\mathfrak{Z} \nsubseteq \mathfrak{A}$, then $N(\mathfrak{A}) \subseteq \mathfrak{M}$.*

Proof. Let $\mathfrak{L} = N(\mathfrak{A})$, $\mathfrak{L}_1 = N_{\mathfrak{M}}(\mathfrak{A})$. Since $A_{\mathfrak{M}}(\mathfrak{A}) = \mathrm{Aut}\,(\mathfrak{A})$, it suffices to show that $C(\mathfrak{A}) \subseteq \mathfrak{M}$. In any case, $C(\mathfrak{A})$ is a 2-group, and $C_{\mathfrak{M}}(\mathfrak{A}) = C_{\mathfrak{M}}(\mathfrak{W}) \lhd \mathfrak{M}$. Since $N(C_{\mathfrak{M}}(\mathfrak{A})) \subseteq \mathfrak{M}$, the lemma follows.

Note next that $\mathfrak{U}$ is the unique normal four-subgroup of $\mathfrak{N}$, and so if $\mathfrak{X}^*$ is a conjugate to $\mathfrak{X}$ in $\mathfrak{G}$, define $\mathfrak{U}^*(\mathfrak{X}^*)$ to be the unique normal 4-subgroup of $N(\mathfrak{X}^*)$, so that $\mathfrak{U}^*(\mathfrak{X}) = \mathfrak{U}$.

If $\mathfrak{U}^*$ is a conjugate of $\mathfrak{U}$, and $U \in \mathfrak{U}^{*\#}$, set $\mathfrak{W}_U(\mathfrak{U}^*) = \mathfrak{U}^{*C(U)}$, so that $\mathfrak{W}_U(\mathfrak{U}^*)$ is conjugate to $\mathfrak{W}$.

LEMMA 18.17. *If $\mathfrak{W}^*$ is a conjugate of $\mathfrak{W}$ in $\mathfrak{G}$ and $Z \in \mathfrak{W}^*$, then $[\mathfrak{W}^*, \mathfrak{U}] = 1$.*

Proof. Let $\mathfrak{W}^* = \mathfrak{W}^X$, so that $Z^{X^{-1}} \in \mathfrak{W}$. If $Z^{X^{-1}} = Z$, then $X^{-1} \in C(Z) = \mathfrak{M}$, so $X \in \mathfrak{M}$, $\mathfrak{W}^X = \mathfrak{W}$, and $[\mathfrak{W}, \mathfrak{U}] = 1$. If $Z^{X^{-1}} \neq Z$, then $Z^{X^{-1}} = U^M$ for some M in $\mathfrak{M}$, since $\mathfrak{M}$ is transitive on $\mathfrak{W} - \mathfrak{Z}$. Also, there is N in $\mathfrak{N}$ such that $U = Z^N$, so $Z^{X^{-1}} = Z^{NM}$, whence $NMX \in C(Z) = \mathfrak{M}$. Since $NMX = M_1 \in \mathfrak{M}$, we have $X = M^{-1}N^{-1}M_1$, and $\mathfrak{W}^* = \mathfrak{W}^X = \mathfrak{W}^{N^{-1}M_1}$,

$$\begin{aligned}[\mathfrak{W}^*, \mathfrak{U}] &= [\mathfrak{W}^{N^{-1}M_1}, \mathfrak{U}] \subseteq [\mathfrak{W}^{N^{-1}M_1}, \mathfrak{W}] = [\mathfrak{W}^{N^{-1}}, \mathfrak{W}]^{M_1} \\ &\subseteq [\mathfrak{X}, \mathfrak{X}]^{M_1} = 1\,.\end{aligned}$$

LEMMA 18.18. *If $\mathfrak{W}^* \in \mathrm{ccl}_{\mathfrak{G}}(\mathfrak{W})$ and $\mathfrak{W}^* \cap \mathfrak{U} \neq 1$, then $[\mathfrak{W}^*, \mathfrak{U}] = 1$.*

Proof. Choose $Y \in \mathfrak{W}^* \cap \mathfrak{U}$, $Y \neq 1$. Then $Y^N = Z$ for some N in $\mathfrak{N}$, so that $Z \in \mathfrak{W}^{*N}$, and $1 = [\mathfrak{W}^{*N}, \mathfrak{U}] = [\mathfrak{W}^*, \mathfrak{U}]^N$, so that $[\mathfrak{W}^*, \mathfrak{U}] = 1$, as asserted.

LEMMA 18.19. *If $\mathfrak{W}^* \in \mathrm{ccl}_{\mathfrak{G}}(\mathfrak{W})$, $\mathfrak{U}^* \in \mathrm{ccl}_{\mathfrak{G}}(\mathfrak{U})$, and $\mathfrak{W}^* \cap \mathfrak{U}^* \neq 1$, then $[\mathfrak{W}^*, \mathfrak{U}^*] = 1$.*

Proof. This is an immediate consequence of Lemma 18.18.

LEMMA 18.20. $V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{U}); \mathfrak{T}) \subseteq C(\mathfrak{U}) = \mathfrak{K} = O_2(\mathfrak{N})$.

Proof. If $\mathfrak{U}^* \in \mathrm{ccl}_{\mathfrak{G}}(\mathfrak{U})$, and $\mathfrak{U}^* \subseteq \mathfrak{T}$, then $\mathfrak{U}^*$ acts on $\mathfrak{U}$, and is not faithful on $\mathfrak{U}$. Let $\mathfrak{U}_0^* = C_{\mathfrak{U}^*}(\mathfrak{U})$, and suppose by way of contradiction that $\mathfrak{U}_0^*$ is of order 2. Since $\mathfrak{U}^* \sim \mathfrak{U}$, and since $N(\mathfrak{U})$ is transitive on $\mathfrak{U}^\#$, there is G in $\mathfrak{G}$ such that $\mathfrak{U}^{*G} = \mathfrak{U}$, $\mathfrak{U}_0^{*G} = \mathfrak{Z}$. Thus, $C(\mathfrak{U}_0^*)^G = \mathfrak{M}$, so that $C(\mathfrak{U}_0^*) = \mathfrak{M}^{G^{-1}}$. And so $\mathfrak{U}^{*C(\mathfrak{U}_0^*)} = \mathfrak{W}^{G^{-1}}$.

Now $1 \neq [\mathfrak{U}, \mathfrak{U}^*] \subseteq \mathfrak{W}^{G^{-1}}$, since $\mathfrak{U} \subseteq C(\mathfrak{U}_0^*)$. Thus, $\mathfrak{W}^{G^{-1}} \cap \mathfrak{U} \neq 1$, and so $[\mathfrak{W}^{G^{-1}}, \mathfrak{U}] = 1$, against $\mathfrak{U}^* \subseteq \mathfrak{W}^{G^{-1}}$, $[\mathfrak{U}^*, \mathfrak{U}] \neq 1$. The proof is complete.

LEMMA 18.21. *If $\mathfrak{Y}$ is a four-subgroup of $\mathfrak{W}$, then $C(\mathfrak{Y}) \subseteq \mathfrak{M}$, and if $Z \in \mathfrak{Y}$, then $N(\mathfrak{Y}) \subseteq \mathfrak{M}$.*

Proof. The lemma is clear if $Z \in \mathfrak{Y}$, and if $Z \notin \mathfrak{Y}$, this is just Lemma 18.16.

LEMMA 18.22. $[\mathfrak{X}, \mathfrak{X}^P] \neq 1$, *where* $\mathfrak{P} = \langle P \rangle$, $\mathfrak{M} = \mathfrak{T}\mathfrak{P}$, $P^3 = 1$.

Proof. Suppose false. Then $[\mathfrak{X}^P, \mathfrak{X}^{P^2}] = 1 = [\mathfrak{X}, \mathfrak{X}^{P^2}]$, so that $\mathfrak{Y} = \langle \mathfrak{X}, \mathfrak{X}^P, \mathfrak{X}^{P^2} \rangle$ is elementary abelian. Since $\mathfrak{V} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{X}); \mathfrak{T}) \lhd \mathfrak{N}$ it follows that $\mathfrak{V}$ is not normal in $\mathfrak{M}$, and so $\mathfrak{V} \not\subseteq \mathfrak{H} = O_2(\mathfrak{M})$. There is therefore $\mathfrak{X}^* \in \mathrm{ccl}_{\mathfrak{G}}(\mathfrak{X})$ such that $\mathfrak{X}^* \subseteq \mathfrak{T}$, $\mathfrak{X}^* \not\subseteq \mathfrak{H}$.

Choose X in $\mathfrak{G}$ so that $\mathfrak{X}^* = \mathfrak{X}^X$. Since $\mathfrak{V} \subseteq C(\mathfrak{U})$, we have $\mathfrak{U} \subseteq C(\mathfrak{X}^X)$. Set $\mathfrak{L}^X = \mathfrak{X}^X \cap C(\mathfrak{W})$, so that $\mathfrak{L} \subset \mathfrak{X}$. Since $\mathfrak{X}^X$ stabilizes the chain $\mathfrak{W} \supset \mathfrak{U} \supset 1$, we have $|\mathfrak{X}^X : \mathfrak{L}^X| \leq 4$.

Case 1. $\mathfrak{L}^X \cap \mathfrak{U}^X = 1$.

In this case, $\mathfrak{X}^X = \mathfrak{L}^X \times \mathfrak{U}^X$ so that $\mathfrak{W} \subseteq C(\mathfrak{L}^X)$. Since $\mathfrak{L}^X$ contains a four-group, it follows that $C(\mathfrak{L}^X)$ is a 2-group. Since $C(\mathfrak{L}^X) \cap \mathfrak{N}^X$ is a normal subgroup of $\mathfrak{N}^X$, we conclude that $C(\mathfrak{L}^X) \subseteq \mathfrak{N}^X$. So $\mathfrak{W} \subseteq \mathfrak{N}^X$, and so $\mathfrak{W}$ centralizes $\mathfrak{U}^X$, whence $\mathfrak{W}$ centralizes $\mathfrak{X}^X$, which is false.

Case 2. $\mathfrak{L}^X \cap \mathfrak{U}^X = \mathfrak{U}_0^X$ has order 2.

Now $\mathfrak{W}^X \subseteq \mathfrak{K}$, so $\mathfrak{U}^X \subseteq \mathfrak{K}$, and if $\mathfrak{U}^X \subseteq \mathfrak{H}$, then $\mathfrak{U}^{XP} \subseteq \mathfrak{H} \subseteq \mathfrak{T}$, so that $\mathfrak{U}^{XP} \subseteq \mathfrak{K}$, by Lemma 18.20. But $\mathfrak{U}_0^X$ has order 2, so that $\mathfrak{U}^X$ does not centralize $\mathfrak{W} = \langle \mathfrak{U}, \mathfrak{U}^{P^{-1}} \rangle$. Hence, $\mathfrak{U}^X$ does not centralize $\mathfrak{U}^{P^{-1}}$, and so $[\mathfrak{U}^X, \mathfrak{U}^{P^{-1}}] \neq 1$, which gives $[\mathfrak{U}^{XP}, \mathfrak{U}] \neq 1$. We conclude from this that $\mathfrak{U}^X \not\subseteq \mathfrak{H}$.

Set $\tilde{\mathfrak{W}}^X = (\mathfrak{U}^X)^{C(U_0^X)}$, so that $\tilde{\mathfrak{W}} \in \{\mathfrak{W}, \mathfrak{W}^Q, \mathfrak{W}^{Q^{-1}}\}$. Since $\mathfrak{W} \subseteq C(\mathfrak{U}_0^X)$, $\mathfrak{W}$ normalizes $\tilde{\mathfrak{W}}^X$. Also, $\mathfrak{W} \subseteq \langle \mathfrak{W}, \mathfrak{W}^Q, \mathfrak{W}^{Q^2} \rangle = \mathfrak{X}$, so $\tilde{\mathfrak{W}}^X \subseteq \mathfrak{X}^X \subseteq \mathfrak{T}$, and $\tilde{\mathfrak{W}}^X$ normalizes $\mathfrak{W}$ so that $[\tilde{\mathfrak{W}}^X, \mathfrak{W}] \subseteq \tilde{\mathfrak{W}}^X \cap \mathfrak{W}$. Now $\mathfrak{U}^X \subseteq \tilde{\mathfrak{W}}^X$, and $[\mathfrak{U}^X, \mathfrak{W}] \neq 1$, since $\mathfrak{U}^X \not\subseteq \mathfrak{L}^X$. Thus, $1 \neq [\tilde{\mathfrak{W}}^X, \mathfrak{W}] \subseteq \tilde{\mathfrak{W}}^X \cap \mathfrak{W}$, and we argue that $[\tilde{\mathfrak{W}}^X, \mathfrak{W}] \not\subseteq \mathfrak{Z}$. For if $[\tilde{\mathfrak{W}}^X, \mathfrak{W}] = \mathfrak{Z}$, we get $[\mathfrak{U}^X, \mathfrak{W}] = \mathfrak{Z}$, so that $\mathfrak{U}^X \subseteqq C_{\mathfrak{M}}(\mathfrak{W}/\mathfrak{Z}) = \mathfrak{H}$, which is false. Choose $A \in [\mathfrak{W}, \mathfrak{U}^X] - \mathfrak{Z}$. Since $\mathfrak{U}^X \subseteqq \mathfrak{X}^X \subseteqq \mathfrak{T}$, and since $\mathfrak{T}$ stabilizes $\mathfrak{W} \supset \mathfrak{U} \supset \mathfrak{Z} \supset 1$, we have $[\mathfrak{W}, \mathfrak{U}^X] \subseteqq \mathfrak{U}$, so that $A \in \mathfrak{U}$.

Now $\widetilde{\mathfrak{W}}^X \in \{\mathfrak{W}^X, \mathfrak{W}^{QX}, \mathfrak{W}^{Q^2X}\}$. Choose i such that $\widetilde{\mathfrak{W}}^X = \mathfrak{W}^{Q^iX}$.

Since $A \in \mathfrak{U} - \mathfrak{Z}$, we have $A = Z^{Q^j}$ for some generator Q^j of $\mathfrak{Q}$. And $A \in \mathfrak{W}^X = \mathfrak{W}^{Q^iX}$, whence $Z^{Q^j} \in \mathfrak{W}^{Q^iX}$ and so $Z^{Q^jX^{-1}Q^{-i}} \in \mathfrak{W}$.

If $Z^{Q^jX^{-1}Q^{-i}} = Z$, then $Q^jX^{-1}Q^{-i} \in \mathfrak{M}$, so that $X \in \mathfrak{N}\mathfrak{M}\mathfrak{N}$, $X = N_1M_1N_2$, where $N_1, N_2 \in \mathfrak{N}$, $M_1 \in \mathfrak{M}$, and we find that

$$\begin{aligned} 1 \neq [\mathfrak{W}^X, \widetilde{\mathfrak{W}}] &\subseteqq [\mathfrak{X}^{N_1M_1N_2}, \mathfrak{W}] = [\mathfrak{X}^{M_1N_2}, \mathfrak{W}] \subseteqq [\mathfrak{X}^{M_1N_2}, \mathfrak{X}] \\ &= [\mathfrak{X}^{M_1}, \mathfrak{X}]^{N_2} = 1 . \end{aligned}$$

So suppose then that $Z^{Q^jX^{-1}Q^{-i}} = U^M$, $U \in \mathfrak{U}$, $M \in \mathfrak{M}$. Now $U = Z^N$ for some N in $\mathfrak{N}$, so $Z^{Q^jX^{-1}Q^{-i}} = Z^{NM}$, and thus $\bar{M} = Q^jX^{-1}Q^{-i}M^{-1}N^{-1} \in C(Z) = \mathfrak{M}$, and so $Q^iX = M^{-1}N^{-1}\bar{M}^{-1}Q^j$, and

$$\begin{aligned} [\widetilde{\mathfrak{W}}^X, \mathfrak{W}] = [\mathfrak{W}^{Q^iX}, \mathfrak{W}] &= [\mathfrak{W}^{M^{-1}N^{-1}\bar{M}^{-1}Q^j}, \mathfrak{W}] \\ &= [\mathfrak{W}^{N^{-1}\bar{M}^{-1}Q^j}, \mathfrak{W}] \\ &\subseteqq [\mathfrak{X}^{\bar{M}^{-1}}, \mathfrak{W}^{Q^{-j}}]^{Q^j} \subseteqq [\mathfrak{X}^{\bar{M}^{-1}}, \mathfrak{X}]^{Q^j} = 1 . \end{aligned}$$

Case 3. $\mathfrak{U}^X \subseteqq \mathfrak{L}^X$.

Here we have $\mathfrak{W} \subseteqq C(\mathfrak{U}^X) = \mathfrak{K}^X \lhd N(\mathfrak{X}^X)$, so $[\mathfrak{X}^X, \mathfrak{W}] \subseteqq \mathfrak{U}^X$, the containment holding since $\mathfrak{K}$ stablizes $\mathfrak{X} \supset \mathfrak{U} \supset 1$. (And $\mathfrak{K}$ stabilizes $\mathfrak{X} \supset \mathfrak{U} \supset 1$, since $\mathfrak{K}$ stabilizes each of the chains $\mathfrak{W}^{Q^i} \supset \mathfrak{U} \supset 1$.) On the other hand, $\mathfrak{X}^X \subseteqq \mathfrak{K}$ stabilizes $\mathfrak{W} \supset \mathfrak{U} \supset 1$, and so $[\mathfrak{X}^X, \mathfrak{W}] \subseteqq \mathfrak{U}$.

Thus, $1 \neq [\mathfrak{X}^X, \mathfrak{W}] \subseteqq \mathfrak{U}^X \cap \mathfrak{U}$. Choose $A \in \mathfrak{U}^X \cap \mathfrak{U}$, $A \neq 1$. Then $A^{X^{-1}} \in \mathfrak{U}$ so that $A^{X^{-1}} = Z^N$, $N \in \mathfrak{N}$. Also, $A = Z^{N'}$, $N' \in \mathfrak{N}$, so $Z^{N'X^{-1}} = Z^N$, and $N'X^{-1}N^{-1} = M \in \mathfrak{M}$, so that $X^{-1} = N'^{-1}MN$, $X = N^{-1}M^{-1}N'$, whence

$$[\mathfrak{X}^X, \mathfrak{W}] = [\mathfrak{X}^{N^{-1}M^{-1}N'}, \mathfrak{W}] \subseteqq [\mathfrak{X}^{M^{-1}N'}, \mathfrak{X}] \subseteqq [\mathfrak{X}^{M^{-1}}, \mathfrak{X}]^{N'} = 1 ,$$

the desired contradiction.

LEMMA 18.23. $[\mathfrak{X}, \mathfrak{X}^P] = \mathfrak{Z}$.

Proof. It suffices to show that $[\mathfrak{X}, \mathfrak{X}^P] \subseteqq \mathfrak{Z}$. Now $\mathfrak{W} \subset \mathfrak{X}$, and $\mathfrak{X}' = 1$, so $[\mathfrak{W}, \mathfrak{X}] = 1$, and since $\mathfrak{W} = \mathfrak{W}^P$, we also have $[\mathfrak{W}, \mathfrak{X}^P] = 1$. We claim that $[\mathfrak{X}^P, \mathfrak{X}] \subseteqq \mathfrak{W}^{Q^j}$ for all j. Namely, $\mathfrak{X}^P \subseteqq \mathfrak{K} = C(\mathfrak{U})$, and $[\mathfrak{K}, \mathfrak{W}] \subseteqq \mathfrak{U}$, since $\mathfrak{W} \lhd \mathfrak{K}$, and $|\mathfrak{W} : \mathfrak{U}| = 2$. Since $\mathfrak{Q}$ normalizes both $\mathfrak{K}$ and $\mathfrak{U}$, we have $[\mathfrak{K}, \mathfrak{W}^{Q^i}] \subseteqq \mathfrak{U}$ for all i. Since $\mathfrak{X}$ is generated by its subgroups $\mathfrak{W}^{Q^i}$, we conclude that $[\mathfrak{K}, \mathfrak{X}] \subseteqq \mathfrak{U}$. Since $\mathfrak{X}^P \subseteqq \mathfrak{K}$, we get $[\mathfrak{X}^P, \mathfrak{X}] \subseteqq \mathfrak{U} \subseteqq \mathfrak{W}^{Q^j}$ for all j. Since $\mathfrak{W} \cap \mathfrak{W}^Q = \mathfrak{U}$, we have in fact shown that $[\mathfrak{X}^P, \mathfrak{x}] \subseteqq \mathfrak{U}$. Since $\mathfrak{X}^{P^2} \subseteqq \mathfrak{K}$, symmetry gives $[\mathfrak{X}^{P^2}, \mathfrak{X}] \subseteqq \mathfrak{U}$, and conjugation by P gives $[\mathfrak{X}, \mathfrak{X}^P] \subseteqq \mathfrak{U}^P$, whence $[\mathfrak{X}^P, \mathfrak{X}] \subseteqq \mathfrak{U} \cap \mathfrak{U}^P = \mathfrak{Z}$, and we are done.

LEMMA 18.24. *If* $r, s, t \in \{1, -1\}$, *then* $\mathfrak{M}Q^rP^sQ^t\mathfrak{M} = \mathfrak{M}QPQ\mathfrak{M}$.

Proof. Let $\mathfrak{H}_0 = C(\mathfrak{W})$. Then $\mathfrak{H}/\mathfrak{H}_0$ is a four-group which maps isomorphically onto the stability group of the chain $\mathfrak{W} \supset \mathfrak{Z} \supset 1$. Thus, $\mathfrak{P}$ acts non trivially on $\mathfrak{H}/\mathfrak{H}_0$. Now $\mathfrak{H}_0 \subseteq C(\mathfrak{U}) = \mathfrak{K}$, and $\mathfrak{H} \cap \mathfrak{K}/\mathfrak{H}_0$ is of order 2. Hence, there are elements T_1, T_2 in $\mathfrak{H} - \mathfrak{H} \cap \mathfrak{K}$ such that

$$T_1^{P^{-1}} \in \mathfrak{H} \cap \mathfrak{K}\,, \quad T_2^{P^{-1}} \notin \mathfrak{H} \cap \mathfrak{K}\,, \quad T_2^P \in \mathfrak{H} \cap \mathfrak{K}\,.$$

Since $T_1 \notin \mathfrak{K}$, we have $QT_1 = T_1Q^{-1}K_1'$, where $K_1' \in \mathfrak{K}$, whence $Q^rPQ^tT_1 = Q^rPT_1Q^{-t}K_1 = Q^rT_1^{P^{-1}}PQ^{-t}K_1 = K^*Q^rPQ^{-t}K_1$, where $K_1, K^* \in \mathfrak{K}$. So $\mathfrak{T}Q^rPQ^t\mathfrak{T} = TQ^rPQ^{-t}\mathfrak{T}$. Similarly, $T_2Q^rPQ^t = K_2Q^{-r}PQ^tK_3$, where K_2, $K_3 \in \mathfrak{K}$, so that $\mathfrak{T}Q^rPQ^t\mathfrak{T} = \mathfrak{T}Q^{-r}PQ^t\mathfrak{T}$. Since $\mathfrak{T} \subseteq \mathfrak{M}$, it suffices to show that $\mathfrak{M}QPQ\mathfrak{M} = \mathfrak{M}QP^{-1}Q\mathfrak{M}$. Now $\mathfrak{K} = \mathfrak{K}^{Q^{-1}} \neq \mathfrak{H}$, so we can choose $T \in \mathfrak{K}$ such that $QTQ^{-1} \notin \mathfrak{H}$. Set $U = QTQ^{-1}$. Then $QPQT = QPUQ$. Now $PU = HUP^{-1}$, where $H \in \mathfrak{H}$, and so $QPQT = QHUP^{-1}Q$. Now $HU \in \mathfrak{T}$, and so $\mathfrak{T}Q(HU) = \mathfrak{T}Q$ or $\mathfrak{T}Q^{-1}$, according as $HU \in \mathfrak{K}$ or $HU \in \mathfrak{T} - \mathfrak{K}$. Thus, $\mathfrak{T}QPQ\mathfrak{T} = \mathfrak{T}Q^fP^{-1}Q\mathfrak{T}$, $f \in \{1, -1\}$. By the first part of the argument, the lemma follows.

LEMMA 18.25. *For all* $Q_1, Q_2 \in \mathfrak{Q}^\#$, $P_1 \in \mathfrak{P}^\#$, $[\mathfrak{W}, \mathfrak{W}^{Q_1P_1Q_2}] = \mathfrak{Z}^{Q_2}$.

Proof. $[\mathfrak{W}, \mathfrak{W}^{Q_1P_1Q_2}] \subseteq [\mathfrak{X}, \mathfrak{X}^{Q_1P_1Q_2}] = [\mathfrak{X}, \mathfrak{X}^{P_1}]^{Q_2} \subseteq \mathfrak{Z}^{Q_2}$, and so it suffices to show that $[\mathfrak{W}, \mathfrak{W}^{Q_1P_1Q_2}] \neq 1$.

If $[\mathfrak{W}, \mathfrak{W}^{Q_1P_1Q_2}] = 1$, then for all $M, M' \in \mathfrak{M}$, we have $[\mathfrak{W}, \mathfrak{W}^{MQ_1P_1Q_2M'}] = 1$, whence by the preceding lemma, $[\mathfrak{W}, \mathfrak{W}^{Q^iPQ^{-j}}] = 1$, if $i, j \in \{1, -1\}$. Hence, conjugation by Q^j gives $[\mathfrak{W}^{Q^j}, \mathfrak{W}^{Q^iP}] = 1$.

On the other hand, $\mathfrak{X} = \langle \mathfrak{W}, \mathfrak{W}^Q, \mathfrak{W}^{Q^2} \rangle$, $\mathfrak{X}^P = \langle \mathfrak{W}, \mathfrak{W}^{QP}, \mathfrak{W}^{Q^2P} \rangle$. Since $\mathfrak{X}' = 1$, we have $[\mathfrak{W}^{Q^j}, \mathfrak{W}] = 1$ for all j. By the preceding paragraph, we conclude that if $j \in \{1, -1\}$, then $\mathfrak{W}^{Q^j}$ centralizes $\mathfrak{X}^P$. Since $[\mathfrak{X}, \mathfrak{X}^P] \neq 1$, we conclude that $[\mathfrak{W}, \mathfrak{X}^P] \neq 1$. Since $\mathfrak{W}' = 1$, this forces $[\mathfrak{W}, \mathfrak{W}^{Q^iP}] \neq 1$ for some i. Since $\mathfrak{P}$ normalizes $\mathfrak{W}$, we get $[\mathfrak{W}, \mathfrak{W}^{Q^i}] \neq 1$, against $\langle \mathfrak{W}, \mathfrak{W}^{Q^i} \rangle \subseteq \mathfrak{X}$, and $\mathfrak{X}' = 1$. The proof is complete.

We now begin the construction of the final configuration of this section. Set $\mathfrak{E} = [\mathfrak{W}, \mathfrak{P}]$, so that $\mathfrak{W} = \mathfrak{E} \times \mathfrak{Z}$. Since $\mathfrak{U}$ and $\mathfrak{E}$ are distinct hyperplanes of $\mathfrak{W}$, it follows that $\mathfrak{U} \cap \mathfrak{E} = \langle U \rangle$ is of order 2. Thus, there is a unique generator Q of $\mathfrak{Q}$ such that $Z^Q = U$.

Set $\mathfrak{L} = \langle \mathfrak{W}, \mathfrak{W}^Q, \mathfrak{W}^{QP}, \mathfrak{W}^{QP^2} \rangle$. Since $\mathfrak{P}$ normalizes $\mathfrak{W}$, we have $\langle \mathfrak{W}, \mathfrak{W}^{QP^i} \rangle \subseteq \mathfrak{X}^{P^i}$, and so $\mathfrak{W} \subseteq Z(\mathfrak{L})$, so that $\mathfrak{L} \subseteq C(\mathfrak{W}) \subseteq \mathfrak{H}$, $\mathfrak{L}$ admits $\mathfrak{P}$, and if $\mathfrak{H}_0 = C(\mathfrak{W})$, we have $[\mathfrak{W}^Q, \mathfrak{H}_0] \subseteq [\mathfrak{X}, \mathfrak{H}_0] \subseteq [\mathfrak{X}, \mathfrak{K}] \subseteq \mathfrak{U} \subseteq \mathfrak{W}$. Thus, $\mathfrak{H}_0 \subseteq N(\mathfrak{L})$. If $W \in \mathfrak{W} - \mathfrak{U}$, then $\mathfrak{L} = \langle \mathfrak{W}, W^Q, W^{QP}, W^{QP^2} \rangle$, and $[W^Q, W^{QP}], [W^Q, W^{QP^2}], [W^{QP}, W^{QP^2}] \in \mathfrak{Z}$. Hence, $|\mathfrak{L}| \leqq 2^6$, and $\mathfrak{L}' = \mathfrak{Z}$. Thus, $\mathfrak{E}$ is a direct factor of $\mathfrak{L}$, so $\mathfrak{L} = \mathfrak{E} \times \mathfrak{L}_0$, where $\mathfrak{L}_0$ admits $\mathfrak{P}$. If $|\mathfrak{L}_0| = 8$, then $\mathfrak{L}_0 \cong D_8$, and so $\mathfrak{P}$ centralizes $\mathfrak{L}_0$. This is false, since $C(\mathfrak{P})$ contains no four-group. So $|\mathfrak{L}_0| = 16$. Since $\mathfrak{L}_0$ is generated by involutions, we have $C_{\mathfrak{L}_0}(\mathfrak{P}) = \mathfrak{L}^1 \supset \mathfrak{Z}$, so that $\mathfrak{L}^1$ is cyclic of order 4. Let $\mathfrak{L}_1 = [\mathfrak{L}_0, \mathfrak{P}]$, so that $\mathfrak{L}_1' = \mathfrak{Z}$, and $\mathfrak{P}$ acts faithfully on $\mathfrak{L}_1$. Hence, $\mathfrak{L}_1$ is a quaternion group. Also, $\mathfrak{L}_0$ is a central product of $\mathfrak{L}_1$ and $\mathfrak{L}^1$.

Now $[\mathfrak{W}, \mathfrak{W}^{QPQ}] = \mathfrak{Z}^Q \neq \mathfrak{Z}$, and so $\mathfrak{W}^{QPQ} \not\subseteq \mathfrak{H}$, as $\mathfrak{H}$ stabilizes $\mathfrak{W} \supset \mathfrak{Z} \supset 1$. Since $\mathfrak{W}^{QP} \subseteq \mathfrak{L} \subseteq \mathfrak{H} \lhd \mathfrak{T}$, we have $\mathfrak{W}^{QP} \subseteq \mathfrak{K}$, and so $\mathfrak{W}^{QPQ} \subseteq \mathfrak{K}$.

The crucial step is to show that $\mathfrak{W}^{QPQ}$ normalizes $\mathfrak{L}$. Recall that $\mathfrak{W} = \mathfrak{E} \times \mathfrak{Z}$, where $\mathfrak{E} = [\mathfrak{W}, \mathfrak{P}]$. Let $\mathfrak{M}_1 = \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{E})$. The number of conjugates of $\mathfrak{E}$ in $\mathfrak{M}$ is 4, and so $|\mathfrak{M}: \mathfrak{M}_1| = 4$. Also $\mathfrak{H}_0 = \boldsymbol{C}(\mathfrak{W}) \subseteq \mathfrak{M}_1$, and $\mathfrak{P} \subseteq \mathfrak{M}_1$, and so $|\mathfrak{M}_1: \mathfrak{P}\mathfrak{H}_0| = 2$.

Now $[\mathfrak{E}, \mathfrak{W}^{QPQ}] \subseteq [\mathfrak{W}, \mathfrak{W}^{QPQ}] = \mathfrak{Z}^Q \subseteq \mathfrak{E}$, by our choice of Q, and so $\mathfrak{W}^{QPQ} \subseteq \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{E})$. Since $\mathfrak{W}^{QPQ} \not\subseteq \mathfrak{H}$, we conclude that $\mathfrak{M}_1 = \mathfrak{H}_0\mathfrak{W}^{QPQ}\mathfrak{P}$. Since $\mathfrak{W}^{QPQ} \subseteq \mathfrak{K}$, it follows that $\mathfrak{W}^{QPQ}$ normalizes both $\mathfrak{W}$ and $\mathfrak{W}^Q$. Thus, $\langle \mathfrak{W}, \mathfrak{W}^Q \rangle^{\mathfrak{M}_1} = \langle \mathfrak{W}, \mathfrak{W}^Q \rangle^{\mathfrak{H}_0\mathfrak{P}} = \langle \mathfrak{W}, \mathfrak{W}^Q \rangle^{\mathfrak{P}} = \mathfrak{L}$. So $\mathfrak{L} \lhd \mathfrak{M}_1$, and $\mathfrak{W}^{QPQ}$ normalizes $\mathfrak{L}$.

Choose $I \in \mathfrak{W}^{QPQ} - \mathfrak{U}^{QPQ}$. Now $\mathfrak{U}^{QPQ} = \mathfrak{U}^{PQ} \subseteq \mathfrak{W}^Q \subseteq \mathfrak{H}$, and so $\mathfrak{U}^{QPQ} = \mathfrak{W}^{QPQ} \cap \mathfrak{H}$, and $I \in \mathfrak{T} - \mathfrak{H}$. Since $\mathfrak{P} \subseteq \boldsymbol{N}(\boldsymbol{Z}(\mathfrak{L}))$, it follows that $\boldsymbol{O}_2(\boldsymbol{N}(\boldsymbol{Z}(\mathfrak{L}))) = \mathfrak{H} \cap \boldsymbol{N}(\boldsymbol{Z}(\mathfrak{L}))$. Hence, I inverts some S_3-subgroup $\mathfrak{P}^*$ of $\boldsymbol{N}(\boldsymbol{Z}(\mathfrak{L}))$. Set $\mathfrak{D} = \langle \mathfrak{P}^*, I \rangle$. Thus, $\mathfrak{D}$ acts on $\boldsymbol{Z}(\mathfrak{L})$, and $\boldsymbol{Z}(\mathfrak{L})$ is of type (2, 2, 4). Since $C_{\mathfrak{L}}(\mathfrak{P}) = \mathfrak{L}^1$ is cyclic of order 4, it follows that $C_{\mathfrak{L}}(\mathfrak{P}^*) = \langle L^* \rangle$, where L^* is of order 4. Also, as I inverts $\mathfrak{P}^*$, I normalizes $\langle L^* \rangle$. Since $\boldsymbol{N}(\mathfrak{P})$ has no non cyclic abelian subgroup of order 8, it follows that I inverts L^*. Since $L^{*2} = Z$, we have $[I, L^*] = Z$. On the other hand, $\mathfrak{U}^{QPQ} \subseteq \mathfrak{W}^Q \subseteq \mathfrak{L}$, so $\boldsymbol{Z}(\mathfrak{L}) \subseteq \boldsymbol{C}(\mathfrak{U}^{QPQ}) = \mathfrak{K}^{QPQ}$. Since $\mathfrak{K}$ stabilizes $\mathfrak{W} \supset \mathfrak{U} \supset 1$, we conclude that $[\boldsymbol{Z}(\mathfrak{L}), \mathfrak{W}^{QPQ}] \subseteq \mathfrak{U}^{QPQ}$. Hence, $Z \in \mathfrak{U}^{QPQ} = \mathfrak{U}^{PQ}$. Also, of course, $Z^{PQ} = Z^Q \in \mathfrak{U}^{PQ}$, and so $\mathfrak{U}^{PQ} = \langle Z, Z^Q \rangle = \mathfrak{U}$. Hence, $\mathfrak{U} = \mathfrak{U}^P$, which is false. This completes a proof that

(18.1) $\mathscr{M}(\mathfrak{T})$ contains an element of order $> |\mathfrak{T}|\cdot 3$.

19. Another exceptional case.

HYPOTHESIS 19.1. If $\mathfrak{L}$ is a solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{T}$ properly, then $f(\mathfrak{L}) \leq 1$.

All results of this section are proved under Hypothesis 19.1.

LEMMA 19.1. (a) *If $\mathfrak{L}$ is a 2-local subgroup of G, then $|\mathfrak{L}|_{2'}$ divides* 15.

(b) *There is precisely one element of $\mathscr{M}(\mathfrak{T})$ of order divisible by* 5.

Proof. Lemmas 5.53, 5.54, and Hypothesis 19.1 imply that (a) holds. By (a) and the results of § 18, there is at least one element of $\mathscr{M}(\mathfrak{T})$ of order divisible by 5.

Suppose $\mathfrak{P}$ is a subgroup of $\mathfrak{G}$ of order 5 and $\mathfrak{L} = \mathfrak{T}\mathfrak{P}$ is a group.

By Lemma 5.53 and its proof, we have

$$\mathfrak{L} = (\mathfrak{L} \cap C(Z(\mathfrak{T})))\cdot(\mathfrak{L} \cap N(J(\mathfrak{T}))) = (\mathfrak{L} \cap C(Z(J_1(\mathfrak{T}))))\cdot(\mathfrak{L} \cap N(J(\mathfrak{T}))) .$$

Thus, if $J(\mathfrak{T}) \ntriangleleft \mathfrak{L}$, then $f(\mathfrak{L}) \geqq 2$. By Hypothesis 19.1, we conclude that $J(\mathfrak{T}) \lhd \mathfrak{L}$. So $M(N(J(\mathfrak{T})))$ is the unique element of $\mathscr{M}(\mathfrak{T})$ of order divisible by 5.

From now on, let $\mathfrak{M}$ be the unique element of $\mathscr{M}(\mathfrak{T})$ of order divisible by 5, and let $\mathfrak{H} = O_2(\mathfrak{M})$. Let $\mathfrak{D}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$, so that $|\mathfrak{D}| = 5$ or 15. Let $\mathfrak{E} = \Omega_1(R_2(\mathfrak{M}))$. Let $\mathfrak{P}$ be the subgroup of $\mathfrak{D}$ of order 5, and let $\mathfrak{E}_0 = [\mathfrak{E}, \mathfrak{P}]$.

LEMMA 19.2. (a) $\mathfrak{P} \subseteq N(Z(J(\mathfrak{T})))$.
(b) $\mathfrak{P} \nsubseteq N(Z(\mathfrak{T}))$, $\mathfrak{P} \nsubseteq N(Z(J_1(\mathfrak{T})))$.
(c) $|\mathfrak{E}_0| = 2^4$.

Proof. (a) and (b) follow from the proof of Lemma 19.1 (b). Since (b) holds, we have $\mathfrak{E}_0 \neq 1$. Since $\mathfrak{H}\mathfrak{P} \lhd \mathfrak{M}$, (b) also implies that $J_1(\mathfrak{T}) \nsubseteq O_2(\mathfrak{T}\mathfrak{P})$, which then implies that (c) holds.

LEMMA 19.3. *If $\mathfrak{N} \in \mathscr{M}(\mathfrak{T})$, and $\mathfrak{N} \neq \mathfrak{M}$, then $\mathfrak{N}$ has an element of order* 6.

Proof. Let $\mathfrak{Q}$ be a $S_{2'}$-subgroup of $\mathfrak{N}$. Then $|\mathfrak{Q}| = 3$, by Lemma 19.1. Let $\mathfrak{K} = O_2(\mathfrak{N})$. Suppose by way of contradiction that $C_{\mathfrak{N}}(\mathfrak{Q}) = \mathfrak{Q}$.

Since $N(J(\mathfrak{T})) \subseteq \mathfrak{M}$, we have $J(\mathfrak{T}) \nsubseteq \mathfrak{K}$. Hence, $Z(\mathfrak{K})$ is a four-group, and so $\mathfrak{K}$ is special. Hence, $\mathfrak{T}/\mathfrak{T}'$ is elementary abelian. This implies that $C_{\mathfrak{T}}(\mathfrak{E}_0)$ has index 2 in $\mathfrak{T}$. Hence, $\mathfrak{E}_0 \subseteq \mathfrak{K}$. But $\mathfrak{K}$ is special, $Z(\mathfrak{K})$ is a four-group, and $\mathfrak{Q}$ acts without fixed points on $\mathfrak{K}$, so that $|\mathfrak{K}: C_{\mathfrak{K}}(U)| = 4$ for every non central involution U of $\mathfrak{K}$. This contradiction completes the proof.

LEMMA 19.4. *One of the following holds*:
(a) $\mathfrak{P} = \mathfrak{D}$ *is of order* 5.
(b) $\mathscr{M}(\mathfrak{T}) = \{\mathfrak{M}, \mathfrak{N}\}$, *where* $\mathfrak{N} = C(Z(\mathfrak{T}))$, $Z(\mathfrak{T})$ *is cyclic, and* $C(\mathfrak{B})$ *is a 2-group for every four-subgroup* $\mathfrak{B}$ *of* $\mathfrak{G}$.

Proof. Suppose $\mathfrak{P} \subset \mathfrak{D}$, so that $\mathfrak{D} = \mathfrak{P} \times \mathfrak{Q}$, where $|\mathfrak{Q}| = 3$. We argue that $\mathfrak{M}$ has an element of order 6. Suppose false. Then $\mathfrak{M}/\mathfrak{H}$ is a dihedral group of order 30 which acts faithfully on $\mathfrak{E}_0$. This is false, since elements of GL(4, 2) of order 15 are not real. So $\mathfrak{M}$ has an element of order 6.

Choose $\mathfrak{N} \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{N} \neq \mathfrak{M}$, so that $\mathfrak{N} = \mathfrak{T}\mathfrak{R}$, $|\mathfrak{R}| = 3$. Let $\mathfrak{K} = O_2(\mathfrak{N})$, and let $\mathfrak{H}_0 = O_2(\mathfrak{T}\mathfrak{Q})$. Thus, $\mathfrak{H}_0 \in N^*(\mathfrak{Q}; 2)$, $\mathfrak{K} \in N^*(\mathfrak{R}; 2)$, $C_{\mathfrak{H}_0}(\mathfrak{Q}) \neq 1$, $C_{\mathfrak{K}}(\mathfrak{R}) \neq 1$. Since $\mathfrak{H}_0$ and $\mathfrak{K}$ are not $\mathfrak{G}$-conjugate, we conclude that

S_3-subgroups of $\mathfrak{G}$ are not cyclic. Hence, $C(\mathfrak{V})$ is a 2-group for every four-subgroup $\mathfrak{V}$ of $\mathfrak{G}$. In particular, if $|\mathfrak{E}_0 : \mathfrak{E}_1| = 2$, then $C(\mathfrak{E}_1)$ is a 2-group. Since $C_{\mathfrak{M}}(\mathfrak{E}_1) = C(\mathfrak{E}_0) = \mathfrak{H}$, and $N(\mathfrak{H}) = \mathfrak{M}$, we conclude that $C(\mathfrak{E}_1) = C_{\mathfrak{M}}(\mathfrak{E}) = \mathfrak{H}$.

Let $\mathfrak{W} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{E}_0); \mathfrak{T})$. Since $C(\mathfrak{E}_0) = C(\mathfrak{E}_1)$ for every hyperplane $\mathfrak{E}_1$ of $\mathfrak{E}_0$, we get $\mathfrak{W} \lhd \mathfrak{M}$. Hence, $\mathfrak{W} \ntriangleleft \mathfrak{N}$, and so $\mathfrak{N}$ centralizes $R_2(\mathfrak{N})$, whence $\mathfrak{N} = C(Z(\mathfrak{T}))$. Since $C(\mathfrak{V})$ is a 2-group for every four-subgroup $\mathfrak{V}$ of $\mathfrak{G}$, $Z(\mathfrak{T})$ is cyclic. The proof is complete.

LEMMA 19.5. $\mathfrak{P} = \mathfrak{D}$.

Proof. Suppose false. Then $\mathfrak{D} = \mathfrak{P} \times \mathfrak{Q}$, where $|\mathfrak{Q}| = 3$, and $\mathfrak{D}$ acts faithfully on $\mathfrak{E}_0$. Hence, $\mathfrak{H} = C(\mathfrak{E}_0)$, and since $Z(\mathfrak{T})$ is cyclic, $|\mathfrak{T}/\mathfrak{H}| > 2$. Since element of $\mathrm{GL}(4,2)$ of order 15 are not real, it follows that $\mathfrak{T}/\mathfrak{H}$ is cyclic of order 4. Thus, there is $T \in \mathfrak{T}$ such that

$$P^T = P^2\,, \quad Q^T = Q^{-1}\,, \quad \mathfrak{P} = \langle P \rangle\,, \quad \mathfrak{Q} = \langle Q \rangle\,.$$

Let $\mathfrak{N} = C(Z(\mathfrak{T})) = \mathfrak{T}\mathfrak{R}$, $\mathfrak{R} = \langle R \rangle$, $R^3 = 1$. Let $\mathfrak{U} \in \mathscr{U}(\mathfrak{T})$, $\mathfrak{U} \subset \mathfrak{E}_0$, and let $\mathfrak{W} = \mathfrak{U}^N = \mathfrak{U}^{\mathfrak{R}} = \langle \mathfrak{U}, \mathfrak{U}^R, \mathfrak{U}^{R^2} \rangle$. Also, let

$$\mathfrak{K} = O_2(\mathfrak{N})\,, \quad \mathfrak{V} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{E}_0); \mathfrak{T})\,.$$

We argue that $\mathfrak{W}$ is elementary of order 2^3. In any case, since $\mathfrak{Z} = \Omega_1(Z(\mathfrak{T})) \cap \mathfrak{U} \subseteq C(\mathfrak{R})$, and since $C(\mathfrak{U})$ is a 2-group, we have $\mathfrak{U} \subset \mathfrak{W}$, $|\mathfrak{W}| \leqq 2^4$. If $\mathfrak{W}$ is abelian, then since $C_{\mathfrak{W}}(\mathfrak{R})$ contains no four-group, we have $|\mathfrak{W}| = 2^3$. Suppose $\mathfrak{W}' \neq 1$. Thus, $[\mathfrak{U}, \mathfrak{U}^R] \neq 1$. Since $[\mathfrak{K}, \mathfrak{U}] = \mathfrak{Z}$, so also $[\mathfrak{K}, \mathfrak{U}^R] = \mathfrak{Z}$. Choose $U \in \mathfrak{U}^R - \mathfrak{Z}$. Then $|\mathfrak{K} : C_{\mathfrak{K}}(U)| = 2$, so $|\mathfrak{T} : C_{\mathfrak{T}}(U)| \leqq 2^2$, as $|\mathfrak{T} : \mathfrak{K}| = 2$. But $U^2 = 1$, and so $U \in \langle \mathfrak{H}, T^2 \rangle$, whence $U \in C(\mathfrak{U})$. This contradiction shows that $\mathfrak{W}' = 1$, $|\mathfrak{W}| = 2^3$.

The crucial step is to show that $C(\mathfrak{F}) \subseteq \mathfrak{M}$ for every 4-subgroup $\mathfrak{F}$ of $\mathfrak{E}_0$. In any case, $C(\mathfrak{F}) \supseteq \mathfrak{H}$, $C(\mathfrak{F})$ is a 2-group, and $C(\mathfrak{F})$ is not a S_2-subgroup of $\mathfrak{G}$, as $Z(\mathfrak{T})$ is cyclic. So $\mathfrak{H} \lhd C(\mathfrak{F})$. Since $N(\mathfrak{H}) = \mathfrak{M}$, we have $C(\mathfrak{F}) \subseteq \mathfrak{M}$.

Since $C(\mathfrak{E}_1) = C(\mathfrak{E}_0)$ for every hyperplane $\mathfrak{E}_1$ of $\mathfrak{E}_0$, we have $\mathfrak{V} \lhd \mathfrak{M}$. Hence, $\mathfrak{V} \ntriangleleft \mathfrak{N}$, so there is G in $\mathfrak{G}$ such that $\mathfrak{E}_0^G \subseteq \mathfrak{E}^* \subseteq \mathfrak{T}$, $\mathfrak{E}^* \nsubseteq \mathfrak{K}$.

Let $\mathfrak{K}_0 = C(\mathfrak{W})$. Thus, $\mathfrak{K}/\mathfrak{K}_0$ is a four-group, and $\mathfrak{N}/\mathfrak{K}_0 \cong \Sigma_4$, since $\mathfrak{K}/\mathfrak{K}_0$ maps isomorphically onto the stability group of the chain $\mathfrak{W} \supset \mathfrak{Z} \supset 1$. Let $\mathfrak{K}_1/\mathfrak{K}_0 = Z(\mathfrak{T}/\mathfrak{K}_0)$. Since $\mathfrak{T} = \mathfrak{K} \cdot \mathfrak{E}^*$, it follows that $\mathfrak{E}^* \cap \mathfrak{K} \subseteq \mathfrak{K}_1$. Hence, $\mathfrak{F}^* = \mathfrak{E}^* \cap \mathfrak{K}_0$ contains a four-group. Since $C(\mathfrak{F}^*) \supseteq \mathfrak{W}$, and $C(\mathfrak{E}^*) \nsupseteq \mathfrak{W}$, $\mathfrak{F}^*$ is a four-group. Let $\mathfrak{L}^* = \mathfrak{E}^* \cap \mathfrak{K}$, so that $\mathfrak{E}^* \supset \mathfrak{L}^* \supset \mathfrak{F}^*$. Hence, $[\mathfrak{W}, \mathfrak{L}^*] = \mathfrak{Z}$, and $[\mathfrak{W}, \mathfrak{E}^*] = \mathfrak{W}^*$ is a four-group. Since $\mathfrak{W} \subseteq C(\mathfrak{F}^*) \subseteq \mathfrak{M}^G$, we get $\mathfrak{W}^* \subseteq \mathfrak{E}^*$, and so $\mathfrak{W}^* = \mathfrak{F}^*$. Hence $\mathfrak{K}_0 \subseteq C(\mathfrak{F}^*) \subseteq \mathfrak{M}^G$, and so $[\mathfrak{K}_0, \mathfrak{E}^*] = \mathfrak{F}^*$. This implies that $\mathfrak{E}^*$ centralizes $\mathfrak{K}_0/\mathfrak{W}$, and so $[\mathfrak{R}, \mathfrak{K}_0] \subseteq \mathfrak{W}$, whence $[\mathfrak{R}, \mathfrak{K}_0] = [\mathfrak{R}, \mathfrak{W}] = \mathfrak{W}_1$ is a four-group, and so $\mathfrak{K}_0 = \mathfrak{K}^0 \times \mathfrak{W}_1$, where $\mathfrak{K}^0 = C_{\mathfrak{K}_0}(\mathfrak{R})$. Since $\mathfrak{K}^0$ contains no four-group, it is either cyclic or generalized quaternion. We assume without loss of generality that $E^* \in \mathfrak{E}^* - \mathfrak{L}^*$, and that E^* inverts $\mathfrak{R}$. Since $[\mathfrak{K}^0, E^*] \subseteq \mathfrak{K}^0 \cap \mathfrak{F}^*$, we get $[\mathfrak{K}^0, E^*] \subseteq \mathfrak{Z}$. Since $N(\mathfrak{R})$ contains no non cyclic abelian subgroup of order 8, it follows that $|\mathfrak{K}^0| \leqq 2^2$. Hence, $|\mathfrak{T}| = 2 \cdot |\mathfrak{K}| = 2^3 \cdot |\mathfrak{K}_0| = 2^5 |\mathfrak{K}^0| \leqq 2^7$, and $|\mathfrak{H}| \leqq 2^5$. Thus, $\mathfrak{Q}$ centralizes $\mathfrak{H}/\mathfrak{E}_0$, and so $\mathfrak{H} = \mathfrak{E}_0 \times C_{\mathfrak{H}}(\mathfrak{Q})$. Since $Z(\mathfrak{T})$ is cyclic, we have $\mathfrak{H} = \mathfrak{E}_0$, $|\mathfrak{T}| = 2^6$. Hence, $\mathfrak{Z} = \mathfrak{K}^0$ and $\mathfrak{T}/\mathfrak{T}'$ is elementary abelian. This is false, since $\mathfrak{T}/\mathfrak{H} \cong Z_4$. The proof is complete.

LEMMA 19.6. *If $\mathfrak{F}$ is any hyperplane of $\mathfrak{E}_0$, there is $P \in \mathfrak{P}^\#$ such that $C_{\mathfrak{M}}(\mathfrak{F} \cap \mathfrak{F}^P) = \mathfrak{H}$.*

Proof. Let $\mathscr{I}$ be the set of involutions of $\mathfrak{M}/\mathfrak{H}$. Thus, $|\mathscr{I}| = 5$, and if $I \in \mathscr{I}$, then $C_{\mathfrak{E}_0}(I)$ is a four-group. Since $C_{\mathfrak{E}_0}(\mathfrak{P}) = 1$, it follows that $C_{\mathfrak{E}_0}(I) \cap C_{\mathfrak{E}_0}(J) = 1$ if I, J are distinct elements of $\mathscr{I}$.

Let $\mathfrak{P} = \langle P \rangle$, and set $\mathfrak{F}_i = \mathfrak{F} \cap \mathfrak{F}^{P^i}$, $i = 1, 2, 3, 4$. Then the $\mathfrak{F}_i$ are four-subgroups of $\mathfrak{F}$ and so $\mathfrak{F}_i \cap \mathfrak{F}_j \neq 1$, $1 \leqq i, j \leqq 4$. Thus, if $\mathfrak{F}_i = C_{\mathfrak{E}_0}(I_i)$ where $I_i \in \mathscr{I}$, then $\mathfrak{F}_1 = \mathfrak{F}_2 = \cdots \mathfrak{F}_4$. But then

$$\bigcap_{j=0} \mathfrak{F}^{P^j}$$

is a four-subgroup of $\mathfrak{E}_0$ which admits $\mathfrak{P}$. This is false, since $C_{\mathfrak{E}_0}(\mathfrak{P}) = 1$. The proof is complete.

LEMMA 19.7. $C(\mathfrak{F}) = C(\mathfrak{E}_0) = \mathfrak{H}$ *for every hyperplane $\mathfrak{F}$ of $\mathfrak{E}_0$.*

Proof. Since $C_{\mathfrak{M}}(\mathfrak{F}) = \mathfrak{H}$ and $N(\mathfrak{H}) = \mathfrak{M}$, it follows that $\mathfrak{H}$ is a S_2-subgroup of $\mathfrak{C} = C(\mathfrak{F})$. Suppose $\mathfrak{C} \supset \mathfrak{H}$. Since $\mathfrak{M} = N(\mathfrak{X})$ for every non identity characteristic subgroup of $\mathfrak{H}$, it follows that $\mathfrak{C} = \mathfrak{H}\mathfrak{S}$, where $|\mathfrak{S}| = 3$. Since $\mathfrak{F}$ contains a four-group, S_3-subgroups of $\mathfrak{G}$ are of order 3.

Choose $P \in \mathfrak{P}^\#$ such that $C_{\mathfrak{M}}(\mathfrak{F} \cap \mathfrak{F}^P) = \mathfrak{H}$. It follows that $C(\mathfrak{F}) = C(\mathfrak{F}^P) = C(\mathfrak{F} \cap \mathfrak{F}^P)$. Hence, $P \in N(\mathfrak{C})$. Let $\mathfrak{L} = \mathfrak{C}\mathfrak{P}$, $\mathfrak{L}_0 = O_2(\mathfrak{L})$. Then $[\mathfrak{L}_0, \mathfrak{P}] = [\mathfrak{H}, \mathfrak{P}]$, and $[\mathfrak{H}, \mathfrak{P}] \lhd \mathfrak{M}$, $[\mathfrak{L}_0, \mathfrak{P}] \lhd \mathfrak{L}$. Hence, $\mathfrak{L} \subseteq \mathfrak{M}$, the desired contradiction.

LEMMA 19.8. $\mathscr{M}(\mathfrak{T}) = \{\mathfrak{M}, \mathfrak{N}\}$, *where* $\mathfrak{N} = C(Z(\mathfrak{T}))$.

Proof. Let $\mathfrak{B} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{E}_0); \mathfrak{T})$. Since $C(\mathfrak{F}) = C(\mathfrak{E}_0)$ for every hyperplane $\mathfrak{F}$ of $\mathfrak{E}_0$, we have $\mathfrak{B} \lhd \mathfrak{M}$.

Choose $\mathfrak{N} \in \mathscr{M}(\mathfrak{T})$, $\mathfrak{N} \neq \mathfrak{M}$. Since $\mathfrak{B} \lhd \mathfrak{M}$, we have $\mathfrak{B} \ntriangleleft \mathfrak{N}$. Also, $|\mathfrak{N}| = 3\,|\mathfrak{T}|$. Choose $G \in \mathfrak{G}$ such that $\mathfrak{E}_0^G \subseteq \mathfrak{T}$, $\mathfrak{E}_0^G \nsubseteq \mathfrak{K} = O_2(\mathfrak{N})$. Since $|\mathfrak{T} : \mathfrak{K}| = 2$, it follows that $Z(\mathfrak{K}) \subseteq C(\mathfrak{K} \cap \mathfrak{E}_0^G) = C(\mathfrak{E}^G)$. So $Z(\mathfrak{K}) = Z(\mathfrak{N})$. The proof is complete.

LEMMA 19.9. $\boldsymbol{O}_2(\mathfrak{N}) \not\supseteq \boldsymbol{O}_2(\mathfrak{M})$.

Proof. Set $\mathfrak{K} = \boldsymbol{O}_2(\mathfrak{N})$, $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{M})$. Since $|\mathfrak{T}: \mathfrak{K}| = 2$, and since $\mathfrak{K} \neq \mathfrak{H}$, we are done if $|\mathfrak{T}: \mathfrak{H}| = 2$. Suppose $|\mathfrak{T}: \mathfrak{H}| > 2$, so that $\mathfrak{T}/\mathfrak{H} \cong Z_4$. Suppose by way of contradiction that $\mathfrak{K} \supset \mathfrak{H}$. Since $\boldsymbol{J}(\mathfrak{T}) \lhd \mathfrak{M}$, we have $\boldsymbol{J}(\mathfrak{T}) = \boldsymbol{J}(\mathfrak{H}) = \boldsymbol{J}(\mathfrak{K}) \lhd \langle \mathfrak{M}, \mathfrak{N} \rangle$, which is false. The proof is complete.

LEMMA 19.10. *If* $\mathfrak{T} \supseteq \mathfrak{L} \supseteq \mathfrak{H}$, *then* $\boldsymbol{N}(\mathfrak{L}) \subseteq \mathfrak{M}$.

Proof. If $\mathfrak{L} = \mathfrak{T}$ or $\mathfrak{L} = \mathfrak{H}$, the lemma clearly holds. Suppose $\mathfrak{T} \supset \mathfrak{L} \supset \mathfrak{H}$. Then $\mathfrak{L} \neq \mathfrak{K}$, by Lemma 19.10, and $\mathfrak{T} \subseteq \boldsymbol{N}(\mathfrak{L})$. This lemma now follows from Lemma 19.8.

LEMMA 19.11. *If* $I \in \mathfrak{E}_0^\#$, *and* $\mathfrak{T}_0$ *is a 2-subgroup of* $\boldsymbol{C}(I)$, *then* $\mathfrak{T}_0 \cap \mathfrak{M}$ *is of index at most 2 in* $\mathfrak{T}_0$.

Proof. Let $\mathfrak{T}_1$ be a S_2-subgroup of $\boldsymbol{C}_{\mathfrak{M}}(I)$. By Lemma 19.10, $\mathfrak{T}_1$ is a S_2-subgroup of $\boldsymbol{C}(I)$. If $\boldsymbol{C}(I) \subseteq \mathfrak{M}$, the lemma obviously holds. Suppose $\boldsymbol{C}(I) \not\subseteq \mathfrak{M}$. Thus, $\mathfrak{C} = \boldsymbol{C}(I)$ has order $|\mathfrak{T}_1| \cdot d$, where $d = 3, 5$ or 15.

Case 1. $d = 3$.

Since $|\mathfrak{T}_1: \boldsymbol{O}_2(\mathfrak{C})| \leqq 2$, and $\boldsymbol{O}_2(\mathfrak{C}) \subseteq \mathfrak{M}$, we get $|\mathfrak{T}_0: \mathfrak{T}_0 \cap \boldsymbol{O}_2(\mathfrak{C})| \leqq 2$, and the lemma follows.

Case 2. $d = 5$ or 15.

Let $\mathfrak{P}_1$ be a subgroup of $\mathfrak{C}$ of order 5, and let $\mathfrak{L}_1 = \mathfrak{T}_1\mathfrak{P}_1$. Let $\mathfrak{L}$ be a maximal 2,5-subgroup of $\mathfrak{G}$ containing $\mathfrak{L}_1$. Thus, $\mathfrak{L}$ and $\mathfrak{M}$ are $\mathfrak{G}$-conjugate. By Lemma 19.10, it follows that $\mathfrak{L} \cap \mathfrak{M}$ contains a S_2-subgroup of $\mathfrak{L}$ and of $\mathfrak{M}$. Since $\mathfrak{T} = \boldsymbol{N}(\mathfrak{T})$, we get that $\mathfrak{L} = \mathfrak{M}$.

Since $\mathfrak{L}_1 \subseteq \mathfrak{L} = \mathfrak{M}$, and since $\mathfrak{C} \not\subseteq \mathfrak{M}$, it follows that $|\mathfrak{C}: \mathfrak{L}_1| = 3$. Now $\mathfrak{T}_1$ is permutable with $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{P}_1)$, and $\mathfrak{M} = \mathfrak{T}_1 \cdot \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{P}_1)$, since $\mathfrak{T}_1 \supseteq \mathfrak{H}$. Also, of course, $\mathfrak{T}_1$ is permutable with $\boldsymbol{N}_{\mathfrak{C}}(\mathfrak{P}_1)$, and $\mathfrak{C} = \mathfrak{T}_1 \cdot \boldsymbol{N}_{\mathfrak{C}}(\mathfrak{P}_1)$. Let $\mathfrak{K} = \langle \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{P}_1), \boldsymbol{N}_{\mathfrak{C}}(\mathfrak{P}_1) \rangle$, so that $\mathfrak{K}$ is a solvable subgroup of $\mathfrak{G}$ permutable with $\mathfrak{T}_1$. Hence, $\mathfrak{K}^* = \mathfrak{K}\mathfrak{T}_1$ is a group. By a standard argument, $\mathfrak{K}^*$ is also solvable, and so $\mathfrak{K}^* = \mathfrak{M} \supseteq \mathfrak{C}$. The proof is complete.

It is now easy to show that Hypothesis 19.1 is not satisfied. We preserve the previous notation. Choose G in $\mathfrak{G}$ such that $\mathfrak{E}_0^G \subseteq \mathfrak{N}$, $\mathfrak{E}_0^G \not\subseteq \mathfrak{K} = \boldsymbol{O}_2(\mathfrak{N})$. Let $\mathfrak{F}_0^G = \mathfrak{E}_0^G \cap \mathfrak{K}$, a hyperplane of $\mathfrak{E}_0^G$. Let $\mathfrak{N} = \mathfrak{T}\mathfrak{Q}$, $\mathfrak{Q} = \langle Q \rangle$, $Q^3 = 1$, and let $\mathfrak{F} = \mathfrak{F}_0^{GQ}$. Thus, $\mathfrak{F} \subset \mathfrak{K} \subset \mathfrak{T}$, and since $\mathfrak{N} = \langle \mathfrak{T}, \mathfrak{E}_0^{GQ} \rangle$, it follows that

$$\mathfrak{M} \cap \mathfrak{E}_0^{GQ} = \mathfrak{F} \,. \tag{19.1}$$

Let $\mathfrak{F}_1 = \mathfrak{F} \cap \mathfrak{H}$. Since $\mathfrak{T}/\mathfrak{H}$ is cyclic, $|\mathfrak{F}:\mathfrak{F}_1| \leqq 2$. If $\mathfrak{F} = \mathfrak{F}_1$, then $\mathfrak{E}_0 \subseteqq \boldsymbol{C}(\mathfrak{F}) = \boldsymbol{C}(\mathfrak{E}_0^{GQ})$, so that $\mathfrak{E}_0^{GQ} \subseteqq \boldsymbol{C}(\mathfrak{E}_0) \subseteqq \mathfrak{M}$, against (19.1). So $|\mathfrak{F}:\mathfrak{F}_1| = 2$.

Let $\mathfrak{F}_1 \times \langle F \rangle = \mathfrak{F}$. Thus, $\mathfrak{T}_0 = \mathfrak{E}_0\mathfrak{F}$ is a 2-subgroup of $\boldsymbol{C}(\mathfrak{F}_1)$. By Lemma 19.11, $\mathfrak{E}_0$ has a subgroup $\mathfrak{E}_1$ of order 8 such that $\mathfrak{E}_1 \subseteqq \boldsymbol{N}(\mathfrak{E}_0^{GQ})$. Hence, $[\mathfrak{F}, \mathfrak{E}_1] \subseteqq \mathfrak{E}_0^{GQ} \cap \mathfrak{E}_0 \subseteqq \mathfrak{F}$. Since $F \notin \mathfrak{H}$, there is an involution I in $[F, \mathfrak{E}_1]$. By Lemma 19.11 applied to I, $\mathfrak{H}$ has a subgroup $\mathfrak{H}_0$ of index 2 which normalizes $\mathfrak{E}_0^G$. Hence, $[\mathfrak{H}_0, \mathfrak{F}] \subseteqq \mathfrak{H} \cap \mathfrak{F} \subseteqq \mathfrak{F}_1$, and so $|[\mathfrak{H}, \mathfrak{F}]| \leqq 2^3$. This implies that $[\mathfrak{P}, \mathfrak{H}] = \mathfrak{E}_0$, so $\mathfrak{H} = \mathfrak{H}^0 \times \mathfrak{E}_0$, where $\mathfrak{H}^0 = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{P})$. Since $\mathfrak{N} = \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{T}))$, and $\mathfrak{H}^0 \lhd \mathfrak{M}$, we conclude that $\mathfrak{H}^0 = 1$. So $|\mathfrak{T}| = 2^4 |\mathfrak{T}:\mathfrak{H}|$.

If $|\mathfrak{T}/\mathfrak{H}| = 2$, then $\mathfrak{T}/\boldsymbol{Z}(\mathfrak{T})$ is abelian and so $\boldsymbol{C}(\boldsymbol{Z}(\mathfrak{T})) = \mathfrak{N}$ is 2-closed. This is false, and so $\mathfrak{T}/\mathfrak{H} \cong Z_4$.

Let $\mathfrak{E}^0 = [\mathfrak{E}_0, \mathfrak{T}]$, $\mathfrak{T}_1 = \boldsymbol{N}_{\mathfrak{T}}(\mathfrak{P})$. Thus, $\mathfrak{T}_1 \cong \mathfrak{T}/\mathfrak{H}$, $\mathfrak{T} = \mathfrak{E}_0\mathfrak{T}_1$, and $\boldsymbol{D}(T) = \langle \mathfrak{E}^0, T^2 \rangle$, where $\mathfrak{T}_1 = \langle T \rangle$. Since $|\mathfrak{T}:\mathfrak{K}| = 2$, we have $\mathfrak{K} \supset \boldsymbol{D}(\mathfrak{T})$, $|\mathfrak{K}:\boldsymbol{D}(\mathfrak{T})| = 2$. Hence, $\mathfrak{K}$ is either $\langle \mathfrak{E}_0, T^2 \rangle$, $\langle \mathfrak{E}^0, T \rangle$, or $\langle \mathfrak{E}^0, T^2, TE \rangle$ for some $E \in \mathfrak{E}_0 - \mathfrak{E}^0$. Since $\mathfrak{E}_0 = \boldsymbol{J}(\mathfrak{T}) \ntriangleleft \mathfrak{N}$, the first possibility is excluded. Each of the remaining two possibilities has the property that the commutator quotient group is of type $(2, 4)$, so these two groups do not have automorphisms of order 3. This contradiction shows that

(19.2) $\mathfrak{G}$ contains a solvable subgroup $\mathfrak{L}$ which contains $\mathfrak{T}$ properly, and satisfies $f(\mathfrak{L}) \geqq 2$.

20. **The construction of ${}^2F_4(2)'$.** From § 19, there is a solvable subgroup $\mathfrak{L}$ of $\mathfrak{G}$ which contains $\mathfrak{T}$ properly and satisfies $f(\mathfrak{L}) \geqq 2$. Set $\mathfrak{M} = \boldsymbol{M}(\mathfrak{L})$.

LEMMA 20.1. *If* $\mathfrak{N} \in \mathscr{M}(\mathfrak{T})$ *and* $\mathfrak{N} \neq \mathfrak{M}$, *then* $|\mathfrak{N}|_{2'}$ *divides* 15.

Proof. Suppose false for $\mathfrak{N}$. Then $\mathfrak{N}$ has a subgroup $\mathfrak{L}_1$ which contains $\mathfrak{T}$ properly and satisfies $f(\mathfrak{L}_1) \geqq 2$. Thus, there is $i \in \{0, 1, 2\}$ such that $\mathfrak{Z}_i \lhd \langle \mathfrak{L}, \mathfrak{L}_1 \rangle = \mathfrak{K}$, say. Hence, $\mathfrak{N} = \boldsymbol{M}(\mathfrak{K}) = \mathfrak{M}$. The proof is complete.

Set $\mathfrak{Z} = \Omega_1(\boldsymbol{R}_2(\mathfrak{M}))$. The first task is the usual one: to show that $|\mathfrak{Z}| \leqq 2^2$.

LEMMA 20.2. *One of the following holds*:

(a) $|\mathfrak{Z}| \leqq 4$.

(b) $\boldsymbol{C}(\mathfrak{Y}) \subseteqq \mathfrak{M}$ *for every hyperplane* $\mathfrak{Y}$ *of* $\mathfrak{Z}$.

Proof. Suppose $|\mathfrak{Z}| \geqq 8$, and $\mathfrak{Y}$ is a hyperplane of $\mathfrak{Z}$ with $\mathfrak{C} = \boldsymbol{C}(\mathfrak{Y}) \nsubseteqq \mathfrak{M}$. Set $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{M}$. Let $\mathfrak{S}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$.

Case 1. $\mathfrak{S}$ centralizes $\mathfrak{Z}$.

Since $\mathfrak{Z}$ is 2-reducible in $\mathfrak{M}$, we conclude that $\mathfrak{Z} \subseteq \boldsymbol{Z}(\mathfrak{M})$, and the lemma follows.

Case 2. $[\mathfrak{E}, \mathfrak{Z}] \neq 1$ and $\mathfrak{E}_0 = \boldsymbol{C}(\mathfrak{Z})$.

Let $\mathfrak{T}_0 = \mathfrak{T} \cap \mathfrak{E}_0$. Then $\mathfrak{E}_0 \lhd \mathfrak{M}$, and $\mathfrak{M}/\mathfrak{E}_0$ is not a 2-group. Thus, $\mathfrak{M} = \mathfrak{E}_0 \cdot \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{T}_0)$, and $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{T}_0) \supset \mathfrak{T}$, whence $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{T}_0) \in \mathscr{M}^*$. Hence, $\boldsymbol{N}(\mathfrak{D}) \subseteq \mathfrak{M}$ for all non identity characteristic subgroups $\mathfrak{D}$ of $\mathfrak{T}_0$. In particular, $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{E}$. By Lemmas 5.53 and 5.54, we conclude that $|\mathfrak{E}: \mathfrak{E}_0| = 3$.

Let $\mathfrak{A}$ be a S_3-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{T}_0$ and let $\mathfrak{T}_1 = \boldsymbol{O}_2(\mathfrak{T}_0\mathfrak{A})$. Then $\mathfrak{T}_1$ is not characteristic in $\mathfrak{T}_0$, and so $|\mathfrak{T}_0: \mathfrak{T}_1| = 2$. Since $\boldsymbol{N}(\mathfrak{T}_0) \in \mathscr{M}^*$, it follows that there is X in $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{T}_0)$ with $X \notin \boldsymbol{N}(\mathfrak{T}_1)$. Let $\mathfrak{K} = \langle \mathfrak{E}, \mathfrak{E}^X \rangle \subseteq \boldsymbol{C}(\mathfrak{Y} \cap \mathfrak{Y}^X)$. Thus, S_3-subgroups of $\mathfrak{K}$ are cyclic. Let $\mathfrak{P} = \Omega_1(\mathfrak{A})$. Then $\mathfrak{T}_1 \in \boldsymbol{N}_{\mathfrak{K}}(\mathfrak{P})$, $\mathfrak{T}_1^X \in \boldsymbol{N}_{\mathfrak{K}}(\mathfrak{P}^X)$. Since S_3-subgroups of $\mathfrak{K}$ are cyclic, this implies that $\langle \mathfrak{T}_1, \mathfrak{T}_1^X \rangle \subseteq \boldsymbol{O}_{3'}(\mathfrak{K})$. But $\langle \mathfrak{T}_1, \mathfrak{T}_1^X \rangle = \mathfrak{T}_0$, and since $\mathfrak{T}_0\mathfrak{A}$ is not 2-closed, we have a contradiction.

Case 3. $[\mathfrak{E}, \mathfrak{Z}] \neq 1$, and $\mathfrak{E}_0 \neq \boldsymbol{C}(\mathfrak{Z})$.

We regard $\mathfrak{E}_0/\boldsymbol{C}(\mathfrak{Z})$ as a group of automorphisms of $\mathfrak{Z}$ which stabilizes the chain $\mathfrak{Z} \supset \mathfrak{Y} \supset 1$. Hence $\mathfrak{E}_0/\boldsymbol{C}(\mathfrak{Z})$ is elementary abelian. Set $\mathfrak{D} = \boldsymbol{C}(\mathfrak{Z})$, and for each subset $\mathfrak{X}$ of $\mathfrak{M}$, set $\bar{\mathfrak{X}} = \mathfrak{X}\mathfrak{D}/\mathfrak{D}$. Then $\boldsymbol{O}_2(\bar{\mathfrak{M}}) = 1$. Since $\bar{\mathfrak{E}}$ is cyclic, we have $\bar{\mathfrak{E}} \lhd \bar{\mathfrak{M}}$, $1 \neq \bar{\mathfrak{E}}$. Choose $\bar{X} \in \bar{\mathfrak{E}}_0^\#$. Since $\bar{X}$ centralizes $\mathfrak{Y}$ and acts faithfully on $\bar{\mathfrak{E}}$, it follows that $[\bar{\mathfrak{E}}, \bar{X}] = \bar{\mathfrak{P}}$ is of order 3, and $\mathfrak{Z} = \mathfrak{Z}^0 \times \mathfrak{Z}^1$, where $\mathfrak{Z}^0 = \boldsymbol{C}_{\mathfrak{Z}}(\bar{\mathfrak{P}})$, $\mathfrak{Z}^1 = [\mathfrak{Z}, \mathfrak{P}]$, and $\mathfrak{Z}^1$ is a four-group. Also, $\mathfrak{Y} = \mathfrak{Z}^0 \times \mathfrak{Y}^1$, where $\mathfrak{Y}^1 \subset \mathfrak{Z}^1$, $|\mathfrak{Y}^1| = 2$.

Since $\bar{\mathfrak{P}}$ is the only subgroup of $\bar{\mathfrak{E}}$ of order 3, we conclude that $|\mathfrak{E}_0: \boldsymbol{C}(\mathfrak{Z})| = 2$. Since $\bar{\mathfrak{P}}$ char $\bar{\mathfrak{E}}$, we have $\mathfrak{Z}^0 \lhd \mathfrak{M}$. Since $\boldsymbol{C}(\mathfrak{Y}) \subseteq \boldsymbol{C}(\mathfrak{Z}^0)$, we get $\boldsymbol{C}(\mathfrak{Y}) \subseteq \mathfrak{M}$. The proof is complete.

LEMMA 20.3. *Suppose* $|\mathfrak{Z}| \geqq 8$, $\mathfrak{Y}$ *is a hyperplane of* $\mathfrak{Z}$ *and* $\boldsymbol{C}(\mathfrak{Y}) \supset \boldsymbol{C}(\mathfrak{Z})$. *Then*:

(a) $|\boldsymbol{C}(\mathfrak{Y}): \boldsymbol{C}(\mathfrak{Z})| = 2$.

(b) *If* $\mathfrak{E}$ *is a* $S_{2'}$*-subgroup of* $\mathfrak{M}$, *then* $[\mathfrak{E}, \boldsymbol{C}(\mathfrak{Y})]\, \boldsymbol{C}(\mathfrak{Z})/\boldsymbol{C}(\mathfrak{Z}) = \mathfrak{D}$ *has order* 3.

(c) $\mathfrak{Z} = \mathfrak{Z}^0 \times \mathfrak{Z}^1$, $\mathfrak{Z}^0 = \boldsymbol{C}_{\mathfrak{Z}}(\mathfrak{D})$, $\mathfrak{Z}^1 = [\mathfrak{Z}, \mathfrak{D}]$, $\mathfrak{Z}^1$ *is a four-group and* $\mathfrak{Z}^i \lhd \mathfrak{M}$, $i = 0, 1$.

(d) $\mathfrak{Y} = \mathfrak{Z}^0 \times \mathfrak{Y}^1$, *where* $\mathfrak{Y}^1 = [\mathfrak{Z}^1, \boldsymbol{C}(\mathfrak{Y})]$ *is of order* 2.

This lemma was proved in the course of the preceding lemma, and is simply recorded here.

LEMMA 20.4. *Suppose* $|\mathfrak{Z}| \geqq 8$, *and* $\mathfrak{K} = \mathfrak{T}\mathfrak{P}$ *is a solvable subgroup* *of* $\mathfrak{G}$, *where* $\mathfrak{P}$ *is a cyclic p-group. Assume that* $\mathbf{Z}(\mathfrak{T}) \not\triangleleft \mathfrak{R}$. *Then* $\mathfrak{Z} \subseteq \mathfrak{R}_0 = \boldsymbol{O}_2(\mathfrak{R})$.

Proof. Suppose false. Since $\mathfrak{R}/\mathfrak{R}_0$ has a normal cyclic S_p-subgroup on which $\mathfrak{T}/\mathfrak{R}_0$ act faithfully, it follows that $\mathfrak{T}/\mathfrak{R}_0$ is cyclic. Hence, $\mathfrak{Y} = \mathfrak{R}_0 \cap \mathfrak{Z}$ is a hyperplane of $\mathfrak{Z}$. Set $\mathfrak{U} = \Omega_1(\boldsymbol{Z}(\mathfrak{T}))^{\mathfrak{R}}$. Since $\boldsymbol{Z}(\mathfrak{T}) \not\triangleleft \mathfrak{R}$, we have $\mathfrak{U} \supset \Omega_1(\boldsymbol{Z}(\mathfrak{T}))$, and so $\mathfrak{U}_0 = [\mathfrak{U}, \mathfrak{P}] \neq 1$, and $\mathfrak{U}_0 \triangleleft \mathfrak{R}$.

Choose $Z \in \mathfrak{Z} - \mathfrak{Y}$. Without loss of generality, we assume that Z inverts $\mathfrak{P}$. Since $\mathfrak{U}_0 \subseteq \boldsymbol{C}(\mathfrak{Y})$, we conclude that $|\mathfrak{U}_0 : \boldsymbol{C}_{\mathfrak{U}_0}(Z)| = 2$, and so $p = 3$, while $\mathfrak{U}_0$ is a four-group.

Choose $U \in \mathfrak{U}_0 - \boldsymbol{C}_{\mathfrak{U}_0}(Z)$, so that $\mathfrak{U}_0 = \langle U \rangle \times \langle [U, Z] \rangle$. We assume that we have chosen Z in $\mathfrak{Z} - \mathfrak{Y}$ such that $\langle Z \rangle^{\mathfrak{M}}$ is of minimal order. By what we have just shown, we have $\langle Z \rangle \not\triangleleft \mathfrak{M}$.

Now $\mathfrak{U} \subseteq \boldsymbol{O}_2(\mathfrak{R}) \subseteq \mathfrak{T} \subseteq \mathfrak{M}$, and $\mathfrak{U}$ centralizes a hyperplane $\mathfrak{Y}$ of $\mathfrak{Z}$. By our choice of Z, we conclude that $\langle Z \rangle^{\mathfrak{M}}$ is a four-group. Now $\mathfrak{D} = \langle \mathfrak{P}, Z \rangle$ acts faithfully on $\mathfrak{R}_0$, and since $[\mathfrak{P}, \mathfrak{R}_0] = \mathfrak{U}_0$, it follows that $|\mathfrak{P}| = 3$, and that $\mathfrak{R}_0 = \mathfrak{U}_0 \times \mathfrak{R}^0$, where $\mathfrak{R}^0 = \boldsymbol{C}_{\mathfrak{R}_0}(\mathfrak{P})$.

We now use our element U. Since $\mathfrak{U}_0$ is a normal four-subgroup of $\mathfrak{T}\mathfrak{P}$, it follows that U inverts a subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ of order 3, and that if $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{M})$, then $[\mathfrak{H}, \mathfrak{Q}] = \langle Z \rangle \times \langle [Z, U] \rangle$, whence $\mathfrak{H} = [\mathfrak{H}, \mathfrak{Q}] \times \mathfrak{H}_0$, where $\mathfrak{H}_0 = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{Q})$. Let $\mathfrak{L} = \mathfrak{T}\mathfrak{Q}$, $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{L})$. First, suppose that $\mathfrak{R}^0$ contains a four-group. In this case, S_3-subgroups of $\mathfrak{G}$ are cyclic, and since $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{Q}) \neq 1$, $\boldsymbol{C}_{\mathfrak{R}_0}(\mathfrak{P}) \neq 1$ and since $\mathfrak{T} = \boldsymbol{N}(\mathfrak{T})$, we conclude that $\mathfrak{R}_0 = \mathfrak{H}$. This is false, since $\mathfrak{Z} \triangleleft \mathfrak{T}\mathfrak{Q}$, $\mathfrak{Z} \not\triangleleft \mathfrak{T}\mathfrak{P}$. So $\mathfrak{R}^0$ is either cyclic or generalized quaternion. So $\mathfrak{T}$ is the direct product of a D_8 and a group which is either cyclic or generalized quaternion. Thus $|\boldsymbol{N}(\mathfrak{X})|_{2'} \leqq 3$ for every non identity subgroup of $\mathfrak{X}$ of $\mathfrak{T}$. This violates §18, and completes the proof.

LEMMA 20.5. *Suppose* $|\mathfrak{Z}| \geqq 8$ *and* $\mathfrak{R} = \mathfrak{T}\mathfrak{P}$ *is a solvable subgroup of* $\mathfrak{G}$, *where* $\mathfrak{P}$ *is a p-group. Then one of the following holds*:

(a) $\boldsymbol{Z}(\mathfrak{T}) \triangleleft \mathfrak{R}$.

(b) $\mathfrak{W} \triangleleft \mathfrak{R}$, *where* $\mathfrak{W} = \boldsymbol{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{Z}); \mathfrak{T})$.

Proof. Set $\mathfrak{R}_0 = \boldsymbol{O}_2(\mathfrak{R})$, $\mathfrak{U} = \Omega_1(\boldsymbol{Z}(\mathfrak{T})^{\mathfrak{R}})$, $\mathfrak{U}_0 = [\mathfrak{U}, \mathfrak{P}]$. Suppose (a) and (b) fail. Then $\mathfrak{U}_0 \neq 1$, and $\mathfrak{W} \not\subseteq \mathfrak{R}_0$. So there is G in $\mathfrak{G}$ such that $\mathfrak{Z}^* = \mathfrak{Z}^{G} \subseteq \mathfrak{T}$, $\mathfrak{Z}^* \not\subseteq \mathfrak{R}_0$. Let $\mathfrak{Z}^*_0 = \mathfrak{Z}^* \cap \mathfrak{R}_0$, so that $\mathfrak{Z}^*_0$ is a hyperplane of $\mathfrak{Z}^*$. Since $\mathfrak{U}_0 \subseteq \boldsymbol{C}(\mathfrak{Z}^*_0)$, and $\mathfrak{U}_0 \not\subseteq \boldsymbol{C}(\mathfrak{Z}^*)$, we conclude that $\boldsymbol{C}_{\mathfrak{U}_0}(\mathfrak{Z}^*)$ is a hyperplane of $\mathfrak{U}_0$, whence $p = 3$, and $\mathfrak{U}_0$ is a four-group. Set $\mathfrak{M}^* = \mathfrak{M}^{G}$. Thus, $\mathfrak{U}_0 \subseteq \mathfrak{M}^*$, $\mathfrak{U}_0 = \langle U \rangle \times \langle Z \rangle$, where $\langle Z \rangle = [\mathfrak{U}_0, \mathfrak{Z}^*] \subset \mathfrak{Z}^*$. Also, $\langle Z^{\mathfrak{M}^*} \rangle$ is a four-group, since $\langle Z \rangle = [\mathfrak{Z}^*, \mathfrak{U}]$.

Case 1. $\boldsymbol{C}(Z) \subseteq \mathfrak{M}^*$. In this case, since $\langle Z \rangle \triangleleft \mathfrak{T}$, we find that $\mathfrak{T} \subseteq \mathfrak{M}^*$. Since $\mathfrak{T} = \boldsymbol{N}(\mathfrak{T})$, this forces $\mathfrak{M}^* = \mathfrak{M}$, which violates Lemma

20.4, as $\mathfrak{Z}^* \not\subseteq \mathfrak{K}_0$.

Case 2. $C(Z) \not\subseteq \mathfrak{M}^*$. Since $\langle Z^{\mathfrak{M}^*}\rangle$ is a four-group, it follows that $C_{\mathfrak{M}^*}(Z)$ is of index 3 in $\mathfrak{M}^*$, whence $|\mathfrak{M}^*| = 3|\mathfrak{T}|$.

Since $Z(\mathfrak{T}) \ntriangleleft \mathfrak{M}$, it follows from the construction of $\mathfrak{M}$ that $Z(J((\mathfrak{T}))$ and $Z(J_1(\mathfrak{T}))$ are normal subgroups of $\mathfrak{M}$. From §18, there is a solvable subgroup $\mathfrak{L}_0$ of $\mathfrak{G}$ which contains $\mathfrak{T}$ properly and such that $\mathfrak{L}_0 = \mathfrak{T}\mathfrak{P}_0$, where $\mathfrak{P}_0$ is a p-group of order larger than 3. If $p \geqq 5$, then Lemmas 5.53, 5.54 imply that $\mathfrak{L}_0 \subseteq \mathfrak{M}$, against $|\mathfrak{M}| = 3|\mathfrak{T}|$. So $p = 3$, and $|\mathfrak{P}_0| \geqq 3^2$. By Lemma 5.54, we get $\mho^1(\mathfrak{P}_0) \subseteq \mathfrak{M}$, whence $\mathfrak{M} = \mathfrak{T}\mho^1(\mathfrak{P}_0)$. This is absurd, since $\mathfrak{M} \in \mathscr{M}(\mathfrak{T})$, and $\mathfrak{M} \subset \mathfrak{L}_0$, while $\mathfrak{L}_0$ is solvable. The proof is complete.

LEMMA 20.6. *One of the following holds*:
(a) $|\mathfrak{Z}| \leqq 4$.
(b) $\mathscr{M}(\mathfrak{T}) = \{C(Z(\mathfrak{T})), N(\mathfrak{W})\}$, *where* $\mathfrak{W} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{Z}); \mathfrak{T})$.

Proof. Since $\mathfrak{T} = N(\mathfrak{T})$, it follows that $N(Z(\mathfrak{T})) = C(Z(\mathfrak{T}))$. Since $|\mathscr{M}(\mathfrak{T})| \geqq 2$, this lemma is an immediate consequence of Lemma 20.5.

LEMMA 20.7. *Suppose* $|\mathfrak{Z}| \geqq 8$. *Then the following hold*:
(a) $|\mathfrak{Z}| = 16$.
(b) $\mathfrak{T}/O_2(\mathfrak{M})$ *is cyclic*.
(c) $|\mathfrak{E}| = 3, 5$ *or* 15, *where* $\mathfrak{E}$ *is a* $S_{2'}$*-subgroup of* $\mathfrak{M}$.
(d) $\mathfrak{Z}\mathfrak{E}$ *is a Frobenius group*.

Proof. Suppose $1 \subset \mathfrak{E}_0 \subseteq \mathfrak{E}$. Let $\mathfrak{Z}^0 = C_{\mathfrak{Z}}(\mathfrak{E}_0)$. Thus, $\mathfrak{Z}^0 \triangleleft \mathfrak{M}$. Suppose $\mathfrak{Z}^0 \neq 1$, and Z is an involution of $\mathfrak{Z}^0 \cap Z(\mathfrak{T})$. Hence, $C(Z) \supseteq C(Z(\mathfrak{T}))$, and so $C(Z) = C(Z(\mathfrak{T}))$, since $C(Z(\mathfrak{T})) \in \mathscr{M}(\mathfrak{T})$. Since $C(Z(\mathfrak{T})) \cap \mathfrak{M} \supset \mathfrak{T}$, we conclude that $\mathfrak{M} = C(Z(\mathfrak{T}))$, and so $\mathfrak{Z} = \Omega_1(Z(\mathfrak{T}))$.

Since $\mathfrak{W} \ntriangleleft \mathfrak{M}$, there is G in $\mathfrak{G}$ such that $\mathfrak{Z}^{\mathfrak{G}} \subseteq \mathfrak{T}$, $\mathfrak{Z}^{\mathfrak{G}} \not\subseteq O_2(\mathfrak{M})$. Hence, $\mathfrak{E}$ has a subgroup $\mathfrak{F}$ of prime order such that $\mathfrak{Z}^G$ does not centralize $O_2(\mathfrak{M})\mathfrak{F}/O_2(\mathfrak{M})$. Let $\mathfrak{Y} = \mathfrak{Z}^G \cap C(\bar{\mathfrak{F}})$, where $\bar{\mathfrak{F}} = O_2(\mathfrak{M})\mathfrak{F}/O_2(\mathfrak{M})$, and let $\mathfrak{F} = \langle F\rangle$. Thus, $\mathfrak{T}\mathfrak{F} = \langle \mathfrak{T}, \mathfrak{Z}^{GF}\rangle$, and so if we set $\mathfrak{N} = N(\mathfrak{W})$, then $\mathfrak{N} \cap \mathfrak{Z}^{GF} = \mathfrak{Y}^F$.

Since $|\mathfrak{Z}| \geqq 8$, we have $|\mathfrak{Y}^F| \geqq 4$. Let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}$. Since $\mathfrak{M} = C(Z(\mathfrak{T}))$, it follows that $C(\mathfrak{U}) = O_2(\mathfrak{N})$. If $Y \in \mathfrak{Y}^{F\#}$ centralizes $\mathfrak{U}$, then since $C(Y) = C(\mathfrak{Z}^{GF})$, we get $\mathfrak{Z}^{GF} \subseteq C(\mathfrak{U}) \triangleleft \mathfrak{N}$, which is false. Hence, $\mathfrak{Y}^F$ acts faithfully on $\mathfrak{U}$, and $C_{\mathfrak{U}}(\mathfrak{Y}^F) = C_{\mathfrak{U}}(Y)$ for all $Y \in \mathfrak{Y}^{F\#}$. This is impossible since $\mathfrak{N}$ is solvable. We conclude that $\mathfrak{Z}\mathfrak{E}$ is a Frobenius group.

Since $\mathfrak{Z}\mathfrak{E}$ is a Frobenius group, it follows that if $\mathfrak{X}$ is any subgroup of $\mathfrak{M}/C(\mathfrak{Z})$ of order 2, then $\mathfrak{Z}$ is a free $F_2\mathfrak{X}$-module. In particular, $|\mathfrak{Z}| = 2^{2z}$ for some integer $z \geqq 2$.

Suppose $z \geqq 3$. Let $\mathfrak{Z}^1$ be a subgroup of $\mathfrak{Z}$ of index 4. We will show that $C(\mathfrak{Z}^1) = C(\mathfrak{Z})$. In any case, $C_{\mathfrak{M}}(\mathfrak{Z}^1)$ is a 2-group, since $\mathfrak{Z}\mathfrak{E}$ is a Frobenius group. Furthermore, by the preceding paragraph, we conclude that $C_{\mathfrak{M}}(\mathfrak{Z}^1) = C(\mathfrak{Z}) = O_2(\mathfrak{M})$. Since $N(\mathfrak{X}) = \mathfrak{M}$ for every non identity characteristic subgroup $\mathfrak{X}$ of $O_2(\mathfrak{M})$, it follows that $O_2(\mathfrak{M})$ is a S_2-subgroup of $C(\mathfrak{Z}^1)$, and then Lemmas 5.53 and 5.54 imply that $|C(\mathfrak{Z}^1): O_2(\mathfrak{M})| = 1$ or 3. Set $\mathfrak{C} = C(\mathfrak{Z}^1)$, $\mathfrak{C}_1 = O_2(\mathfrak{C})$, and suppose that $\mathfrak{C} \supset O_2(\mathfrak{M})$. In this case, $\mathfrak{C}_1$ has index 2 in $O_2(\mathfrak{M})$, and $\mathfrak{C}_1$ is not normal in $\mathfrak{M}$. Choose M in $\mathfrak{M}$ with $M \notin N(\mathfrak{C}_1)$. Then $\langle \mathfrak{C}, \mathfrak{C}^M \rangle \subseteqq C(\mathfrak{Z}^1 \cap \mathfrak{Z}^{1M})$, and since $\mathfrak{Z}^1 \cap \mathfrak{Z}^{1M} \neq 1$, $\langle \mathfrak{C}, \mathfrak{C}^M \rangle$ is solvable. Let $\mathfrak{A}$ be a S_3-subgroup of $\mathfrak{C}$, so that $\mathfrak{C}_1 \in N(\mathfrak{A}; 2)$, $\mathfrak{C}_1^M \in N(\mathfrak{A}^M; 2)$. Since S_3-subgroups of $\langle \mathfrak{C}, \mathfrak{C}^M \rangle$ are cyclic, we conclude that $\langle \mathfrak{C}_1, \mathfrak{C}_1^M \rangle \subseteqq O_{3'}(\langle \mathfrak{C}, \mathfrak{C}^M \rangle)$, which is false, since $\mathfrak{C}$ is not 2-closed. So $C(\mathfrak{Z}) = C(\mathfrak{Z}^1)$ for every subgroup $\mathfrak{Z}^1$ of $\mathfrak{Z}$ of index 4, provided $z \geqq 3$.

Let $\mathfrak{W}_1 = \langle V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{Y}); \mathfrak{T}) \mid |\mathfrak{Z}: \mathfrak{Y}| = 2 \rangle$, and continue to suppose that $z \geqq 3$. By the preceding argument, we conclude that $\mathfrak{W}_1 \subseteqq C(\mathfrak{Z})$, and so, if $G \in \mathfrak{G}$, and $|\mathfrak{Z}^G: \mathfrak{Z}^G \cap \mathfrak{M}| \leqq 2$, then $\mathfrak{Z}^G \subseteqq \mathfrak{M}$.

Set $\mathfrak{N} = C(Z(\mathfrak{T}))$. Since $\mathfrak{W} \ntriangleleft \mathfrak{N}$, we can choose G in $\mathfrak{G}$ such that $\mathfrak{Z}^G \subseteqq \mathfrak{T}$, $\mathfrak{Z}^G \nsubseteqq O_2(\mathfrak{N})$, and then we can choose N in $\mathfrak{N}$ such that $\mathfrak{T} \cap \mathfrak{Z}^{GN}$ is of index 2 in $\mathfrak{Z}^{GN}$. By the above argument, we get $\mathfrak{Z}^{GN} \subseteqq \mathfrak{M}$, and so $\langle \mathfrak{T}, \mathfrak{Z}^{GN} \rangle \subseteqq \mathfrak{M} \cap \mathfrak{N} = \mathfrak{T}$, the desired contradiction. So $z = 2$. Since $\mathfrak{M}/O_2(\mathfrak{M})$ acts faithfully on $\mathfrak{Z}$, it follows that $\mathfrak{T}/O_2(\mathfrak{M})$ is cyclic. The proof is complete.

LEMMA 20.8. *Suppose* $|\mathfrak{Z}| \geqq 8$. *Then the following hold*:

(a) *For some* G *in* $\mathfrak{G}$, $|\mathfrak{Z}^G \cap \mathfrak{M}| = 8$, *and* $|\mathfrak{Z}^G \cap O_2(\mathfrak{M})| = 4$.

(b) $\mathfrak{Z}$ *contains a four-group* $\mathfrak{Z}^1$ *with* $C(\mathfrak{Z}^1) \nsubseteqq \mathfrak{M}$.

(c) $|\mathfrak{N}| = 3|\mathfrak{T}|$, *where* $\mathfrak{N} = C(Z(\mathfrak{T}))$.

Proof. Since $C_{\mathfrak{M}}(Z(\mathfrak{T})) = \mathfrak{T}$, the construction of $\mathfrak{M}$ implies that $N(Z(J(\mathfrak{T}))) \subseteqq \mathfrak{M}$, $N(Z(J_1(\mathfrak{T}))) \subseteqq \mathfrak{M}$. By Lemmas 5.53 and 5.54, it follows that (c) holds.

Since $\mathfrak{W} \lhd \mathfrak{M}$, it follows that there is X in $\mathfrak{G}$ such that $\mathfrak{Z}^X \subseteqq \mathfrak{T}$, $\mathfrak{Z}^X \nsubseteqq O_2(\mathfrak{N})$. By (c), it follows that $\mathfrak{Y} = O_2(\mathfrak{N}) \cap \mathfrak{Z}^X$ is a hyperplane of $\mathfrak{Z}^X$, so of order 8. Let P be an element of $\mathfrak{N}$ of order 3, and set $G = XP$. Thus, $\mathfrak{Y}^P \subseteqq O_2(\mathfrak{N}) \subseteqq \mathfrak{T}$, and $\mathfrak{N} = \langle \mathfrak{T}, \mathfrak{Z}^G \rangle$. Hence, $\mathfrak{M} \cap \mathfrak{Z}^G = \mathfrak{Y}^P$.

Since $\mathfrak{T}/O_2(\mathfrak{M})$ is cyclic, we get that $|\mathfrak{Y}^P \cap O_2(\mathfrak{M})| \geqq 4$. Suppose $\mathfrak{Y}^P \subseteqq O_2(\mathfrak{M})$. But $C(\mathfrak{Y}^P) \subseteqq \mathfrak{M}^G$, and we find that $\mathfrak{Z} \subseteqq \mathfrak{M}^G$, whence $[\mathfrak{Z}, \mathfrak{Z}^G] = 1$, against $\mathfrak{Z}^G \nsubseteqq \mathfrak{M}$. So (a) holds. Let $\mathfrak{U} = \mathfrak{Y}^P \cap O_2(\mathfrak{M})$. The argument just given shows that $C(\mathfrak{U}) \nsubseteqq \mathfrak{M}^G$, and so (b) holds. The proof is complete.

LEMMA 20.9. *One of the following holds*:

(a) $|\mathfrak{Z}| \leqq 4$.

(b) $|\mathfrak{M}| = 5|\mathfrak{T}|$.

Proof. Suppose $|\mathfrak{Z}| \geqq 8$. Since $|\mathfrak{N}| = 3|\mathfrak{T}|$, it follows that $|\mathfrak{M}| = |\mathfrak{T}|d$, where $d = 5$ or 15. Suppose $d = 15$.

Since $\mathfrak{M}/\boldsymbol{O}_2(\mathfrak{M})$ acts faithfully on $\mathfrak{Z}$, and $|\mathfrak{Z}| = 16$, while $\mathfrak{M}/\mathfrak{M}'$ is a 2-group, it follows that $\mathfrak{T}/\boldsymbol{O}_2(\mathfrak{M})$ is cyclic of order 4. Let $\mathfrak{E} = \mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{E}$ is a $S_{2'}$-subgroup of $\mathfrak{M}$, and $|\mathfrak{A}| = 5$, $|\mathfrak{B}| = 3$. Let $\mathfrak{N} = \mathfrak{T}\mathfrak{Q}$, where $|\mathfrak{Q}| = 3$, and let $\mathfrak{K} = \boldsymbol{O}_2(\mathfrak{M})$, $\mathfrak{K}_0 = \boldsymbol{O}_2(\mathfrak{T}\mathfrak{B})$. Thus, $\mathfrak{K}$ and $\mathfrak{K}_0$ are of index 2 in $\mathfrak{T}$, and are maximal elements of $\mathscr{N}(\mathfrak{Q}; 2)$, $\mathscr{N}(\mathfrak{B}; 2)$, respectively. Since $\mathfrak{N} = \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{T}))$, we have $\boldsymbol{C}_{\mathfrak{K}}(\mathfrak{Q}) \neq 1$, and since $\mathfrak{T}/\boldsymbol{O}_2(\mathfrak{M})$ is of order 4, we have $\boldsymbol{C}_{\mathfrak{K}_0}(\mathfrak{B}) \neq 1$. Since $\mathfrak{K}$ and $\mathfrak{K}_0$ are not $\mathfrak{G}$-conjugate, it follows that S_3-subgroups of $\mathfrak{G}$ are non cyclic. Hence, $\boldsymbol{C}(\mathfrak{U})$ is a $3'$-group for every 4-subgroup $\mathfrak{U}$ of $\mathfrak{G}$. On the other hand, there is a four-subgroup $\mathfrak{U}$ of $\mathfrak{Z}$ such that $\boldsymbol{C}(\mathfrak{U}) \not\subseteq \mathfrak{M}$. Set $\mathfrak{C} = \boldsymbol{C}(\mathfrak{U})$, $\mathfrak{C}_0 = \boldsymbol{C}_{\mathfrak{M}}(\mathfrak{U})$. Then $\mathfrak{C}_0$ is a 2-group, since $\mathfrak{Z}\mathfrak{E}$ is a Frobenius group. If $\mathfrak{C}_0 = \boldsymbol{O}_2(\mathfrak{M})$, then since $\mathfrak{C}$ is a $3'$-group, our factorizations imply that $\mathfrak{C} \subseteq \mathfrak{M}$. Hence, $\mathfrak{C}_0 \supset \boldsymbol{O}_2(\mathfrak{M})$, and so $\mathfrak{C}_0$ is of index 2 in a S_2-subgroup of $\mathfrak{M}$, and so $\mathfrak{C}_0$ is the unique subgroup of $\mathfrak{T}$ of index 2 which contains $\boldsymbol{O}_2(\mathfrak{M})$ (assuming as we may that $\mathfrak{C}_0 \subseteq \mathfrak{T}$). But then $\mathfrak{T}\mathfrak{B} \subseteq \boldsymbol{N}(\mathfrak{C}_0)$, and so $\boldsymbol{N}(\mathfrak{X}) \subseteq \mathfrak{M}$ for every non identity characteristic subgroup $\mathfrak{X}$ of $\mathfrak{C}_0$. Once again, since $\mathfrak{C}$ is a $3'$-group, our factorizations complete the proof.

THEOREM 20.1. $|\mathfrak{Z}| \leqq 4$.

Proof. Suppose false, so that the preceding results may be applied. Thus, $\mathfrak{M}/\boldsymbol{O}_2(\mathfrak{M})$ is a Frobenius group of order 10 or 20. It follows that

$$(\ast)\qquad |\mathfrak{M}: \boldsymbol{C}_{\mathfrak{M}}(Z)| = 2^a.5\,, \quad a \leqq 1\,, \quad \text{for all } Z \in \mathfrak{Z}^{\#}\,.$$

We will show that $\boldsymbol{C}_{\mathfrak{M}}(Z)$ is a S_2-subgroup of $\boldsymbol{C}(Z)$ for all $Z \in \mathfrak{Z}^{\#}$. We may assume that $\boldsymbol{C}_{\mathfrak{M}}(Z) = \mathfrak{T}_0$ is of index 2 in $\mathfrak{T}$. This implies that $\mathfrak{T}/\boldsymbol{O}_2(\mathfrak{M})$ is of order 4, since $\boldsymbol{C}_{\mathfrak{M}}(Z)$ properly contains $\boldsymbol{O}_2(\mathfrak{M})$ for every Z in $\mathfrak{Z}$. Since $\boldsymbol{O}_2(\mathfrak{M}) \subset \mathfrak{T}_0 \subset \mathfrak{T}$, it follows that $\boldsymbol{Z}(\mathfrak{T})$ is cyclic, and that if $\Omega_1(\boldsymbol{Z}(\mathfrak{T})) = \langle Z_0 \rangle$, then $\langle Z, Z_0 \rangle = \Omega_1(\boldsymbol{Z}(\mathfrak{T}_0))$. Let $\mathfrak{T}_1$ be a S_2-subgroup of $\boldsymbol{C}(Z)$ which contains $\mathfrak{T}_0$, and suppose by way of contradiction that $\mathfrak{T}_0 \subset \mathfrak{T}_1$. Thus, $\langle \mathfrak{T}, \mathfrak{T}_1 \rangle \subseteq \boldsymbol{N}(\langle Z, Z_0 \rangle)$ and $\mathfrak{T}_1 \not\subseteq \mathfrak{M}$, whence $\langle \mathfrak{T}, \mathfrak{T}_1 \rangle = \mathfrak{N}$, against $\mathfrak{N} = \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{T}))$. So

$$(\ast\ast)\qquad \boldsymbol{C}_{\mathfrak{M}}(Z) \text{ is a } S_2\text{-subgroup of } \boldsymbol{C}(Z) \text{ for all } Z \in \mathfrak{Z}^{\#}\,.$$

Since $\boldsymbol{Z}(\mathfrak{T}) \ntriangleleft \mathfrak{M}$, and since $\mathfrak{T}$ is a maximal subgroup of $\mathfrak{M}$, the construction of $\mathfrak{M}$ implies that $\mathfrak{M} = \boldsymbol{N}(\boldsymbol{Z}(\boldsymbol{J}(\mathfrak{T}))) = \boldsymbol{N}(\boldsymbol{Z}(\boldsymbol{J}_1(\mathfrak{T})))$. Since $\mathfrak{Z}\mathfrak{E}$ is a Frobenius group, we even get $\mathfrak{M} = \boldsymbol{N}(\boldsymbol{J}(\mathfrak{T})) = \boldsymbol{N}(\boldsymbol{J}_1(\mathfrak{T}))$. Hence, $\boldsymbol{J}(\mathfrak{T}) = \boldsymbol{J}(\mathfrak{T}_0)$, $\boldsymbol{J}_1(\mathfrak{T}) = \boldsymbol{J}_1(\mathfrak{T}_0)$. By $(\ast\ast)$, we conclude that

$$(\ast\ast\ast)\qquad |\boldsymbol{C}(Z): \boldsymbol{C}_{\mathfrak{M}}(Z)| = 1 \text{ or } 3 \text{ for all } Z \in \mathfrak{Z}^{\#}\,.$$

Choose G in $\mathfrak{G}$ such that $\mathfrak{Z}^G \cap \mathfrak{M} = \mathfrak{Y}$ is of order 8, and such that $\mathfrak{Y}_0 = \mathfrak{Y} \cap \boldsymbol{O}_2(\mathfrak{M})$ has order 4, as in Lemma 20.8. Choose $Y \in \mathfrak{Y}_0^\#$. Thus, $\mathfrak{Z} \subseteq \boldsymbol{C}(Y)$, and $\boldsymbol{O}_2(\boldsymbol{C}(Y))$ normalizes $\mathfrak{Z}^G$, by (**). Since $\mathfrak{Z} \cdot \boldsymbol{O}_2(\boldsymbol{C}(Y))$ is a 2-group, it follows that $\mathfrak{Z} \cap \boldsymbol{O}_2(\boldsymbol{C}(Y))$ is of index at most 2 in $\mathfrak{Z}$, by (***). Now choose $X \in \mathfrak{Y} - \mathfrak{Y}_0$. Then $[X, \mathfrak{Z} \cap \boldsymbol{O}_2(\boldsymbol{C}(Y))] \neq 1$, and so we may assume that our element Y has been chosen in $\mathfrak{Z} \cap \mathfrak{Y}_0$. Hence, $\boldsymbol{O}_2(\boldsymbol{C}(Y)) \cap \boldsymbol{O}_2(\mathfrak{M})$ is of index at most 2 in $\boldsymbol{O}_2(\mathfrak{M})$.

We may assume that X inverts $\mathfrak{E}$. Set $\mathfrak{T}^0 = \boldsymbol{O}_2(\boldsymbol{C}(Y)) \cap \boldsymbol{O}_2(\mathfrak{M})$, so that $|\boldsymbol{O}_2(\mathfrak{M}) \colon \mathfrak{T}^0| \leqq 2$, and $[\mathfrak{T}^0, X] \subseteq \mathfrak{Y}_0$. This implies that $\mathfrak{Z} = [\boldsymbol{O}_2(\mathfrak{M}), \mathfrak{E}]$, and since $\mathfrak{Z} \supseteq \Omega_1(\boldsymbol{Z}(\mathfrak{T}))$, it follows that $\boldsymbol{O}_2(\mathfrak{M}) = \mathfrak{Z}$. This in turn implies that $\mathfrak{Z} \subseteq \boldsymbol{O}_2(\mathfrak{N})$, whence $\mathfrak{Z} \triangleleft \mathfrak{N}$, the desired contradiction. The proof is complete.

LEMMA 20.10. $\mathfrak{Z} \subseteq \boldsymbol{Z}(\mathfrak{M})$.

Proof. If $|\mathfrak{Z}| = 2$, the lemma is obvious. Suppose $|\mathfrak{Z}| = 4$, and that $\mathfrak{Z} \not\subseteq \boldsymbol{Z}(\mathfrak{M})$. In this case, $\mathfrak{M}/\boldsymbol{C}(\mathfrak{Z}) \cong \Sigma_3$, and $\mathfrak{M}$ is transitive on $\mathfrak{Z}^\#$. We argue that

$$\boldsymbol{C}(Z) \subseteq \mathfrak{M} \quad \text{for all} \quad Z \in \mathfrak{Z}^\# . \tag{20.1}$$

This is clear if $|\mathfrak{M}| \neq 3|\mathfrak{T}|$, so suppose $|\mathfrak{M}| = 3|\mathfrak{T}|$. In this case, $\boldsymbol{N}_{\mathfrak{M}}(\boldsymbol{Z}(\mathfrak{T})) = \mathfrak{T}$, and so $\boldsymbol{J}(\mathfrak{T}) \triangleleft \mathfrak{M}$, $\boldsymbol{J}_1(\mathfrak{T}) \triangleleft \mathfrak{M}$. Since $\mathscr{M}(\mathfrak{T})$ contains an element $\mathfrak{N}$ with $|\mathfrak{N}| > 3|\mathfrak{T}|$, the usual factorizations yield a contradiction. So (20.1) holds.

Let $\mathfrak{P}$ be a subgroup of odd prime power order permutable with $\mathfrak{T}$ and not contained in $\mathfrak{M}$. Since $\mathfrak{P} \cap \mathfrak{M} = 1$, and since $\boldsymbol{J}(\mathfrak{T}) \triangleleft \mathfrak{M}$, $\boldsymbol{J}_1(\mathfrak{T}) \triangleleft \mathfrak{M}$, it follows that $|\mathfrak{P}| = 3$. Hence, if $\mathfrak{U}$ is a minimal normal subgroup of $\mathfrak{T}\mathfrak{P}$, then $\mathfrak{U}$ is a four-group. Let $\mathfrak{W} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{Z}); \mathfrak{T})$. It is a straightforward consequence of (20.1) that $\mathfrak{W} \subseteq \boldsymbol{C}(\mathfrak{U})$; $\mathfrak{W} \subseteq \boldsymbol{C}(\mathfrak{Z})$. Since $\mathfrak{M}/\boldsymbol{C}(\mathfrak{Z}) \cong \Sigma_3$, it follows that $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{W}) \supset \mathfrak{T}$, and so $\boldsymbol{N}(\mathfrak{W}) \subseteq \mathfrak{M}$. Since $\mathfrak{T}\mathfrak{P}/\boldsymbol{O}_2(\mathfrak{T}\mathfrak{P}) \cong \Sigma_3$, we conclude that $\mathfrak{W} \triangleleft \mathfrak{T}\mathfrak{P}$, whence $\mathfrak{P} \subseteq \mathfrak{M}$, the desired contradiction. The proof is complete.

LEMMA 20.11. *If $\mathfrak{X}$ is a normal four-subgroup of $\mathfrak{M}$, then $\boldsymbol{C}(X) \subseteq \mathfrak{M}$ for all $X \in \mathfrak{X}^\#$.*

Proof. We may assume that $X \notin \boldsymbol{Z}(\mathfrak{M})$. Hence, if $\mathfrak{X} \cap \mathfrak{Z} = \mathfrak{U}$, then $\mathfrak{X} = \mathfrak{U} \times \langle X \rangle$, and $\boldsymbol{C}_{\mathfrak{M}}(X)$ is of index 2 in $\mathfrak{M}$. Let $\mathfrak{T}_0 = \boldsymbol{C}_{\mathfrak{T}}(X)$, so that $|\mathfrak{T} \colon \mathfrak{T}_0| = 2$, and $\mathfrak{M} = \mathfrak{T}_0 \cdot \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{S})$, where $\mathfrak{S}$ is a $S_{2'}$-subgroup of $\mathfrak{M}$. Set $\mathfrak{C} = \boldsymbol{C}(Z)$. Then $\mathfrak{C} = \mathfrak{T}_0 \cdot \boldsymbol{N}_{\mathfrak{C}}(\mathfrak{S})$, and so $\mathfrak{T}_0$ is permutable with the solvable group $\langle \boldsymbol{N}_{\mathfrak{C}}(\mathfrak{S}), \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{S}) \rangle = \mathfrak{K}$. By a standard argument, $\mathfrak{T}_0\mathfrak{K}$ is solvable, and so $\mathfrak{M} = \mathfrak{T}_0\mathfrak{K} \supseteq \mathfrak{C}$. The proof is complete.

Set

$$\mathscr{F}_0 = \{\mathfrak{X} \mid \mathfrak{X} \text{ is an elementary abelian normal 2-subgroup of } \mathfrak{M} \text{ of order} \geqq 8\} .$$

By Lemma 20.11 and § 13, it follows that $\mathscr{F}_0 \neq \emptyset$. Let $\mathscr{F}_1 = \{\mathfrak{X} \in \mathscr{F}_0 \mid \mathfrak{Z} \subseteqq \mathfrak{X}\}$. Since $\mathfrak{Z} \subseteqq \boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{M}))$, it follows that $\mathscr{F}_1 \neq \emptyset$. Next, set

$$\mathscr{F} = \{\mathfrak{F} \in \mathscr{F}_1, \text{ there is a normal subgroup } \mathfrak{E} \text{ of } \mathfrak{M} \text{ such that } \mathfrak{Z} \subseteqq \mathfrak{E} \subset \mathfrak{F}, \text{ and such that } |\mathfrak{E}| \leqq 4, \text{ while } \mathfrak{F}/\mathfrak{E} \text{ is a chief factor of } \mathfrak{M}\}.$$

Thus, $\mathscr{F} \neq \emptyset$.

Choose $\mathfrak{F} \in \mathscr{F}$. We subject $\mathfrak{F}$ to the same examination which was built up in § 13.

LEMMA 20.12. *$\boldsymbol{C}(\mathfrak{F}_0) \subseteqq \mathfrak{M}$ for every hyperplane $\mathfrak{F}_0$ of $\mathfrak{F}$.*

Proof. If $|\mathfrak{E}| = 4$, then $\mathfrak{F}_0 \cap \mathfrak{E} \neq 1$, so we are done by Lemma 20.11. We may assume that $|\mathfrak{E}| = 2$, and that $\mathfrak{E} \not\subseteq \mathfrak{F}_0$. Hence, $\boldsymbol{C}_{\mathfrak{M}}(\mathfrak{F}_0) = \boldsymbol{C}(\mathfrak{F})$. Let $\mathfrak{K} = \boldsymbol{C}_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{E}) \supset \mathfrak{L} = \boldsymbol{C}(\mathfrak{F})$. Thus, $\mathfrak{K}/\mathfrak{L}$ and $\mathfrak{F}/\mathfrak{Z} (\mathfrak{Z} = \mathfrak{E}!)$ are paired into $\mathfrak{Z}$, and so are in duality. Hence, $\mathfrak{K}$ permutes transitively the hyperplanes of $\mathfrak{F}$ which do not contain $\mathfrak{Z}$, so that

$$\mathfrak{M} = \mathfrak{K} \cdot \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{F}_0) , \quad |\mathfrak{M}: \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{F}_0)| = |\mathfrak{F}: \mathfrak{Z}| .$$

Let $\mathfrak{C} = \boldsymbol{C}(\mathfrak{F}_0) \supseteqq \boldsymbol{C}_{\mathfrak{M}}(\mathfrak{F}_0) = \boldsymbol{C}(\mathfrak{F}) = \mathfrak{L}$, and let $\mathfrak{T}_0 = \mathfrak{T} \cap \mathfrak{L}$, so that $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{L}$. Since $\mathfrak{F}/\mathfrak{Z}$ is a chief factor of $\mathfrak{M}$ of order >2, it follows that $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{T}_0) \supset \mathfrak{T}$, and so $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{T}_0) \in \mathscr{M}^*$. Hence, $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{C}$, and $\boldsymbol{N}(\mathfrak{D}) \subseteqq \mathfrak{M}$ for every non identity characteristic subgroup $\mathfrak{D}$ of $\mathfrak{T}_0$.

Let $\mathfrak{Q}$ be a $S_{2'}$-subgroup of $\mathfrak{L}$ and let $\mathfrak{R}$ be a $S_{2'}$-subgroup of $\mathfrak{C}$ which contains $\mathfrak{Q}$. We may assume that $\mathfrak{R} \supset \mathfrak{Q}$. By Lemmas 5.53 and 5.54, we get that $|\mathfrak{C}: \mathfrak{L}| = 3$. Let $\mathfrak{S}$ be a S_3-subgroup of $\mathfrak{C}$ permutable with $\mathfrak{T}_0$, and let $\mathfrak{C}^* = \mathfrak{T}_0\mathfrak{S}$. Set $\mathfrak{T}_1 = \boldsymbol{O}_2(\mathfrak{C}^*)$, so that $\mathfrak{T}_1 \ntriangleleft \mathfrak{T}$. Choose $T \in \mathfrak{T}$, $T \notin \boldsymbol{N}(\mathfrak{T}_1)$. Let $\mathfrak{C}^{**} = \langle \mathfrak{C}^*, \mathfrak{C}^{*T} \rangle$. Thus, $\mathfrak{C}^{**}$ normalizes $\mathfrak{F}_0 \cap \mathfrak{F}_0^T \neq 1$, and so $\mathfrak{C}^{**}$ is solvable. Since S_3-subgroups of $\mathfrak{C}^{**}$ are cyclic, it follows that $\langle \mathfrak{T}_1, \mathfrak{T}_1^T \rangle \subseteqq \boldsymbol{O}_{3'}(\mathfrak{C}^{**})$. This is false, since $\langle \mathfrak{T}_1, \mathfrak{T}_1^T \rangle = \mathfrak{T}_0$, and $\mathfrak{T}_0 \ntriangleleft \mathfrak{C}^*$. The proof is complete.

LEMMA 20.13. *If J is an involution of $\mathfrak{M}$ and $\boldsymbol{C}_{\mathfrak{F}}(J) = \mathfrak{F}_0$ is a hyperplane of $\mathfrak{F}$, then one of the following holds*:

(a) $[\mathfrak{F}, J] \subseteqq \mathfrak{E}$.

(b) $|\mathfrak{F}/\mathfrak{E}| = 4$.

This lemma follows from a standard argument, and is in fact an immediate consequence of the following lemma.

LEMMA 20.14. *If J is an involution of $\mathfrak{M}$ and $[\mathfrak{F}, J] \not\subseteq \mathfrak{E}$, then $\mathfrak{F}/\mathfrak{E}$ is a free $F_2\langle J\rangle$-module.*

Proof. Let $\mathfrak{D} = C_{\mathfrak{M}}(\mathfrak{F}/\mathfrak{E})$, and for every subset $\mathfrak{S}$ of $\mathfrak{M}$, let $\bar{\mathfrak{S}} = \mathfrak{S}\mathfrak{D}/\mathfrak{D}$. Thus, $\bar{J} \neq 1$. Since $\mathfrak{F}/\mathfrak{E}$ is a chief factor of $\mathfrak{M}$, it follows that $O_2(\bar{\mathfrak{M}}) = 1$, and so $F(\bar{\mathfrak{M}})$ is a cyclic group of odd order. Thus, J inverts a normal subgroup $\bar{\mathfrak{P}}$ of $\bar{\mathfrak{M}}$ of odd prime order. Since $\mathfrak{F}/\mathfrak{E}$ is a chief factor of $\mathfrak{M}$, it follows that $\bar{\mathfrak{P}}$ has no fixed point on $\mathfrak{F}/\mathfrak{E}$, and the lemma follows.

LEMMA 20.15. *If $|\mathfrak{E}| = 4$, then $\mathscr{M}(\mathfrak{T}) = \{\mathfrak{M}, \mathfrak{N}_1\}$, where $\mathfrak{N}_1 = N(\mathfrak{V}_1)$, $\mathfrak{V}_1 = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{E}); \mathfrak{T})$.*

Proof. Let $\mathfrak{P}$ be a p-group permutable with $\mathfrak{T}$, and set $\mathfrak{L} = \mathfrak{T}\mathfrak{P}$. Suppose $\mathfrak{L} \not\subseteq \mathfrak{M}$, so that $\mathfrak{L} \cap \mathfrak{M} = \mathfrak{T}$. Let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{L}$, and let $\mathfrak{E}^G = \mathfrak{E}^*$ be a conjugate of $\mathfrak{E}$ which is contained in $\mathfrak{T}$. Let $\mathfrak{E}_1^G = \mathfrak{E}_1^* = \mathfrak{E}^* \cap O_2(\mathfrak{L})$. Since $\mathfrak{T}/O_2(\mathfrak{L})$ is cyclic, we have $\mathfrak{E}_1^* \neq 1$. Since $\mathfrak{P} \not\subseteq \mathfrak{M}$, it follows that $[\mathfrak{P}, \mathfrak{U}] \neq 1$. Hence, $\mathfrak{U} \subseteq C(\mathfrak{E}_1^*) \subseteq \mathfrak{M}^G$. Suppose $\mathfrak{E}_1^* \subset \mathfrak{E}^*$. In this case, $[\mathfrak{U}, \mathfrak{E}^*] \neq 1$, and so $[\mathfrak{U}, \mathfrak{E}^*] = \mathfrak{E}_1^* \subseteq Z(\mathfrak{U})$. This implies that $[O_2(\mathfrak{L}), \mathfrak{E}^*] = \mathfrak{E}_1^*$, and so $\mathfrak{U}$ is a four-group, $|\mathfrak{P}| = 3$, and $O_2(\mathfrak{L}) = \mathfrak{U} \times (O_2(\mathfrak{L}) \cap C(\mathfrak{P}))$. Since $\mathfrak{P} \not\subseteq \mathfrak{M}$, it follows that $Z(\mathfrak{T}) \cap C(\mathfrak{P}) = 1$, and so $\mathfrak{U} = O_2(\mathfrak{L})$, against $2 \in \pi_4$. So $\mathfrak{V}_1 \subseteq O_2(\mathfrak{L})$, whence $\mathfrak{L} \subseteq N(\mathfrak{V}_1)$. The proof is complete.

LEMMA 20.16. *If $|\mathfrak{E}| = 4$, then $|\mathfrak{F}| = 8$, 16 or 64.*

Proof. Suppose $|\mathfrak{F}| > 8$.

Since $\mathscr{M}(\mathfrak{T}) = \{\mathfrak{M}, \mathfrak{N}_1\}$, it follows that $N_{\mathfrak{M}}(\mathfrak{V}_1) = \mathfrak{T}$.

Let $\mathfrak{C} = C(\mathfrak{F})$, $\mathfrak{D}/\mathfrak{C} = O_2(\mathfrak{M}/\mathfrak{C})$, and let $\mathfrak{P}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$. Then $\mathfrak{D}\mathfrak{P} \lhd \mathfrak{M}$, since $\mathfrak{P}$ is cyclic. Since $|\mathfrak{F}| > 8$, it follows that $\mathfrak{P} \not\subseteq \mathfrak{C}$. Let $\mathfrak{T}_0 = \mathfrak{T} \cap \mathfrak{D}$. Thus, $\mathfrak{M} = \mathfrak{D}\cdot N_{\mathfrak{M}}(\mathfrak{T}_0)$, and so $N_{\mathfrak{M}}(\mathfrak{T}_0) \supset \mathfrak{T}$. Hence, $\mathfrak{V}_1 \not\subseteq \mathfrak{T}_0$.

Choose G in $\mathfrak{G}$ such that $\mathfrak{E}^* = \mathfrak{E}^G \subseteq \mathfrak{T}$, $\mathfrak{E}^* \not\subseteq \mathfrak{T}_0$. Let

$$\mathfrak{E}_0^G = \mathfrak{E}_0^* = \mathfrak{E}^* \cap \mathfrak{T}_0 .$$

Case 1. $\mathfrak{E}^* \cap \mathfrak{F} = 1$.

If $E \in \mathfrak{E}^{*\#}$, then $C_{\mathfrak{F}}(E) \subseteq \mathfrak{M}^G$, and so $[C_{\mathfrak{F}}(E), \mathfrak{E}^*] \subseteq \mathfrak{E}^* \cap \mathfrak{F} = 1$. Hence, $C_{\mathfrak{F}}(E) = C_{\mathfrak{F}}(\mathfrak{E}^*)$ for all $E \in \mathfrak{E}^{*\#}$. Since $\mathfrak{M}$ is solvable, we conclude that $|\mathfrak{E}_0^*| = 2$. Thus, $\mathfrak{E}_0^*$ stabilizes the chain $\mathfrak{F} \supset \mathfrak{E} \supset 1$, and so $|\mathfrak{F}: C_{\mathfrak{F}}(\mathfrak{E}_0^*)| \leq 4$. Choose $J \in \mathfrak{E}^* - \mathfrak{E}_0^*$. Then $\mathfrak{F}/\mathfrak{E}$ is a free $F_2\langle J\rangle$-module, and so $|\mathfrak{F}: \mathfrak{E}| = 2^{2f}$, $f \leq 2$.

Case 2. $\mathfrak{E}^* \cap \mathfrak{F} \neq 1$.

In this case, we get $\mathfrak{E}^* \cap \mathfrak{F} = \mathfrak{E}_0^*$, and $\mathfrak{F} \subseteq \boldsymbol{C}(\mathfrak{E}_0^*) \subseteq \mathfrak{M}^G$. Hence, $[\mathfrak{F}, \mathfrak{E}^*] = \mathfrak{E}_0^*$, and $|\mathfrak{F}: \mathfrak{E}| = 4$.

The proof is complete.

The next lemma is very important, and is a repetition of an earlier argument, with slight alterations.

LEMMA 20.17. *If $\mathfrak{P}$ is a subgroup of $\mathfrak{M}$ of odd prime order then $\mathfrak{F}\mathfrak{P} \in \mathscr{M}^*$.*

Proof. Suppose false. Let

$$\mathscr{S}_0 = \{\mathfrak{S} \mid \mathfrak{F}\mathfrak{P} \subseteq \mathfrak{S} \subseteq \mathfrak{G}, \mathfrak{S} \not\subseteq \mathfrak{M}, \mathfrak{S} \text{ is solvable}\}.$$

Thus, $\mathscr{S}_0 \neq \emptyset$. For each $\mathfrak{S}$ in $\mathscr{S}_0$, let $t(\mathfrak{S}) = |\mathfrak{S} \cap \mathfrak{M}|_2$, and let $t = \max t(\mathfrak{S})$, where $\mathfrak{S}$ ranges over $\mathscr{S}_0$. Set

$$\mathscr{S} = \{\mathfrak{S} \in \mathscr{S}_0 \mid t(\mathfrak{S}) = t\}.$$

Choose $\mathfrak{S}$ in $\mathscr{S}$ of minimal order. Let $\mathfrak{T}_0$ be a S_2-subgroup of $\mathfrak{S} \cap \mathfrak{M}$, and let $\mathfrak{T}_1$ be a S_2-subgroup of $\mathfrak{S}$ which contains $\mathfrak{T}_0$. Since $S_{2'}$-subgroups of $\mathfrak{S}$ are cyclic, it follows that $\mathfrak{T}_1\mathfrak{P} = \mathfrak{S}_1$ is a group.

Case 1. $\mathfrak{S} = \mathfrak{T}_1\mathfrak{P}$.

In this case, $\mathfrak{T}_1 \not\subseteq \mathfrak{M}$, so $\mathfrak{T}_0 \subset \mathfrak{T}_1$. We assume without loss of generality that $\mathfrak{T}_0 \subset \mathfrak{T}$. Let $\mathfrak{L} = \mathfrak{T}\mathfrak{P}$, $\mathfrak{T}^0 = \boldsymbol{O}_2(\mathfrak{L})$. If $\mathfrak{T}^0 \not\subseteq \mathfrak{T}_0$, then set $\mathfrak{T}_0 \cap \boldsymbol{O}_2(\mathfrak{S}) = \mathfrak{T}_{00}$. We find that $\boldsymbol{N}(\mathfrak{T}_{00}) \not\subseteq \mathfrak{M}$, and $|\boldsymbol{N}(\mathfrak{T}_{00}) \cap \mathfrak{M}|_2 > t$, against the definition of t. So $\mathfrak{T}^0 \subseteq \mathfrak{T}_0$. Since $\mathfrak{T}\mathfrak{P} \subseteq \boldsymbol{N}(\mathfrak{T}^0)$, we have $\boldsymbol{N}(\mathfrak{T}^0) \subseteq \mathfrak{M}$. Since $\mathfrak{T}^0 = \mathfrak{T}_0 \cap \boldsymbol{O}_2(\mathfrak{S})$, we conclude that $\mathfrak{S} \subseteq \mathfrak{M}$, which is false.

Case 2. $\mathfrak{S} \neq \mathfrak{T}_1\mathfrak{P}$.

By minimality of $\mathfrak{S}$, we have $\mathfrak{T}_1 \subseteq \mathfrak{M}$.

Let $\mathfrak{Q}$ be a S_p-subgroup of $\mathfrak{S}$ which contains $\mathfrak{P}$.

Case 2(a). $\mathfrak{S} = \mathfrak{T}_1\mathfrak{Q}$.

By maximality of t, we have $\boldsymbol{O}_2(\mathfrak{S}) \in \boldsymbol{\mathsf{N}}^*_{\mathfrak{M}}(\mathfrak{P}; 2)$, and so $\boldsymbol{N}(\boldsymbol{O}_2(\mathfrak{S})) \subseteq \mathfrak{M}$, which gives $\mathfrak{S} \subseteq \mathfrak{M}$, a contradiction.

Case 2(b). $\mathfrak{S} \neq \mathfrak{T}_1\mathfrak{Q}$.

By minimality of $\mathfrak{S}$, we have $\mathfrak{T}_1\mathfrak{Q} \subseteq \mathfrak{M}$. Thus, $\mathfrak{S} = \mathfrak{T}_1\mathfrak{Q}\mathfrak{R}$, where $\mathfrak{R}$ is a cyclic r-group centralizing $\mathfrak{Q}$, and r is an odd prime $\neq p$.

Let $\mathfrak{K} = O_{p'}(\mathfrak{S})$, and let $\mathfrak{K}_2 = \mathfrak{K} \cap \mathfrak{T}_1$, so that $\mathfrak{K} = \mathfrak{K}_0\mathfrak{R}$. First, suppose $r \neq 3$. In this case, there is $\mathfrak{X}$ char $\mathfrak{K}_0$, $\mathfrak{X} \neq 1$, with $\mathfrak{X} \lhd \mathfrak{K}$, whence $\mathfrak{X} \lhd \mathfrak{S}$. By maximality of t, we have $\mathfrak{K}_2 \in \mathsf{N}^*_{\mathfrak{M}}(\mathfrak{P}; 2)$, and so $N_{\mathfrak{M}}(\mathfrak{X}) \supseteq \mathfrak{T}\mathfrak{P}$, whence $\mathfrak{S} \subseteq \mathfrak{M}$. This is false, and so $r = 3$.

Let $\mathfrak{L} = O_{3'}(\mathfrak{S})$, $\mathfrak{L}_2 = \mathfrak{L} \cap \mathfrak{T}_1$, so that $\mathfrak{L} = \mathfrak{L}_2\mathfrak{Q}$. Since $r = 3$ we have $p > 3$, and so there is $\mathfrak{Y}$ char $\mathfrak{L}_2$, $\mathfrak{Y} \neq 1$, with $\mathfrak{Y} \lhd \mathfrak{L}$, whence $\mathfrak{Y} \lhd \mathfrak{S}$. Let $\mathfrak{S}^* = N(\mathfrak{Y})$. Thus, $\mathfrak{S}^* \in \mathscr{S}$, so by what we have already shown, it follows that $\mathfrak{S}^* \cap \mathfrak{M}$ contains a S_2-subgroup of $\mathfrak{S}^*$, whence $\mathfrak{T}_1$ is a S_2-subgroup of $\mathfrak{M}$, and so $\mathfrak{S} \subseteq \mathfrak{M}$. The proof is complete.

LEMMA 20.18. *If* $|\mathfrak{E}| = 4$, *then* $\mathfrak{T}/O_2(\mathfrak{N}_1)$ *is cyclic.*

Proof. $\mathfrak{T}/O_2(\mathfrak{N}_1)$ acts faithfully on $\bar{\mathfrak{Q}} = O_2(\mathfrak{N}_1)\mathfrak{Q}/O_2(\mathfrak{N}_1)$, where $\mathfrak{Q}$ is a $S_{2'}$-subgroup of $\mathfrak{N}_1$. Thus, the lemma holds if $|\mathfrak{Q}| = 3$ or 5. Suppose $|\mathfrak{Q}| = 15$.

Let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}_1$. Thus, $C(\mathfrak{U}) = O_2(\mathfrak{N}_1)$. Since either $N(Z(J(\mathfrak{T}))) \subseteq \mathfrak{M}$ or $N(Z(J_1(\mathfrak{T}))) \subseteq \mathfrak{M}$, it follows that $J_1(\mathfrak{T}) \not\subseteq O_2(\mathfrak{N}_1)$. This in turn forces $|\mathfrak{U}| = 2^4$. Since elements of GL(4, 2) of order 15 are non real, it follows that no element to $\mathfrak{T}/O_2(\mathfrak{N}_1)$ inverts $\mathfrak{Q}$, whence $\mathfrak{T}/O_2(\mathfrak{N}_1)$ is cyclic (of order 4). The proof is complete.

LEMMA 20.19. *If* $|\mathfrak{E}| = 4$ *and* $|\mathfrak{F}| = 64$, *then* $\mathfrak{V}_2 \lhd \mathfrak{N}_1$, *where* $\mathfrak{V}_2 = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}); \mathfrak{T})$.

Proof. Let $\mathfrak{K} = O_2(\mathfrak{N}_1)$, and let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}_1$, so that $\mathfrak{K} = C(\mathfrak{U})$.

Suppose $G \in \mathfrak{G}$ and $\mathfrak{F}^G = \mathfrak{F}^* \subseteq \mathfrak{T}$. We will show that $\mathfrak{F}^* \subseteq \mathfrak{K}$. Let $\mathfrak{F}_1^G = \mathfrak{F}_1^* = \mathfrak{F}^* \cap \mathfrak{K}$, and suppose that $\mathfrak{F}_1^* \subset \mathfrak{F}^*$. By Lemma 20.18, we have $|\mathfrak{F}^*: \mathfrak{F}_1^*| = 2$. Hence, $\mathfrak{U} \subseteq C(\mathfrak{F}_1^*) \subseteq \mathfrak{M}^G$, and so $[\mathfrak{U}, \mathfrak{F}^*] \subseteq \mathfrak{F}^*$. Since $\mathfrak{U} \subseteq C(\mathfrak{F}_1^*)$, Lemma 20.13 implies that $1 \subset [\mathfrak{U}, \mathfrak{F}^*] \subseteq \mathfrak{E}^G$. Hence, $\mathfrak{E}^G \cap Z(\mathfrak{T}) \neq 1$. Since $\mathfrak{E}$ is a T.I. set in $\mathfrak{G}$, and since $\mathfrak{E} \supseteq \Omega_1(Z(\mathfrak{T})) = \mathfrak{Z}$, we have $\mathfrak{E} = \mathfrak{E}^G$, so that $\mathfrak{F} = \mathfrak{F}^G$.

Let $\mathfrak{F}_0 = \mathfrak{F} \cap \mathfrak{K}$, so that $\mathfrak{F} = \mathfrak{F}_0 \times \langle F \rangle$, and F inverts a subgroup $\mathfrak{P}$ of $\mathfrak{N}_1$ of odd prime order p. Let $\mathfrak{K}_1 = [\mathfrak{P}, \mathfrak{K}]$, so that $\mathfrak{K}_1 = [\mathfrak{K}_1, F] \times [\mathfrak{K}_1, F]^P$, where $\mathfrak{P} = \langle P \rangle$, and $[\mathfrak{K}_1, F] \subseteq \mathfrak{F}_0$. Let $\mathfrak{K}_2 = C_{\mathfrak{K}}(\mathfrak{P})$. A standard argument shows that $\mathfrak{F}_0 = \mathfrak{F}_0 \cap \mathfrak{K}_2 \times \mathfrak{F}_0 \cap \mathfrak{K}_1$, that $\mathfrak{F}_0 \cap \mathfrak{K}_1 = [\mathfrak{K}_1, F]$, and that $\mathfrak{F}_0 \subseteq C(\mathfrak{K}_1)$. Since each element of $[\mathfrak{K}_1, F]^P$ centralizes the hyperplane $\mathfrak{F}_0$ of $\mathfrak{F}$, Lemma 20.13 implies that $[\mathfrak{K}_1, F] \subseteq \mathfrak{E}$. Since $2 \in \pi_4$, we conclude that $[\mathfrak{K}_1, F] = \mathfrak{E}$, $|\mathfrak{K}_1| = 2^4$. Since $1 \subset \mathfrak{F}_0 \cap \mathfrak{K}_2 \subseteq C_{\mathfrak{K}_2}(\mathfrak{K}_1) \lhd \mathfrak{T}\mathfrak{P}$, it follows that $C(\mathfrak{P}) \cap Z(\mathfrak{T}) \neq 1$, the desired contradiction. The proof is complete.

LEMMA 20.20. *If* $|\mathfrak{E}| = 4$ *and* $|\mathfrak{F}| = 64$, *then the following hold*:

(a) $\mathfrak{F}$ *is not a* T.I. *set in* $\mathfrak{G}$.
(b) $|\mathfrak{M}|_{2'} = 5$.
(c) $|\mathfrak{N}_1|_{2'} = 5$ *or* 15.

Proof. Let $\mathfrak{C} = \boldsymbol{C}(\mathfrak{F})$, $\mathfrak{D}/\mathfrak{C} = \boldsymbol{O}_2(\mathfrak{M}/\mathfrak{C})$, $\mathfrak{T}_0 = \mathfrak{T} \cap \mathfrak{D}$. Since $N_{\mathfrak{M}}(\mathfrak{T}_0) \supset \mathfrak{T}$, it follows from Lemma 20.19 that $\mathfrak{V}_2 \not\subseteq \mathfrak{T}_0$. Choose G in $\mathfrak{G}$ such that $\mathfrak{F}^G \subseteq \mathfrak{T}$, $\mathfrak{F}^G \not\subseteq \mathfrak{T}_0$. If (a) is false, then since $\mathfrak{F} \neq \mathfrak{F}^G$, we conclude that $[\boldsymbol{C}_{\mathfrak{F}}(X), \mathfrak{F}] \subseteq \mathfrak{F} \cap \mathfrak{F}^G = 1$ for all $X \in \mathfrak{F}^{G\#}$. Since $|\mathfrak{F}| = |\mathfrak{F}^G|$, this forces $\mathfrak{F} \subseteq \boldsymbol{C}(\mathfrak{F}^G)$, against $\mathfrak{F}^G \not\subseteq \mathfrak{D}$. So (a) holds.

Let $\mathfrak{P}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$, and let $\mathfrak{P}_0 = \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{F}) = \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{F}/\mathfrak{C})$. If $\mathfrak{P}_0 \neq 1$, then Lemma 20.17 implies that $\mathfrak{F}$ is a T.I. set in $\mathfrak{G}$, against (a). So $\mathfrak{P}_0 = 1$. Since $\mathfrak{P}\mathfrak{F}/\mathfrak{C}$ is a Frobenius group, and $\mathfrak{F}/\mathfrak{C}$ is a chief factor of $\mathfrak{M}$, we have $|\mathfrak{P}| = 5$ or 15. If $|\mathfrak{P}| = 15$, then $\mathfrak{T}/\boldsymbol{O}_2(\mathfrak{M})$ is cyclic of order 4. In this case, let $\mathfrak{T}^0$ be the unique subgroup of $\mathfrak{T}$ of index 2 which contains $\boldsymbol{O}_2(\mathfrak{M})$. Then $\mathfrak{T}^0 \supseteq \Omega_1(\mathfrak{T})$, and so $\mathfrak{V}_2 \subseteq \mathfrak{T}^0$. Hence, $N_{\mathfrak{M}}(\mathfrak{V}_2) \supset \mathfrak{T}$, against Lemma 20.19. So (b) holds.

To prove (c), it suffices to show that a $S_{2'}$-subgroup $\mathfrak{Q}$ of $\mathfrak{N}_1$ is not of order 3. Suppose by way of contradiction that $|\mathfrak{Q}| = 3$.

Let $\mathfrak{U}$ be minimal normal subgroup of $\mathfrak{N}_1$, so that $|\mathfrak{U}| = 4$, and $\mathfrak{U} \cap \boldsymbol{Z}(\mathfrak{T}) = \mathfrak{U}_0$ is of order 2. Let $\mathfrak{W} = \mathfrak{U}^{\mathfrak{M}} = \mathfrak{U}^{\mathfrak{P}}$. Let $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{M})$, $\mathfrak{K} = \boldsymbol{O}_2(\mathfrak{N}_1)$, so that $\mathfrak{M}/\mathfrak{H}$ is a Frobenius group of order 10 or 20 and $\mathfrak{N}_1/\mathfrak{K} \cong \Sigma_3$.

We argue that $\mathfrak{W}' = 1$. Namely, $\mathfrak{U} \lhd \mathfrak{H}$, and so if $\mathfrak{P} = \langle P \rangle$, then $\mathfrak{U}^{P^i} \lhd \mathfrak{H}$ for all i. Suppose $[\mathfrak{U}, \mathfrak{U}^{P^i}] \neq 1$. Then $\mathfrak{U}^{P^i} = \mathfrak{U}_0 \times \langle U \rangle$, and $[\mathfrak{U}, U] = \mathfrak{U}_0$. Hence, $\mathfrak{T} = \mathfrak{K}\langle U \rangle$, and $[U, \mathfrak{K} \cap \mathfrak{H}] \subseteq [\mathfrak{U}, U] = \mathfrak{U}_0$. Since $\mathfrak{K}/\mathfrak{K} \cap \mathfrak{H}$ is cyclic, it follows that $[\mathfrak{K}', U] \subseteq \mathfrak{U}_0$. We assume without loss of generality that U inverts $\mathfrak{Q}$.

Case 1. $\mathfrak{K}' = 1$.

Since $\boldsymbol{J}_1(\mathfrak{T}) \not\subseteq \mathfrak{K}$, and since $\boldsymbol{Z}(\mathfrak{T}) \cap \boldsymbol{C}(\mathfrak{Q}) = 1$, it follows that $\mathfrak{K}$ is generated by 4 elements. Hence every abelian subgroup of $\mathfrak{T}$ is generated by 5 elements, against $|\mathfrak{F}| = 64$.

Case 2. $\mathfrak{K}' \neq 1$.

Since $[\mathfrak{K}', U] \subseteq \mathfrak{U}_0$, it follows that $\mathfrak{K}' = \mathfrak{U} \times \boldsymbol{C}_{\mathfrak{K}'}(\mathfrak{Q})$, and U centralizes $\boldsymbol{C}_{\mathfrak{K}'}(\mathfrak{Q})$. Since $\boldsymbol{Z}(\mathfrak{T}) \cap \boldsymbol{C}(\mathfrak{P}) = 1$, $\boldsymbol{C}_{\mathfrak{K}'}(\mathfrak{Q})$ contains no non identity characteristic subgroup of $\mathfrak{K}'$, and so $\mathfrak{K}'$ is elementary abelian.

Let $\mathfrak{K}_0 = \boldsymbol{C}_{\mathfrak{K}}(\mathfrak{Q})$, $\mathfrak{K}^0 = \mathfrak{K}_0\mathfrak{U} = \mathfrak{K}_0 \times \mathfrak{U}$. Since $\mathfrak{K}^0 \supseteq \mathfrak{K}'$, and $\mathfrak{K}^0$ admits U, we have $\mathfrak{K}^0 \lhd \mathfrak{T}$. Since $\mathfrak{K}_0$ contains no non identity characteristic subgroup of $\mathfrak{K}^0$, it follows that $\mathfrak{K}_0$ is elementary abelian.

Let $\bar{\mathfrak{K}} = \mathfrak{K}/\mathfrak{K}^0$, $\bar{\mathfrak{R}} = (\mathfrak{H} \cap \mathfrak{K})\mathfrak{K}^0/\mathfrak{K}^0$. Thus, $\bar{\mathfrak{K}}/\bar{\mathfrak{R}}$ is cyclic of order at most 4, and $[\bar{\mathfrak{R}}, U] = 1$. Since $\boldsymbol{C}_{\bar{\mathfrak{K}}}(\mathfrak{Q}) = 1$, it follows that one of the following holds:

(a) $\bar{\mathfrak{K}}$ is a four-group.

(b) $\bar{\mathfrak{K}}$ is the direct product of 2 cyclic groups of order 4.

Since $\mathfrak{K}^0 = \mathfrak{K}_0\mathfrak{U}$, it follows that $[\mathfrak{K}, \mathfrak{K}^0] \subseteq \mathfrak{U}$. Hence, if $K \in \mathfrak{K}$, then $C_{\mathfrak{K}^0}(K)$ is of index at most 4 in $\mathfrak{K}^0$. Since $\mathfrak{K} = \langle \mathfrak{K}^0, K_1, K_2\rangle$ for suitable K_1, K_2, it follows that $|\mathfrak{K}^0| \leqq 64$, whence $|\mathfrak{K}| \leqq 64.4^2 = 2^{10}$. So $|\mathfrak{T}| \leqq 2^{11}$. If $|\mathfrak{T}| \leqq 2^{10}$, then $|\mathfrak{H}| \leqq 2^9$, and since $|\mathfrak{F}| = 2^6$, it follows that $[\mathfrak{H}, \mathfrak{P}] = [\mathfrak{F}, \mathfrak{P}] \lhd \mathfrak{M}$, against $|\mathfrak{Z}| \leqq 4$. So $|\mathfrak{T}| = 2^{11}$, and this forces $|\mathfrak{K}_0| = 2^4$. Since $\mathfrak{K}_0 \cap \boldsymbol{Z}(\mathfrak{K}) = 1$, we may identify $\mathfrak{K}_0$ with a subgroup of $\mathrm{Hom}(\bar{\mathfrak{K}}, \mathfrak{U})$. Since $\mathrm{Hom}(\bar{\mathfrak{K}}, \mathfrak{U}) \cong \mathrm{Hom}(\bar{\mathfrak{K}}/\boldsymbol{D}(\bar{\mathfrak{K}}), \mathfrak{U})$ is of order 2^4, we get $\mathfrak{K}_0 \cong \mathrm{Hom}(\bar{\mathfrak{K}}, \mathfrak{U})$. This is false, since $\mathfrak{Q}$ centralizes $\mathfrak{K}_0$ and does not centralize $\mathrm{Hom}(\bar{\mathfrak{K}}, \mathfrak{U})$. We conclude that $\mathfrak{W}' = 1$.

Since $[\mathfrak{T}, \mathfrak{U}] = \mathfrak{U}_0$, we have $[\mathfrak{H}, \mathfrak{W}] \subseteq \mathfrak{U}_0$. Since $|\mathfrak{Z}| = 4$, and since $\mathfrak{W} \supset \mathfrak{U}$, we have $[\mathfrak{H}, \mathfrak{W}] = \mathfrak{U}_0$. Thus, $\mathfrak{W}_1 = [\mathfrak{W}, \mathfrak{P}]$ is of order 2^4, and $\mathfrak{W}^* = \mathfrak{W}_1\mathfrak{U}_0 = \mathfrak{W}_1 \times \mathfrak{U}_0 \lhd \mathfrak{M}$, $|\mathfrak{W}^*| = 2^5$.

Since $\mathfrak{N}_1 = \boldsymbol{N}(\mathfrak{V}_1)$, we can choose G in $\mathfrak{G}$ such that $\mathfrak{E}^G = \mathfrak{E}^* \subseteq \mathfrak{T}$, $\mathfrak{E}^* \not\subseteq \mathfrak{H}$. Thus, $\mathfrak{E}^* \cap \mathfrak{H} = \mathfrak{E}_1^G = \mathfrak{E}_1^*$ is of order 2, and since $[\mathfrak{W}^*, \mathfrak{E}_1^*] \subseteq \mathfrak{U}_0$, we have $|\mathfrak{W}^*: C_{\mathfrak{W}^*}(\mathfrak{E}_1^*)| \leqq 2$. Let $\mathfrak{W}_0^* = C_{\mathfrak{W}^*}(\mathfrak{E}_1^*)$. Choose $\mathfrak{E} \in \mathfrak{E}^* - \mathfrak{E}_1^*$, so that $\mathfrak{W}^*/\mathfrak{U}_0$ is a free $F_2\langle E\rangle$-module. Hence, $[\mathfrak{W}_0^*, E] \neq 1$. Since $\mathfrak{W}_0^* \subseteq \mathfrak{M}^G$, we conclude that $[\mathfrak{W}_0^*, E] = \mathfrak{E}_1^*$. Hence, $\mathfrak{W}_0^* = \mathfrak{W}^*$, and so $[\mathfrak{W}^*, E] = \mathfrak{E}_1^*$. This is false, since $\mathfrak{W}^*/\mathfrak{U}_0$ is a free $F_2\langle E\rangle$-module of order 2^4. The proof is complete.

We now complete the analysis of the case $|\mathfrak{E}| = 4$ and $|\mathfrak{F}| = 64$. Let $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{M})$, $\mathfrak{K} = \boldsymbol{O}_2(\mathfrak{N}_1)$, and let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}_1$. Let $\mathfrak{P}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$, $\mathfrak{Q}$ a $S_{2'}$-subgroup of $\mathfrak{N}_1$. Choose $T \in \mathfrak{T} - \mathfrak{K}$, $T^2 \in \mathfrak{K}$, so that $\mathfrak{U}_1 = C_{\mathfrak{U}}(T)$ is a four-group, and $|\mathfrak{U}| = 2^4$. It is important to show that

$$\text{(20.2)} \qquad \mathfrak{U}_1 \lhd \mathfrak{M}, \qquad \mathfrak{P} \subseteq C(\mathfrak{U}_1).$$

Suppose (20.2) is false. In this case, $\mathfrak{U}_1 \not\subseteq \boldsymbol{Z}(\mathfrak{T})$, and so $\mathfrak{T}/\mathfrak{K}$ is cyclic of order 4, and $\mathfrak{U}_0 = \mathfrak{U}_1 \cap \boldsymbol{Z}(\mathfrak{T})$ is of order 2. Let $\mathfrak{T}_0 = \langle T, \mathfrak{K}\rangle$. Then $\mathfrak{U}_1 \subseteq \boldsymbol{Z}(\mathfrak{T}_0)$, and $\mathfrak{T}_0 \supseteq \Omega_1(\mathfrak{T})$. Since $\mathfrak{U}_1 \lhd \mathfrak{T}$, we have $\mathfrak{U}_1 \lhd \mathfrak{H}$, and so $\mathfrak{W} = \mathfrak{U}_1^{\mathfrak{M}} = \mathfrak{U}_1^{\mathfrak{P}} \lhd \mathfrak{M}$. Since $\mathfrak{U}_1 \subseteq \boldsymbol{Z}(\Omega_1(\mathfrak{T}))$, we see that $\mathfrak{W}' = 1$.

Let $\mathfrak{W}_1 = [\mathfrak{W}, \mathfrak{P}]$, so that $|\mathfrak{W}_1| = 2^4$ and $\mathfrak{W}^* = \mathfrak{W}_1\mathfrak{U}_0 = \mathfrak{W}_1 \times \mathfrak{U}_0 \lhd \mathfrak{M}$, $|\mathfrak{W}^*| = 2^5$. The argument of Lemma 20.20 can be applied to yield a contradiction. So (20.2) holds.

Let $\mathfrak{U}_1 \subset \mathfrak{U}^1 \subset \mathfrak{U}$, with $\mathfrak{U}^1 \lhd \mathfrak{T}$, so that $|\mathfrak{U}^1| = 2^3$. We argue that $\mathfrak{K} = C(\mathfrak{U}^1)$. In any case, $\mathfrak{K} = C_{\mathfrak{N}_1}(\mathfrak{U}^1)$, and since $N(\mathfrak{K}) = \mathfrak{N}_1$, it follows that $\mathfrak{K}$ is a S_2-subgroup of $C(\mathfrak{U}^1)$. Since $\mathfrak{U}^1 \supset \mathfrak{U}_1$ and $\mathfrak{U}_1 \lhd \mathfrak{M}$, we have $C(\mathfrak{U}^1) \subseteq \mathfrak{M}$. Thus, if $C(\mathfrak{U}^1) \supset \mathfrak{K}$, then $C(\mathfrak{U}^1) \supseteq \mathfrak{P}$, so that $C(\mathfrak{U}^1) = \mathfrak{K}\mathfrak{P}$. Since $|\mathfrak{P}| = 5$, there is $\mathfrak{X} \in \{\boldsymbol{Z}(\mathfrak{K}), \boldsymbol{J}(\mathfrak{K})\}$ with $\mathfrak{X} \lhd \mathfrak{K}\mathfrak{P}$. Thus, $\mathfrak{P} \subseteq N(\mathfrak{X}) = \mathfrak{N}_1$, which is false, since $\mathfrak{M} \cap \mathfrak{N}_1 = \mathfrak{T}$. So $[\mathfrak{P}, \mathfrak{U}^1] \neq 1$, and $C(\mathfrak{U}^1) = \mathfrak{K}$.

Let $\mathfrak{X} = \mathfrak{U}^{1\mathfrak{M}} = \mathfrak{U}^{1\mathfrak{P}}$. Since $\mathfrak{U}^1 \lhd \mathfrak{H}$, we have $\mathfrak{U}^1 = \langle U\rangle \times \mathfrak{U}_1$, and

$\mathfrak{X} = \langle \mathfrak{U}_1, U^{P^i} \mid 0 \leqq i \leqq 4\rangle$. Also, $\mathfrak{U}_1 \subseteqq \boldsymbol{Z}(\mathfrak{X})$, since $\mathfrak{X} \subseteqq \Omega_1(\mathfrak{T})$. We argue that $\mathfrak{X}' = 1$. Suppose false. Then $[\mathfrak{U}^1, U^{P^i}] \neq 1$ for some i. Set $V = U^{P^i}$. Thus, $V \in \mathfrak{K}$, and $[\mathfrak{H} \cap \mathfrak{K}, V] \subseteqq [\mathfrak{H}, V] \subseteqq \mathfrak{U}_1$. Since $\mathfrak{U}_1 \lhd \mathfrak{M}$, it follows that $|\mathfrak{Q}| = 5$. We may assume that V inverts $\mathfrak{Q}$. Since $\mathfrak{K}/\mathfrak{K} \cap \mathfrak{H}$ is cyclic, it follows that $\mathfrak{Q}$ centralizes $\mathfrak{K}/\mathfrak{U}$, and since $\boldsymbol{Z}(\mathfrak{T}) \cap \boldsymbol{C}(\mathfrak{Q}) = 1$, we get $\mathfrak{K} = \mathfrak{U}$. This violates $|\mathfrak{F}| = 64$. So $\mathfrak{X}$ is abelian.

Let $\mathfrak{M}_0 = \boldsymbol{C}(\mathfrak{U}_1)$ so that $|\mathfrak{M} : \mathfrak{M}_0| \leqq 2$. Let $\mathfrak{H}_0 = \boldsymbol{O}_2(\mathfrak{M}_0) = \mathfrak{M}_0 \cap \mathfrak{H}$. Thus, $\mathfrak{H}_0$ stabilizes the chain $\mathfrak{U}^1 \supset \mathfrak{U}_1 \supset 1$, and so $\mathfrak{H}_0$ stabilizes the chain $\mathfrak{W} \supset \mathfrak{U}_1 \supset 1$, whence $\boldsymbol{D}(\mathfrak{H}_0) \subseteqq \boldsymbol{C}(\mathfrak{W})$. Since $\mathfrak{H}_0 \cap \mathfrak{K}$ centralizes $\mathfrak{U}^1$, and since $\mathfrak{H}_0/\mathfrak{H}_0 \cap \mathfrak{K}$ is cyclic, it follows that $|\mathfrak{H}_0 : \boldsymbol{C}_{\mathfrak{H}_0}(\mathfrak{U}^1)| \leqq 2$. Let $\mathfrak{U}(1) = [\mathfrak{U}^1, \mathfrak{H}_0]$, so that $|\mathfrak{U}(1)| = 2$, $\mathfrak{U}(1) \subset \mathfrak{U}_1 \subseteqq \boldsymbol{C}(\mathfrak{P})$. Since $\mathfrak{W} = \mathfrak{U}^{1\mathfrak{P}}$, it follows that $[\mathfrak{W}, \mathfrak{H}_0] = \mathfrak{U}(1)$.

Now choose G in $\mathfrak{G}$ such that $\mathfrak{E}^G \subseteqq \mathfrak{T}$, $\mathfrak{E}^G \nsubseteqq \mathfrak{H}$; G exists since $\mathfrak{N}_1 = \boldsymbol{N}(\mathfrak{B}_1)$. Since $\mathfrak{T}/\mathfrak{H}$ is cyclic, $\mathfrak{E}^G \cap \mathfrak{H} \neq 1$. Since $\mathfrak{E}$ is a T.I. set in $\mathfrak{G}$, we have $\mathfrak{E}^G \subseteqq \mathfrak{M}_0$, and so $\mathfrak{E}^G \cap \mathfrak{H} = \mathfrak{E}^G \cap \mathfrak{H}_0 = \langle E_0 \rangle$. Thus, $[E_0, \mathfrak{W}] \subseteqq \mathfrak{U}(1)$, and so $|\mathfrak{W} : \boldsymbol{C}_{\mathfrak{W}}(E_0)| \leqq 2$. Choose $E \in \mathfrak{E}^G - \langle E_0 \rangle$. We may assume that E inverts $\mathfrak{P}$. Since $\mathfrak{U}^1 \nsubseteqq \boldsymbol{C}(\mathfrak{P})$, and $\mathfrak{W}$ is elementary of order at most 2^7, it follows that $[\mathfrak{W}, \mathfrak{P}] = \mathfrak{W}_0$ is of order 2^4 and is a free $F_2\langle E \rangle$-module. Hence, E does not centralize $\mathfrak{W}_0 \cap \mathfrak{W}^*$ where $\mathfrak{W}^* = \boldsymbol{C}_{\mathfrak{W}}(E_0)$, and so $[\mathfrak{W}_0 \cap \mathfrak{W}^*, E] = \langle E_0 \rangle$. Since $E_0 \in \mathfrak{W}$, we have $\mathfrak{W}^* = \mathfrak{W}$, and so $[\mathfrak{W}, E] = \langle E_0 \rangle$. This is false, since $\mathfrak{W}_0$ is a free $F_0\langle E \rangle$-module of order 2^4. This completes a proof that if $|\mathfrak{E}| = 4$, then $|\mathfrak{F}| \neq 64$.

LEMMA 20.21. *Suppose* $|\mathfrak{E}| = 4$ *and* $|\mathfrak{F}| = 16$. *Then the following hold*:

(a) $\mathfrak{F}$ *is not a* T.I. *set in* $\mathfrak{G}$.
(b) $|\mathfrak{M}|_{2'} = 3$.
(c) $|\mathfrak{N}_1|_{2'} = 5$.

Proof. Suppose (a) is false. In this case, if $G \in \mathfrak{G} - \mathfrak{M}$ and $\mathfrak{F}^G \subseteqq \mathfrak{T}$, then $[\boldsymbol{C}_{\mathfrak{F}}(F), \mathfrak{F}^G] \subseteqq \mathfrak{F} \cap \mathfrak{F}^G = 1$, for all $F \in \mathfrak{F}^{G\sharp}$, and so $[\mathfrak{F}, \mathfrak{F}^G] = 1$. Set $\mathfrak{B}_2 = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}); \mathfrak{T})$. Thus, $\mathfrak{M} = \boldsymbol{C}(\mathfrak{F}) \cdot \boldsymbol{N}_{\mathfrak{M}}(\mathfrak{B}_2)$ and so $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{B}_2) \supset \mathfrak{T}$, whence $\boldsymbol{N}(\mathfrak{B}_2) \subseteqq \mathfrak{M}$.

Let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}_1$, and let $\mathfrak{K} = \boldsymbol{O}_2(\mathfrak{N}_1)$. Since $\boldsymbol{N}_{\mathfrak{N}_1}(\mathfrak{B}_2) = \mathfrak{T}$, there is G in $\mathfrak{G}$ such that $\mathfrak{F}^G = \mathfrak{F}^* \subseteqq \mathfrak{T}$, $\mathfrak{F}^* \nsubseteqq \mathfrak{K}$.

Since $\mathfrak{F}$ is a T.I. set in $\mathfrak{G}$, and $|\mathfrak{F}| > 2$, if follows that $\mathfrak{F}^* \cap \mathfrak{U} \neq 1$. Hence, $\mathfrak{K} \subseteqq \boldsymbol{C}(\mathfrak{F}^* \cap \mathfrak{U}) \subseteqq \boldsymbol{N}(\mathfrak{F}^*) = \mathfrak{M}^G$. This implies that $\mathfrak{F}^* \cap \mathfrak{K} = \mathfrak{F}_1^*$ is of index 2 in $\mathfrak{F}^*$.

Let $\mathfrak{Q}$ be a subgroup of $\mathfrak{N}_1$ of odd prime order such that $\mathfrak{F}^* = \mathfrak{F}_1^* \times \langle F \rangle$, where F inverts $\mathfrak{Q}$. Let $\mathfrak{K}_1 = [\mathfrak{K}, \mathfrak{Q}]$. Thus, $\mathfrak{K}_1 = [\mathfrak{K}_1 F] \times [\mathfrak{K}_1, F]^Q$, where $\langle Q \rangle = \mathfrak{Q}$. By a standard argument, $\mathfrak{F}^* = \mathfrak{F}^* \cap \mathfrak{K}_2 \times \mathfrak{F}^* \cap \mathfrak{K}_1$, where $\mathfrak{K}_2 = \boldsymbol{C}_{\mathfrak{K}}(\mathfrak{Q})$. Since $\mathfrak{Q} \nsubseteqq \boldsymbol{N}(\mathfrak{F}^*)$, we have $\mathfrak{F}^* \cap \mathfrak{K}_2 = 1$, and so $|\mathfrak{F}^*| = [\mathfrak{K}_1, F]$ is of order 2^3, $|\mathfrak{K}_1| = 2^6$. Hence, $|\mathfrak{Q}| = 3$. Since $C(\mathfrak{Q}) \cap Z(\mathfrak{T}) = 1$, $\mathfrak{K}_2$ acts faithfully on $\mathfrak{K}_1$. Since $|\mathfrak{N}_1|_{2'} = 3$ or 15, and since $C(\mathfrak{Q}_0) \cap Z(\mathfrak{T}) = 1$ for every non identity odd order subgroup $\mathfrak{Q}_0$ of $\mathfrak{N}_1$, it follows that $|\mathfrak{N}_1|_{2'} = 3$. Since $J_1(\mathfrak{T}) \not\subseteq \mathfrak{K}$, it follows that $\mathfrak{K}_2 \neq 1$. Thus, $\mathfrak{K} \in N^*(\mathfrak{Q}; 2)$ and $C_{\mathfrak{K}}(\mathfrak{Q}) \neq 1$. On the other hand, $3 \mid |\mathfrak{M}|$, and if $\mathfrak{S}$ is a S_3-subgroup of $\mathfrak{M}$, then $\mathfrak{E} \subseteq C(\mathfrak{S})$. This implies that $\mathfrak{S}$ is a S_3-subgroup of $\mathfrak{G}$. Let $\mathfrak{L} = \mathfrak{T}\mathfrak{P}$, where $\mathfrak{P} = \Omega_1(\mathfrak{S})$. Then $O_2(\mathfrak{L}) \in N^*(\mathfrak{P}; 2)$ and $O_2(\mathfrak{L}) \cap C(\mathfrak{P}) \neq 1$. By the transitivity theorem, we get that $\mathfrak{K}$ and $O_2(\mathfrak{L})$ are $\mathfrak{G}$-conjugate, hence are equal. This is false, since $N(O_2(\mathfrak{L})) \subseteq \mathfrak{M}$, $N(\mathfrak{K}) = \mathfrak{N}_1 \not\subseteq \mathfrak{M}$. So (a) holds.

Lemma 20.17 and (a) imply (b).

Since some element of $\mathscr{M}(\mathfrak{T})$ has order $> 3 \cdot |\mathfrak{T}|$, it follows that $|\mathfrak{N}_1|_{2'} = 5$ or 15. Suppose $|\mathfrak{N}_1|_{2'} = 15$. Since $J_1(\mathfrak{T}) \not\subseteq O_2(\mathfrak{N}_1)$, it follows that every minimal normal subgroup of $\mathfrak{N}_1$ has order 2^4, and so $\mathfrak{T}/O_2(\mathfrak{N}_1)$ is cyclic of order 4. Let $\mathfrak{T} \supset \mathfrak{T}_0 \supset O_2(\mathfrak{N}_1)$, and let $\mathfrak{Q}$ be a $S_{2'}$-subgroup of $\mathfrak{N}_1$. Then $\mathfrak{Q} = \mathfrak{Q}_3 \times \mathfrak{Q}_5$, where $|\mathfrak{Q}_p| = p$. Also, $\mathfrak{Q}_3$ centralizes $\mathfrak{T}_0/O_2(\mathfrak{N}_1)$, and $\mathfrak{T}_0 \in N^*(\mathfrak{Q}_3; 2)$. Let $\mathfrak{P}$ be a S_3-subgroup of $\mathfrak{M}$. Then $O_2(\mathfrak{M}) \in N^*(\mathfrak{P}; 2)$, and $C(\mathfrak{P}) \cap O_2(\mathfrak{M})$ contains a four-group. So $\mathfrak{P}$ is a S_3-subgroup of $\mathfrak{G}$, and $\mathfrak{T}_0$, $O_2(\mathfrak{M})$ are $\mathfrak{G}$-conjugate, hence are equal. This is false, and so (c) holds.

Let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}_1$. Thus, $|\mathfrak{U}| = 2^4$, $C(\mathfrak{U}) = \mathfrak{K} = O_2(\mathfrak{N}_1)$, and $\mathfrak{N}_1/\mathfrak{K}$ is a Frobenius group of order 10 or 20. Let $\mathfrak{T}_0/\mathfrak{K}$ be the subgroup of $\mathfrak{T}/\mathfrak{K}$ of order 2. Then $\mathfrak{U}_0 = C_{\mathfrak{U}}(\mathfrak{T}_0)$ is a four-group. Let $\mathfrak{Q}$ be a $S_{2'}$-subgroup of $\mathfrak{N}_1$, so that $\mathfrak{Q} = \langle Q \rangle$ is of order 5. Then set $\mathfrak{U}_i = \mathfrak{U}_0^{Q^i}$, so that

$$\mathfrak{U}^\# = \bigcup_{i=0}^{4} \mathfrak{U}_i^\# .$$

Suppose $I \in \mathfrak{U}^\#$. Set $\mathfrak{C} = C(I)$. We argue that $\mathfrak{C} \cap \mathfrak{N}_1$ is a S_2-subgroup of $\mathfrak{C}$. We may assume that $\mathfrak{T}_0 \subseteq \mathfrak{C} \cap \mathfrak{N}_1 \subseteq \mathfrak{T}$. If $\mathfrak{C} \cap \mathfrak{N}_1 = \mathfrak{T}$, we are done, so suppose that $\mathfrak{C} \cap \mathfrak{N}_1 = \mathfrak{T}_0 \subset \mathfrak{T}$. Since $J(\mathfrak{T}) \lhd \mathfrak{K}$, we have $J(\mathfrak{T}) = J(\mathfrak{T}_0) = J(\mathfrak{K}) \lhd \mathfrak{N}_1$, and so $N(\mathfrak{T}_0) \subseteq \mathfrak{N}_1$. So $\mathfrak{C} \cap \mathfrak{N}_1$ is a S_2-subgroup of $\mathfrak{C}$.

Let $\mathfrak{C}_0 = O_2(\mathfrak{C})$, $\mathfrak{T}^0 = \mathfrak{C} \cap \mathfrak{N}_1$. We argue that $\mathfrak{T}^0/\mathfrak{C}_0$ is cyclic. This is obvious if $\mathfrak{T}^0$ is a S_2-subgroup of $\mathfrak{G}$, since $\mathfrak{T}/O_2(\mathfrak{M})$ and $\mathfrak{T}/O_2(\mathfrak{N}_1)$ are cyclic. So suppose $\mathfrak{T}^0 = \mathfrak{T}_0 \subset \mathfrak{T}$ and $\mathfrak{C} = \mathfrak{T}^0 \cdot \mathfrak{R}$, where $|\mathfrak{R}|$ is odd. Thus, $|\mathfrak{R}|$ divides 15. If $|\mathfrak{R}| \neq 15$, then $\mathfrak{T}^0/O_2(\mathfrak{C})$ is certainly cyclic. So suppose $|\mathfrak{R}| = 15$, $\mathfrak{R} = \mathfrak{R}_3 \times \mathfrak{R}_5$, where $|\mathfrak{R}_p| = p$. Let $\mathfrak{D} = O_{3'}(\mathfrak{C})$, so that $\mathfrak{D} = \mathfrak{D}_2\mathfrak{R}_5$, where $\mathfrak{D}_2 = \mathfrak{D} \cap \mathfrak{T}^0$. We can then choose $\mathfrak{X} \in \{Z(\mathfrak{D}_2), J(\mathfrak{D}_2)\}$ such that $\mathfrak{X} \lhd \mathfrak{D}$, whence $\mathfrak{X} \lhd \mathfrak{C}$. Let $\mathfrak{L} = N(\mathfrak{X})$. If $\mathfrak{L} = \mathfrak{C}$, then $\mathfrak{D}_2 \in N^*(\mathfrak{R}_3; 2)$, and $C_{\mathfrak{D}_2}(\mathfrak{R}_3) \neq 1$. This forces $\mathfrak{D}_2$ and $O_2(\mathfrak{M})$ to be $\mathfrak{G}$-conjugate, whence $\mathfrak{D}_2 = \mathfrak{T}^0$. This is false, since $N(\mathfrak{T}^0) \subseteq \mathfrak{N}_1$, and $3 \nmid |\mathfrak{N}_1|$. So $\mathfrak{C} \subset \mathfrak{L}$, whence $|\mathfrak{L} : \mathfrak{C}| = 2$. Thus, $\mathfrak{L}$ contains a S_2-subgroup of $\mathfrak{G}$ and so $\mathfrak{L}$ is conjugate to a subgroup of $\mathfrak{M}$ or of $\mathfrak{N}_1$. This is false, since $15 \mid |\mathfrak{L}|$, $15 \nmid |\mathfrak{M}|$, $15 \nmid |\mathfrak{N}_1|$. So $\mathfrak{T}^0/\mathfrak{C}_0$ is cyclic. Since $\mathfrak{T}^0$ is a S_2-subgroup of $\mathfrak{C}$, it follows that if $\mathfrak{Y}$ is any 2-subgroup of $C(I)$, then $\mathfrak{Y} \cap N(\mathfrak{U}) \lhd \mathfrak{Y}$, and $\mathfrak{Y}/\mathfrak{Y} \cap N(\mathfrak{U})$ is cyclic.

Suppose $\mathfrak{U}_1$ is a hyperplane of $\mathfrak{U}$. Then $C_{\mathfrak{N}_1}(\mathfrak{U}_1) = \mathfrak{R}$. We argue that $C(\mathfrak{U}_1) = \mathfrak{R}$. Suppose false. We may assume that $Z(\mathfrak{T}) \cap \mathfrak{U}_1 \neq 1$, so that $C(\mathfrak{U}_1) \subseteq \mathfrak{M}$. Since $\mathfrak{N}_1 = N(\mathfrak{R})$, it follows that $\mathfrak{R}$ is a S_2-subgroup of $C(\mathfrak{U}_1)$. Hence, $C(\mathfrak{U}_1) = \mathfrak{R}\mathfrak{A}$, where $|\mathfrak{A}| = 3$. Now $\mathfrak{V}_1 \subseteq \mathfrak{R}_1$, since $\mathfrak{V}_1 \lhd \mathfrak{N}_1$. Let $\mathfrak{R}_0 = O_2(\mathfrak{R}\mathfrak{A})$. If $\mathfrak{V}_1 \subseteq \mathfrak{R}_0$, then $\mathfrak{V}_1 \lhd \mathfrak{R}\mathfrak{A}$, whence $\mathfrak{A} \subseteq \mathfrak{N}_1$, against $3 \nmid |\mathfrak{N}_1|$. So $\mathfrak{V}_1 \not\subseteq \mathfrak{R}_0$, and $\mathfrak{R}\mathfrak{A}/\mathfrak{R}_0 \cong \Sigma_3$. Since $\mathfrak{U} \subseteq Z(\mathfrak{R})$, it follows that $\mathfrak{U} \subseteq Z(\mathfrak{L}_0)$. Since $\mathfrak{A} \not\subseteq \mathfrak{N}_1$, $\mathfrak{A}$ does not centralize $Z(\mathfrak{R}_0)$. Let $\mathfrak{X} = [\Omega_1(Z(\mathfrak{R}_0)), \mathfrak{A}] \lhd \mathfrak{R}\mathfrak{A}]$. Choose G in $\mathfrak{G}$ such that $\mathfrak{E}^G \subseteq \mathfrak{R}$, $\mathfrak{E}^G \not\subseteq \mathfrak{R}_0$. Set $\mathfrak{E}^* = \mathfrak{E}^G$, $\mathfrak{E}_1^* = \mathfrak{E}^G \cap \mathfrak{R}_0$. Then $\mathfrak{X} \subseteq C(\mathfrak{E}_1^*) \supseteq \mathfrak{M}^G$, and so $[\mathfrak{X}, \mathfrak{E}^*] = \mathfrak{E}_1^*$, whence $|\mathfrak{X}| = 4$ and $\mathfrak{E}^* \lhd \mathfrak{R}$. This implies that $\mathfrak{R}_0 = \mathfrak{X} \times \mathfrak{Y}$, where $\mathfrak{Y} = C_{\mathfrak{R}_0}(\mathfrak{A}) \subseteq C_{\mathfrak{R}_0}(\mathfrak{E}^*)$. Thus, $\langle \mathfrak{X}, \mathfrak{E}^* \rangle \cong D_8$, and $\mathfrak{R} = \mathfrak{Y} \times \langle \mathfrak{X}, \mathfrak{E}^* \rangle$.

Since $J_1(\mathfrak{T}) \not\subseteq \mathfrak{R}$, and since $\mathfrak{N}_1 = \mathfrak{T}\mathfrak{Q}$, where $Z(\mathfrak{R}) \cap C(\mathfrak{Q}) = 1$, it follows that $\mathfrak{U} = \Omega_1(Z(\mathfrak{R}))$. Since $C(\mathfrak{U}_1) \not\subseteq \mathfrak{N}_1$, it follows that $\mathfrak{Y}$ contains no non identity characteristic subgroup of $\mathfrak{R}$. So $\mathfrak{Y} = \mathfrak{Y}_1 \times \mathfrak{Y}_2 \times \mathfrak{Y}_3$, where each $\mathfrak{Y}_i$ is either a dihedral group of order 8, or it is of order 2. Since $\mathfrak{U} = [\mathfrak{U}, \mathfrak{Q}]$, and $|\mathfrak{Q}| = 5$, we have the desired contradiction. So $C(\mathfrak{U}_1) = \mathfrak{R}$ for every hyperplane $\mathfrak{U}_1$ of $\mathfrak{U}$.

We can now copy the final argument of § 19 and conclude that if $|\mathfrak{E}| = 4$, then $|\mathfrak{F}| \neq 16$.

LEMMA 20.22. *Suppose* $|\mathfrak{E}| = 4$ *and* $|\mathfrak{F}| = 8$. *Then the following hold*:

(a) $\mathfrak{F}$ *is a* T.I. *set in* $\mathfrak{G}$.

(b) *A* $S_{2'}$*-subgroup of* $\mathfrak{M}$ *centralizes* $\mathfrak{F}$.

(c) $|\mathfrak{M}|_{2'} > 3$.

(d) $\mathfrak{V}_2 \lhd \mathfrak{N}_1$, *where* $\mathfrak{V}_2 = \langle V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}_0); \mathfrak{T}) \mid |\mathfrak{F} : \mathfrak{F}_0| = 2 \rangle$.

Proof. Since $\mathfrak{Z} \subseteq Z(\mathfrak{M})$, it follows that a $S_{2'}$-subgroup of $\mathfrak{M}$ centralizes $\mathfrak{E}$, and so (b) holds, as $|\mathfrak{F} : \mathfrak{E}| = 2$. Lemma 20.17 and (b) imply (a).

Let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}_1$, and let $\mathfrak{R} = O_2(\mathfrak{N}_1)$, so that $\mathfrak{R} = C(\mathfrak{U})$. Suppose $\mathfrak{V}_2 \ntriangleleft \mathfrak{N}_1$. Then $\mathfrak{V}_2 \not\subseteq \mathfrak{R}$, so there is G in $\mathfrak{G}$ such that $|\mathfrak{F}^G \cap \mathfrak{T}| \geqq 4$, $\mathfrak{F}^G \cap \mathfrak{T} \not\subseteq \mathfrak{R}$. By Lemma 20.18, we have $\mathfrak{F}^G \cap \mathfrak{R} \neq 1$. Thus, $\mathfrak{U} \subseteq C(\mathfrak{F}^G \cap \mathfrak{R}) \subseteq \mathfrak{M}^G$, and so $1 \subset [\mathfrak{U}, \mathfrak{F}^G \cap \mathfrak{T}] \subseteq \mathfrak{F}^G \cap \mathfrak{R}$.

Let $\mathfrak{F}^G \cap \mathfrak{T} = \mathfrak{F}^G \cap \mathfrak{R} \times \langle F \rangle$. Then $\mathfrak{U}$ is a free $F_2\langle F \rangle$-module. This implies that $[\mathfrak{U}, \mathfrak{F}^G \subset \mathfrak{T}] = C_{\mathfrak{U}}(\mathfrak{F}^G \cap \mathfrak{T})$, and so $[\mathfrak{U}, \mathfrak{F}^G \cap \mathfrak{T}] \supseteq \Omega_1(Z(\mathfrak{T}))$. Since $\Omega_1(Z(\mathfrak{T})) \subseteq \mathfrak{F}$, it follows that $\mathfrak{F}^G \cap \mathfrak{F} \supseteq \Omega_1(Z(\mathfrak{T}))$, whence $\mathfrak{F} = \mathfrak{F}^G$.

Let $\mathfrak{Q} = \langle Q \rangle$ be a subgroup of $\mathfrak{N}_1$ of odd prime order inverted by F. Let $\mathfrak{R}_1 = [\mathfrak{R}, \mathfrak{Q}]$. Thus, $\mathfrak{R}_1 = [\mathfrak{R}_1, F] \times [\mathfrak{R}_1, F]^Q$, and $[\mathfrak{R}_1, F] \subseteq \mathfrak{F} \cap \mathfrak{R} = \mathfrak{F}_0$. Since $|\mathfrak{F}_0| \leqq 4$, we have $|\mathfrak{R}_1| = 2^{2k}$, where $1 \leqq k \leqq 2$. If

$k = 1$, then $\mathfrak{K} = \mathfrak{K}_1 \times C_{\mathfrak{K}}(\mathfrak{Q})$, and so $C_{\mathfrak{K}}(\mathfrak{Q}) \cap \mathfrak{F} \neq 1$, whence $\mathfrak{Q} \subseteq \mathfrak{M}$. This is false, since $\mathfrak{M} \cap \mathfrak{N}_1 = \mathfrak{T}$. So $k = 2$.

Let $\mathfrak{K}_2 = C_{\mathfrak{K}}(\mathfrak{Q})$. Then since $Z(\mathfrak{T}) \cap C(\mathfrak{Q}) = 1$, it follows that $\mathfrak{K}_2$ acts faithfully on $\mathfrak{K}_1$. Since $[\mathfrak{K}_2, F] \subseteq \mathfrak{K}_2 \cap \mathfrak{F} = 1$, it follows that $\mathfrak{K}_2$ acts faithfully on $\mathfrak{F}_0 = [\mathfrak{K}_1, F]$, and so $|\mathfrak{K}_2| \leqq 2$.

Suppose $\mathfrak{K}_2 = 1$. In this case, $|\mathfrak{T}| \leqq 2^6$. Since a $S_{2'}$-subgroup of $\mathfrak{M}$ acts faithfully on $O_2(\mathfrak{M})$ and centralizes $\mathfrak{F}$, we get that $|O_2(\mathfrak{M})| = 2^5$, $|\mathfrak{M}|_{2'} = 3$, $|\mathfrak{T}| = 2^6$. Hence, $\mathfrak{T}/\mathfrak{K}$ is of order 4. This implies that $\mathfrak{T}$ has just one normal elementary subgroup of order 8, and so $\mathfrak{F} \subseteq \mathfrak{K}_1$. Hence, $C_{\mathfrak{T}}(\mathfrak{F}) = \mathfrak{K}_1$, whence $\mathfrak{K}_1 \subseteq O_2(\mathfrak{M})$, and so $\mathfrak{K}_1 = J(O_2(\mathfrak{M})) \lhd \mathfrak{M}$, against $\mathfrak{N}_1 \not\subseteq \mathfrak{M}$. So $|\mathfrak{K}_2| = 2$.

Since $|\mathfrak{K}_2| = 2$, it follows that $\mathfrak{K}_1$ is not a minimal normal subgroup of $\mathfrak{N}_1$, whence $|\mathfrak{N}_1| = 3|\mathfrak{T}|$, and so $|\mathfrak{T}| = 2^6$. But this forces $|\mathfrak{M}| = 3|\mathfrak{T}|$, against § 18. So (d) holds.

Suppose by way of contradiction that $|\mathfrak{M}|_{2'} = 3$. Thus, $|\mathfrak{N}_1|_{2'} = 5$ or 15. Let $\mathfrak{K} = O_2(\mathfrak{N}_1)$, and let $\mathfrak{T}_0/\mathfrak{K}$ be the subgroup of $\mathfrak{T}/\mathfrak{K}$ of order 2. Let $\mathfrak{U} = \Omega_1(Z(\mathfrak{K}))$ so that $|\mathfrak{U}| = 2^4$. Suppose $\mathfrak{U}_1$ is a hyperplane of $\mathfrak{U}$. Then $C_{\mathfrak{N}_1}(\mathfrak{U}_1) = \mathfrak{K}$. We argue that $C(\mathfrak{U}_1) = \mathfrak{K}$. We may assume that $\mathfrak{U}_1 \cap Z(\mathfrak{T}) \neq 1$. Thus, $C(\mathfrak{U}_1) \subseteq \mathfrak{M}$. We may assume that $C(\mathfrak{U}_1) = \mathfrak{K}\mathfrak{A}$, where $|\mathfrak{A}| = 3$. Since $\mathfrak{U} = \Omega_1(Z(\mathfrak{K}))$, we have $\mathfrak{U} \subseteq Z(\mathfrak{K}_0)$, where $\mathfrak{K}_0 = O_2(\mathfrak{K}\mathfrak{A})$. By (a), it follows that $\mathfrak{V}_1 \subseteq \mathfrak{K}_0$, and so $\mathfrak{A} \subseteq N(\mathfrak{V}_1) = \mathfrak{N}_1$, which is false. So $C(\mathfrak{U}_1) = \mathfrak{K}$.

It follows as in an earlier argument that if I is an involution of $\mathfrak{U}$, and $C(I) = \mathfrak{C}$, then for each 2-subgroup $\mathfrak{T}_1$ of $\mathfrak{C}$, $\mathfrak{T}_1 \cap N(\mathfrak{U}) \lhd \mathfrak{T}_1$, and $\mathfrak{T}_1/\mathfrak{T}_1 \cap N(\mathfrak{U})$ is cyclic. The argument of § 19 then yields a contradiction. So (c) holds. The proof is complete.

Let p be an odd prime divisor of $|\mathfrak{M}|$, and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{M}$. Set $\mathfrak{L} = \mathfrak{T}\mathfrak{P}$, $\mathfrak{V} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}); \mathfrak{T})$. Since $\mathfrak{V} \subseteq \mathfrak{V}_2$, we have $\mathfrak{V} \lhd \mathfrak{N}_1$, and so $N_{\mathfrak{L}}(\mathfrak{V}) = \mathfrak{T}$. Choose G in $\mathfrak{G}$ such that $\mathfrak{F}^G \subseteq \mathfrak{T}$, $\mathfrak{F}^G \not\subseteq \mathfrak{H} = O_2(\mathfrak{L})$. Let $\mathfrak{F}_0^G = \mathfrak{F}^G \cap \mathfrak{H}$, so that $\mathfrak{F}^G = \mathfrak{F}_0^G \times \langle F \rangle$, and F inverts a S_p-subgroup of $\mathfrak{L}$, which we may assume is $\mathfrak{P}$. Let $\mathfrak{X}$ be the normal closure of $\mathfrak{F}_0^G$ in $\mathfrak{L}$, and let $\mathfrak{Y} = C_{\mathfrak{L}}(\mathfrak{X})$. Thus, $\mathfrak{X}\mathfrak{Y} \subseteq \mathfrak{H}$, and $\mathfrak{Y} \supseteq \mathfrak{U}$, where $\mathfrak{U} = \Omega_1(Z(\mathfrak{K}))$. Thus, $\mathfrak{P}$ acts faithfully on $\mathfrak{Y}$, as $C(\mathfrak{U}) \cap \mathfrak{P} = 1$. Since $\mathfrak{Y} \subseteq C(\mathfrak{X}) \subseteq C(\mathfrak{F}_0^G) \subseteq \mathfrak{M}^G$, we have $[\mathfrak{Y}, F] \subseteq \mathfrak{F}_0^G$. Let $\mathfrak{Y}_1 = [\mathfrak{Y}, \mathfrak{P}]$, so that $\mathfrak{Y}_1 = [\mathfrak{Y}_1, F] \times [\mathfrak{Y}_1, F]^P$, where $\mathfrak{P} = \langle P \rangle$. Since $[\mathfrak{Y}_1, F]| \leqq 4$, we get that $|\mathfrak{P}| = 3$ or 5. Hence, by (c) of the preceding lemma, we see that $|\mathfrak{M}|_{2'} = 5$ or 15.

Now we take $\mathfrak{P}$ of order 5 in the preceding discussion. In this case, we see that $[\mathfrak{Y}_1, F] = \mathfrak{F}_0^G$ is a four-group and so $\mathfrak{F}_0^G \subseteq O_2(\mathfrak{M})$. Let $\tilde{\mathfrak{X}}$ be the normal closure of $\mathfrak{F}_0^G$ in $\mathfrak{M}$, and let $\tilde{\mathfrak{Y}} = C_{\mathfrak{M}}(\tilde{\mathfrak{X}})$, $\tilde{\mathfrak{Y}}_1 = [\tilde{\mathfrak{Y}}, \mathfrak{P}]$. Suppose $|\mathfrak{M}|_{2'} = 15$, and $\mathfrak{Q} \subseteq C_{\mathfrak{M}}(\mathfrak{P})$, $|\mathfrak{Q}| = 3$. Then $\mathfrak{Q}$ normalizes $\tilde{\mathfrak{Y}}_1$. We argue that $\mathfrak{Q}$ centralizes $\tilde{\mathfrak{Y}}_1$. Suppose false. Then since elements of $\mathrm{GL}(4, 2)$ of order 15 are non real, it follows that F centralizes

$\boldsymbol{O}_2(\mathfrak{M})\mathfrak{Q}/\boldsymbol{O}_2(\mathfrak{M})$. This implies that $\mathfrak{M}$ has a subgroup of order 3 which normalizes but does not centralize $[\mathfrak{Y}_1, F] = \mathfrak{F}_0^G$. This is false, since $\boldsymbol{N}(\mathfrak{A})/\boldsymbol{C}(\mathfrak{A})$ is a 2-group for every subgroup $\mathfrak{A}$ of $\mathfrak{F}$. So $\mathfrak{Q}$ centralizes $\mathfrak{Y}_1$, whence $\mathfrak{Q} \subseteqq \boldsymbol{C}(\mathfrak{F}_0^G) \subseteqq \mathfrak{M}^G$. Thus, $\mathfrak{Q}$ is a S_3-subgroup of $\mathfrak{M}$ and of $\mathfrak{M}^G$. By Lemma 20.17, $\mathfrak{F}\mathfrak{Q} \in \mathscr{M}^*$, $\mathfrak{F}^G\mathfrak{Q} \in \mathscr{M}^*$. Hence, $\mathfrak{M} = \mathfrak{M}^G$, whence $\mathfrak{F} = \mathfrak{F}^G$. This is absurd, since $\mathfrak{F} \subseteqq \boldsymbol{O}_2(\mathfrak{M})$, $\mathfrak{F}^G \not\subseteqq \boldsymbol{O}_2(\mathfrak{M})$. So $|\mathfrak{M}|_{2'} = 5$.

If $|\mathfrak{N}_1|_{2'} = 5$ or 15, then $\mathfrak{U} = \Omega_1(\boldsymbol{Z}(\mathfrak{R}))$ is of order 16, and the final argument of § 19 yields a contradiction. So $|\mathfrak{N}_1|_{2'} = 3$, $|\mathfrak{U}| = 4$.

Let $\mathfrak{U}_0 = \mathfrak{U} \cap \boldsymbol{Z}(\mathfrak{T})$, so that $|\mathfrak{U}_0| = 2$, $\mathfrak{U}_0 \subseteqq \boldsymbol{Z}(\mathfrak{M})$. Let $\mathfrak{W} = \mathfrak{U}^{\mathfrak{M}} = \mathfrak{U}^{\mathfrak{P}}$, where $\mathfrak{P}$ is a $S_{2'}$-subgroup of $\mathfrak{M}$, $|\mathfrak{P}| = 5$. Then $|\mathfrak{W}| \leqq 2^6$, and $\mathfrak{W} \supset \mathfrak{U}$. By a standard argument, $\mathfrak{W}' = 1$. Let $\mathfrak{W}_0 = \boldsymbol{C}_{\mathfrak{W}}(\mathfrak{P})$. Since $[\boldsymbol{O}_2(\mathfrak{M}), \mathfrak{W}] \subseteqq \mathfrak{U}_0 \subseteqq \mathfrak{W}_0$, it follows that $\mathfrak{W}_0 \lhd \mathfrak{M}$. If $|\mathfrak{W}_0| = 4$, then we may let $\mathfrak{W}$ play the role of $\mathfrak{F}$. We have already shown that this case does not occur. So $|\mathfrak{W}_0| = 2$. Since $\boldsymbol{C}_{\mathfrak{W}}(\mathfrak{E})$ admits $\mathfrak{P}$, we get $\mathfrak{W} \subseteqq \boldsymbol{C}(\mathfrak{E})$, so we may let $\mathfrak{W}\mathfrak{E}$ play the role of $\mathfrak{F}$. This contradiction establishes the important

THEOREM 20.2. $|\mathfrak{E}| = 2$.

LEMMA 20.23. (a) $|\mathfrak{M}|_{2'} > 3$.
(b) $|\mathfrak{F}| > 8$.

Proof. Suppose $|\mathfrak{M}|_{2'} = 3$. Since $|\mathfrak{F}| \geqq 8$ and $\mathfrak{F}/\mathfrak{E}$ is a chief factor of $\mathfrak{M}$, it follows that $|\mathfrak{F}| = 8$.

Choose $\mathfrak{N} \in \mathscr{M}(\mathfrak{T})$ with $|\mathfrak{N}|_{2'} > 3$. Let $\mathfrak{U}$ be a minimal normal subgroup of $\mathfrak{N}$, and let $\mathfrak{Q}$ be a $S_{2'}$-subgroup of $\mathfrak{N}$. Thus, $\mathfrak{U} \supset \mathfrak{E}$, and $\mathfrak{Q}$ acts faithfully on $\mathfrak{U}$. Since $\boldsymbol{J}_1(\mathfrak{T}) \not\subseteqq \boldsymbol{O}_2(\mathfrak{N})$, it follows that $|\mathfrak{U}| = 2^4$, $|\mathfrak{Q}| = 5$ or 15.

Let $\mathfrak{T}_0/\boldsymbol{O}_2(\mathfrak{N}_1)$ be the subgroup of $\mathfrak{T}/\boldsymbol{O}_2(\mathfrak{N})$ of order 2. Suppose $I \in \mathfrak{U}^\sharp$, $\mathfrak{C} = \boldsymbol{C}(I)$, $\mathfrak{T}^0 = \boldsymbol{C}_{\mathfrak{N}}(I)$. We will show that $\mathfrak{T}^0$ is a S_2-subgroup of $\mathfrak{C}$. We may assume that $\boldsymbol{O}_2(\mathfrak{N}) \subseteqq \mathfrak{T}_0 \subseteqq \mathfrak{T}^0 \subseteqq \mathfrak{T}$. Thus, $\boldsymbol{J}(\mathfrak{T}) = \boldsymbol{J}(\mathfrak{T}^0) = \boldsymbol{J}(\boldsymbol{O}_2(\mathfrak{N}))$, and so $\mathfrak{N}(\mathfrak{T}^0) \subseteqq \mathfrak{N}$, whence $\mathfrak{T}^0$ is a S_2-subgroup of $\mathfrak{C}$. Since $|\mathfrak{N}^*|_{2'}$ divides 15 for every $\mathfrak{N}^* \in \mathscr{M}(\mathfrak{T})$, it follows that $|\mathfrak{N}^*|_{2'}$ divides 15 for every 2-local subgroup $\mathfrak{N}^*$ of $\mathfrak{G}$, and so $|\mathfrak{C}|_{2'}$ divides 15. We must show that $\mathfrak{T}^0/\boldsymbol{O}_2(\mathfrak{C})$ is cyclic. Suppose false. Then $\mathfrak{T}^0\mathfrak{R} = \mathfrak{C}$, where $|\mathfrak{R}| = 15$. A standard argument shows that $\mathfrak{C} \subset \mathfrak{C}^*$, where $\mathfrak{C}^*$ contains a S_2-subgroup of $\mathfrak{G}$. Since $|\mathfrak{M}|_{2'} = 3$, $\mathfrak{C}^*$ is not conjugate to $\mathfrak{M}$. Since $5 \mid |\mathfrak{C}^*|$, it follows that $\mathfrak{C}^*$ is conjugate to $\mathfrak{N}$. Since $|\mathfrak{C}^* : \mathfrak{C}| = 2$, it follows that the normal closure of I in $\mathfrak{C}^*$ is a four-group centralized by $\mathfrak{R}$. This is false, since $\boldsymbol{C}(\mathfrak{E}) = \mathfrak{M}$ is a $5'$-group. We conclude that $\mathfrak{T}^0/\boldsymbol{O}_2(\mathfrak{C})$ is cyclic. It is straightforward to check that $\boldsymbol{C}(\mathfrak{U}_1) = \boldsymbol{O}_2(\mathfrak{N})$ for every hyperplane $\mathfrak{U}_1$ of $\mathfrak{U}$, and so the argument of § 19 yields a contradiction. This establishes (a).

Suppose $|\mathfrak{F}| = 8$. Since $|\mathfrak{M}|_{2'} > 3$, Lemma 20.17 implies that $\mathfrak{F}$ is a T.I. set in $\mathfrak{G}$. Hence, if $\mathfrak{V} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}); \mathfrak{T})$, it follows that $\mathfrak{V} \subseteq C(\mathfrak{F})$, and so $N_{\mathfrak{M}}(\mathfrak{V}) \supset \mathfrak{T}$, whence $N(\mathfrak{V}) \subseteq \mathfrak{M}$.

Let $\mathfrak{P}$ be a subgroup of $\mathfrak{G}$ of odd prime order which is permutable with $\mathfrak{T}$ and such that $\mathfrak{P} \not\subseteq \mathfrak{M}$. Set $\mathfrak{L} = \mathfrak{T}\mathfrak{P}$. Thus, $N_{\mathfrak{L}}(\mathfrak{V}) = \mathfrak{T}$, so there is G in $\mathfrak{G}$ such that $\mathfrak{F}^G \subseteq \mathfrak{T}$, $\mathfrak{F}^G \not\subseteq \mathfrak{H} = O_2(\mathfrak{L})$. Let $\mathfrak{F}_0^G = \mathfrak{F}^G \cap \mathfrak{H}$, so that $\mathfrak{F}^G = \mathfrak{F}_0^G \times \langle F \rangle$. We assume without loss of generality that F inverts $\mathfrak{P}$. Let $\mathfrak{U} = \Omega_1(Z(\mathfrak{H}))$, so that $\mathfrak{U} = [\mathfrak{U}, \mathfrak{P}]$. Thus, $\mathfrak{U} \subseteq C(\mathfrak{F}_0^G) \subseteq \mathfrak{M}^G$, and so $1 \subset [\mathfrak{U}, \mathfrak{F}] \subseteq \mathfrak{F}_0^G$, whence $\mathfrak{H} \subseteq \mathfrak{M}^G$. Let $\mathfrak{H}_1 = [\mathfrak{H}, \mathfrak{P}]$, so that $\mathfrak{H}_1 = [\mathfrak{H}_1, F] \times [\mathfrak{H}_1, F]^P$, where $\mathfrak{P} = \langle P \rangle$. Since $|\mathfrak{F}| = 8$, and since $C(\mathfrak{P}) \cap \mathfrak{F}^G = 1$, we see that $[\mathfrak{H}_1, F] = \mathfrak{F}_0^G$ is of order 4, $|\mathfrak{H}_1| = 16$. Since $Z(\mathfrak{T}) \subset C(\mathfrak{P}) = 1$, it follows that $\mathfrak{H}_2$ acts faithfully on $\mathfrak{H}_1$, where $\mathfrak{H}_2 = C_{\mathfrak{H}}(\mathfrak{P})$, and so $|\mathfrak{H}_2| \leqq 2$. But $|\mathfrak{M}|_{2'} > 3$, and so $|O_2(\mathfrak{M})| \geqq 2^7$, $|\mathfrak{T}| \geqq 2^8$. This is false, since $|\mathfrak{T}| = |\mathfrak{H}| \cdot |\mathfrak{T} : \mathfrak{H}| \leqq |\mathfrak{H}_2| \cdot 2^4 \cdot 2^2 \leqq 2^7$. So (b) holds.

LEMMA 20.24. *$\mathscr{M}(\mathfrak{T}) = \{\mathfrak{M}, \mathfrak{N}\}$, where $\mathfrak{N} = N(\mathfrak{V})$, $\mathfrak{V} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}); \mathfrak{T})$.*

Proof. Suppose p is an odd prime and $\mathfrak{P}$ is a p-group permutable with $\mathfrak{T}$, and $\mathfrak{P} \not\subseteq \mathfrak{M}$. Set $\mathfrak{L} = \mathfrak{T}\mathfrak{P}$. Thus, $\mathfrak{L} \cap \mathfrak{M} = \mathfrak{T}$. Let $\mathfrak{K} = O_2(\mathfrak{L})$, $\mathfrak{U} = \Omega_1(Z(\mathfrak{K}))$. Thus, $\mathfrak{U} = [\mathfrak{U}, \mathfrak{P}]$, and $\mathfrak{P}$ acts faithfully on $\mathfrak{U}$.

Suppose $G \in \mathfrak{G}$ and $\mathfrak{F}^G \subseteq \mathfrak{T}$, $\mathfrak{F}^G \not\subseteq \mathfrak{K}$. Let $\mathfrak{F}_0^G = \mathfrak{F}^G \cap \mathfrak{K}$, so that $\mathfrak{F}^G = \mathfrak{F}_0^G \times \langle F \rangle$, and we may assume that F inverts $\mathfrak{P}$. Thus, $\mathfrak{U}$ centralizes the hyperplane $\mathfrak{F}_0^G$ of $\mathfrak{F}^G$, and so $[\mathfrak{U}, F] = \mathfrak{Z}^G$. Thus, $|\mathfrak{U}| = 4$, $|\mathfrak{P}| = 3$, and $\mathfrak{Z}^G = \mathfrak{Z}$, whence $\mathfrak{F} = \mathfrak{F}^G$. Let $\mathfrak{K}_1 = [\mathfrak{K}, \mathfrak{P}]$, $\mathfrak{K}_2 = C_{\mathfrak{K}}(\mathfrak{P})$. Then $\mathfrak{K}_1 = [\mathfrak{K}_1, F] \times [\mathfrak{K}_1, F]^P$, where $\mathfrak{P} = \langle P \rangle$, and $\mathfrak{F} \cap \mathfrak{K} = \mathfrak{F} \cap \mathfrak{K}_1 \times \mathfrak{F} \cap \mathfrak{K}_2$. Thus, $[\mathfrak{K}_1, F]^P$ centralizes the hyperplane $\mathfrak{F} \cap \mathfrak{K}$ of $\mathfrak{F}$, and so $[[\mathfrak{K}_1, F]^P, F] = \mathfrak{Z}$, whence $|\mathfrak{K}_1| = 4$. Thus, $\mathfrak{K} = \mathfrak{K}_1 \times \mathfrak{K}_2$, and $Z(\mathfrak{T})$ is non cyclic. This contradiction shows that $\mathfrak{V} \lhd \mathfrak{L}$. The proof is complete.

Let $\widetilde{\mathscr{F}_0} = \{\mathfrak{X} \mid \mathfrak{X} \in \mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}),\ \mathfrak{X} \subseteq \mathfrak{T},\ \mathfrak{X} \not\subseteq O_2(\mathfrak{M})\}$. Since $N_{\mathfrak{M}}(\mathfrak{V}) = \mathfrak{T}$, it follows that $\widetilde{\mathscr{F}_0} \neq \emptyset$. Let $\widetilde{\mathscr{F}} = \{\mathfrak{X} \mid \mathfrak{X} = \mathfrak{F}^G \in \mathscr{F}_0,\ \mathfrak{Z}^G \subseteq O_2(\mathfrak{M})\}$. It is crucial to show that $\widetilde{\mathscr{F}} \neq \emptyset$. We must work for this.

LEMMA 20.25. *If $\widetilde{\mathscr{F}} = \emptyset$, then $\mathfrak{F}$ has a subgroup $\mathfrak{F}_0$ of index 4 such that $C(\mathfrak{F}_0) \not\subseteq \mathfrak{M}$.*

Proof. Choose $\mathfrak{F}^G \in \widetilde{\mathscr{F}_0}$. Let $\mathfrak{F}_1^G = \mathfrak{F}^G \cap O_2(\mathfrak{M})$, and suppose that $\mathfrak{Z}^G \not\subseteq \mathfrak{F}_1^G$. Let $\mathfrak{A}$ be a complement to $\mathfrak{F}_1^G$ in $\mathfrak{F}^G$ which contains $\mathfrak{Z}^G$. Let $\mathfrak{P}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$, and let $\mathfrak{H} = O_2(\mathfrak{M})$, so that $\mathfrak{H}\mathfrak{P} \lhd \mathfrak{M}$ and $\mathfrak{A}$ acts faithfully on $\mathfrak{H}\mathfrak{P}/\mathfrak{H}$. Thus, there is a subgroup $\mathfrak{Q}$ of $\mathfrak{H}\mathfrak{P}$ of odd prime order inverted by Z^G, the generator of $\mathfrak{Z}^G$. Let $\mathfrak{L} = \mathfrak{H}\mathfrak{Q}\mathfrak{F}^G$, $\mathfrak{L}_0 = O_2(\mathfrak{L})$. Thus, $\mathfrak{Y} = \mathfrak{F}^G \cap \mathfrak{L}_0$ is a hyperplane of $\mathfrak{Z}^G$ and $\mathfrak{F}^G = \mathfrak{Y} \times \mathfrak{Z}^G$.

Let $\bar{\mathfrak{F}} = \mathfrak{F}/\mathfrak{Z}$. Thus, we may view $\bar{\mathfrak{F}}$ as a module for $\mathfrak{L}/\mathfrak{H}$. Let $\mathfrak{D} = \langle \mathfrak{Q}, \mathfrak{Z}^G \rangle$, so that $\mathfrak{L}/\mathfrak{H}$ is the direct product of $\mathfrak{Y}\mathfrak{H}/\mathfrak{H}$ and $\mathfrak{D}\mathfrak{H}/\mathfrak{H}$.

By the $\mathfrak{P} \times \mathfrak{Q}$-lemma, $\mathfrak{D}$ acts faithfully on $\bar{\mathfrak{F}}_1 = \mathfrak{F}_1/\mathfrak{Z} = C_{\bar{\mathfrak{F}}}(\mathfrak{Y})$. Since $C_{\bar{\mathfrak{F}}}(\mathfrak{Q}) = 1$, it follows that $\bar{\mathfrak{F}}_1$ is a free $\mathfrak{F}_2\mathfrak{Z}^G$-module. Thus, there is $F \in \mathfrak{F}_1^\#$ such that $[F, Z^G] \neq 1$. Thus, $F \notin \mathfrak{M}^G$. On the other hand, $[F, Y] \subseteq \mathfrak{Z}$, and so $\mathfrak{Y}_0 = C_{\mathfrak{Y}}(F)$ is of index at most 2 in $\mathfrak{Y}$, so of index at most 4 in $\mathfrak{F}^G$. Since $F \in C(\mathfrak{Y}_0)$, we have $C(\mathfrak{Y}_0) \not\subseteq \mathfrak{M}^G$. The proof is complete.

Set

$$\mathscr{B} = \{\mathfrak{B} \mid \mathfrak{B} \subseteq \mathfrak{F}, \, |\mathfrak{F} : \mathfrak{B}| = 4, \, C(\mathfrak{B}) \not\subseteq \mathfrak{M}\} \, .$$

Suppose that $\mathscr{B} \neq \emptyset$. Choose $\mathfrak{U} \subseteq \mathfrak{F}$, $|\mathfrak{U}| = 4$, $\mathfrak{U} \lhd \mathfrak{T}$. As before, let $\mathfrak{P}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$. Let $\mathfrak{F}_0 = [\mathfrak{F}, \mathfrak{P}]$, so that $\mathfrak{F} = \mathfrak{F}_0 \times \mathfrak{Z}$ and $\mathfrak{F}_0\mathfrak{P}$ is a Frobenius group. Since $\mathfrak{F}$ is not 2-reducible in $\mathfrak{M}$, it follows that $\mathfrak{H}/\mathfrak{H}_0$ and $\mathfrak{F}/\mathfrak{Z}$ are in duality, where $\mathfrak{H} = O_2(\mathfrak{M})$, $\mathfrak{H}_0 = C(\mathfrak{F})$. Thus, $\mathfrak{H}$ permutes transitively the set of hyperplanes of $\mathfrak{F}$ which do not contain $\mathfrak{Z}$. Thus, if $\mathscr{B} \neq \emptyset$, then $\mathscr{B}_0 \neq \emptyset$, where

$$\mathscr{B}_0 = \{\mathfrak{B} \mid \mathfrak{B} \in \mathscr{B}, \, \mathfrak{B} \subseteq \mathfrak{F}_0\} \, .$$

Let $\mathfrak{U}_0 = \mathfrak{U} \cap \mathfrak{F}_0 = \langle \mathfrak{U} \rangle$, and choose $\mathfrak{B} \in \mathscr{B}_0$. Since $\mathfrak{B}$ is a hyperplane of $\mathfrak{F}_0$, and $\mathfrak{F}_0\mathfrak{P}$ is a Frobenius group, there is P in $\mathfrak{P}$ such that $U \in \mathfrak{B}^P$. Thus, $\mathscr{B}_1 \neq \emptyset$, where

$$\mathscr{B}_1 = \{\mathfrak{B} \in \mathscr{B}_0 \mid U \in \mathfrak{B}\} \, .$$

We have thus shown that

$$\text{if } \mathscr{B} \neq \emptyset, \text{ then } C(U) \not\subseteq \mathfrak{M}, \text{ where } \langle U, Z \rangle \in \mathscr{U}(\mathfrak{T}), \text{ and } U \in \mathfrak{F} \, .$$

Let us continue with our assumption that $\mathscr{B} \neq \emptyset$. Set $\mathfrak{C} = C(U)$, so that $\mathfrak{C} \not\subseteq \mathfrak{M}$. Since $\mathfrak{F}_0\mathfrak{P}$ is a Frobenius group, it follows that $\mathfrak{C} \cap \mathfrak{M} = \mathfrak{T}_0 = C_{\mathfrak{u}}(U)$ is of index 2 in $\mathfrak{T}$. Since $Z(\mathfrak{T})$ is cyclic, it follows that $\mathfrak{U} = \Omega_1(Z(\mathfrak{T}_0))$. We argue that $\mathfrak{T}_0$ is not a S_2-subgroup of $\mathfrak{C}$. Suppose false. In this case, $\mathfrak{C}$ has a subgroup $\mathfrak{A}$ of odd prime order which is permutable with $\mathfrak{T}_0$. Let $\mathfrak{L} = \mathfrak{T}_0\mathfrak{A}$, $\mathfrak{L}_0 = O_2(\mathfrak{L})$, $\mathfrak{Z}_0 = \Omega_1(Z(\mathfrak{L}_0)) \supseteq \mathfrak{U}$. Since $\mathfrak{A} \not\subseteq \mathfrak{M}$, it follows that $\mathfrak{Z}_1 = [\mathfrak{Z}_0, \mathfrak{A}] \neq 1$. Let $\mathfrak{B}_0 = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{F}); \mathfrak{T}_0)$. We argue that $\mathfrak{B}_0 \subseteq \mathfrak{L}_0$. Suppose this, too, is false. Choose G in $\mathfrak{G}$ such that $\mathfrak{F}^G \subseteq \mathfrak{T}_0$, $\mathfrak{F}^G \not\subseteq \mathfrak{L}_0$. Thus, $\mathfrak{Z}_1 \subseteq C(\mathfrak{F}^G \cap \mathfrak{L}_0)$, and $\mathfrak{F}^G \cap \mathfrak{L}_0$ is a hyperplane of $\mathfrak{F}^G$, whence $[\mathfrak{Z}_1, \mathfrak{F}^G] = \mathfrak{Z}^G$. Thus, $|\mathfrak{Z}_1| = 4$, $|\mathfrak{A}| = 3$, and $\mathfrak{F}^G \lhd \mathfrak{T}_0$. If $\mathfrak{F}^G \neq \mathfrak{F}$, then $\mathfrak{T}_0 \subseteq \mathfrak{M} \cap \mathfrak{M}^G$. Thus, $\mathfrak{M}^G$ has a S_2-subgroup which normalizes $\mathfrak{T}_0$, and so normalizes $\mathfrak{U}$, whence $A_{\mathfrak{G}}(\mathfrak{U}) = \mathrm{Aut}(\mathfrak{U})$. This violates the assumption that $\mathfrak{T}_0$ is a S_2-subgroup of $C(U)$. So $\mathfrak{F}^G = \mathfrak{F}$.

Let $\mathfrak{L}_1 = [\mathfrak{L}_0, \mathfrak{A}]$. Then $\mathfrak{A} = \langle A \rangle$, and $\mathfrak{L}_1 = \langle \mathfrak{L}^1, \mathfrak{L}^{1A} \rangle$, where $\mathfrak{L}^1 = [\mathfrak{L}_1, F]$, and $\mathfrak{F} = \mathfrak{F} \cap \mathfrak{L}_0 \times \langle F \rangle$. Hence $\mathfrak{L}_1' \subseteq \mathfrak{F}$ and so $\mathfrak{A} \subseteq C(\mathfrak{L}_1')$. We argue that $\mathfrak{L}_1' \subseteq \mathfrak{U}$. Namely, $\mathfrak{H} \cap C(U) = \mathfrak{H}_1$ has the property that $C_{\mathfrak{F}}(\mathfrak{H}_1) = \mathfrak{U}$, and so if $L \in \mathfrak{L}_1' - \mathfrak{U}$, there is $H \in \mathfrak{H}_1$ such that $[L, H] = Z$, where $\mathfrak{Z} = \langle Z \rangle$. Hence, we find that $Z \in \mathfrak{L}_1' \subseteqq \boldsymbol{C}(\mathfrak{A})$, against $\mathfrak{A} \not\subseteqq \mathfrak{M}$. So $\mathfrak{L}_1' \subseteqq \mathfrak{U}$. Since $\mathfrak{A}$ centralizes U and $\mathfrak{L}_1'$, while $\mathfrak{A}$ does not centralize Z, it follows that $\mathfrak{L}_1' \subseteqq \langle U \rangle$. It is now straightforward to check that $\mathfrak{F} \cap \mathfrak{L}_0 = \langle \mathfrak{F} \cap \mathfrak{L}_1, \mathfrak{F} \cap \mathfrak{L}_2 \rangle$, and that $\mathfrak{F} \cap \mathfrak{L}_2$ centralizes $\mathfrak{L}_1$. Here $\mathfrak{L}_2 = \boldsymbol{C}_{\mathfrak{L}_0}(\mathfrak{A})$. Suppose $\mathfrak{L}_1' \neq 1$. Choose $X \in \mathfrak{L}^{1A} - \boldsymbol{Z}(\mathfrak{L}_1)$. Thus, $\boldsymbol{C}_{\mathfrak{F}}(X)$ is of index at most 4 in $\mathfrak{F}$ and $[\mathfrak{F}, X] \not\subseteqq \mathfrak{Z}$. We conclude that $|\mathfrak{F}| = 2^5$, as $\mathfrak{F}/\mathfrak{Z}$ is a free $\mathfrak{F}_2\langle X \rangle$-module. Hence, in this case, we have $\mathfrak{L}_1 = \mathfrak{Z}_1 \times \mathfrak{L}_{11}$, where $\mathfrak{L}_{11}$ is the central product of two dihedral groups of order 8. Hence, $\mathfrak{L}_{11}$ contains a four-group $\mathfrak{L}^{11}$ such that $\mathfrak{L}_{11} = (\mathfrak{L}_{11} \cap \mathfrak{F})\mathfrak{L}^{11}$. Since $[\mathfrak{L}^{11}, \mathfrak{F}] \not\subseteqq \mathfrak{Z}$ for all $L^{11} \in \mathfrak{L}^{11}$, it follows that $\mathfrak{T}/\mathfrak{H}$ is not cyclic. This is false, since $|\mathfrak{F}/\mathfrak{Z}| = 2^4$. So $\mathfrak{L}_1$ is abelian. Hence, $\mathfrak{L}_1 = \mathfrak{L}^1 \times \mathfrak{L}^{1A}$. As usual, we get that $\mathfrak{L}_0 \cap \mathfrak{F} \subseteqq \boldsymbol{C}(\mathfrak{L}_1)$. Thus, elements of $\mathfrak{L}^{1A}$ centralize the hyperplane $\mathfrak{L}_0 \cap \mathfrak{F}$ of $\mathfrak{F}$, and so $[\mathfrak{L}^{1A}, F] \subseteqq \mathfrak{Z}$. Hence, $|\mathfrak{L}_1| = 4$, $\mathfrak{L}_0 = \mathfrak{L}_1 \times \mathfrak{L}_2$.

Let $\mathfrak{T} = \langle \mathfrak{T}_0, T \rangle$. Thus, $\langle T, \mathfrak{A} \rangle \subseteqq \boldsymbol{N}(\mathfrak{L}_2 \cap \mathfrak{L}_2^T)$. Suppose $\mathfrak{L}_2 \cap \mathfrak{L}_2^T \neq 1$. Then $\mathfrak{T}\mathfrak{A}$ is a group, and $|\mathrm{ccl}_{\mathfrak{T}\mathfrak{A}}(U)| = 2$, whence $\mathfrak{A} \subseteqq \boldsymbol{C}(\mathfrak{U}) \subseteqq \mathfrak{M}$. This is false, so $\mathfrak{L}_2 \cap \mathfrak{L}_2^T = 1$, and so $\mathfrak{L}_2$ is isomorphic to a subgroup of $\mathfrak{L}_0/\mathfrak{L}_2$. Hence, $|\mathfrak{T}_0| \leqq 2^6$, $|\mathfrak{T}| \leqq 2^7$. This is false, since $|\mathfrak{H}| \geqq 2^9$. This contradiction shows that $\mathfrak{V}_0 \subseteqq \mathfrak{L}_0$, so that $\mathfrak{V}_0 \lhd \mathfrak{L}$. Thus, $\mathfrak{T}\mathfrak{A}$ is a solvable group, whence $\mathfrak{A} \subseteqq \boldsymbol{C}(\mathfrak{U}) \subseteqq \mathfrak{M}$. This is false, and so $\mathfrak{T}_0$ is contained properly in a S_2-subgroup $\mathfrak{T}^*$ of $\mathfrak{C}$. Hence, $\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{U}) = \mathrm{Aut}\,(\mathfrak{U})$, and $\boldsymbol{N}(\mathfrak{U}) = \mathfrak{T}\mathfrak{R}$, where $|\mathfrak{R}| = 3$. Also, of course $\boldsymbol{N}(\mathfrak{U}) \subseteqq \mathfrak{R}$.

Let $\mathfrak{W} = \langle \mathfrak{F}^R \mid R \in \mathfrak{R} \rangle = \mathfrak{F}^{\boldsymbol{N}(\mathfrak{U})}$. Since $\boldsymbol{C}(\mathfrak{U})$ is a 2-group, it follows that $\mathfrak{W} \subseteqq \mathfrak{T}$. Let $\mathfrak{V}^* = \boldsymbol{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{W}); \mathfrak{T})$. Since $\mathfrak{V}^* \subseteqq \mathfrak{V}$, we have $\boldsymbol{N}(\mathfrak{V}^*) = \mathfrak{R}$, and so $\boldsymbol{N}_{\mathfrak{M}}(\mathfrak{V}^*) = \mathfrak{T}$. Note that $\mathfrak{U} \subseteqq \mathfrak{F} \cap \mathfrak{F}^R \cap \mathfrak{F}^{R^2}$. Choose G in $\mathfrak{G}$ such that $\mathfrak{W}^G \subseteqq \mathfrak{T}$, $\mathfrak{W}^G \not\subseteqq \mathfrak{H} = \boldsymbol{O}_2(\mathfrak{M})$.

Case 1. $\mathfrak{U}^G \subseteqq \mathfrak{H}$. Now $\mathfrak{W}^G = \langle \mathfrak{F}^{R^iG}, i = 0, 1, 2 \rangle$, so there is i such that $\mathfrak{F}^{R^iG} \subseteqq \mathfrak{T}$, $\mathfrak{F}^{R^iG} \not\subseteqq \mathfrak{H}$. Hence, $\mathfrak{F}^{R^iG} \in \widetilde{\mathscr{F}}$, and so $\widetilde{\mathscr{F}} \neq \emptyset$.

Case 2. $|\mathfrak{U}^G \cap \mathfrak{H}| = 2$.

In this case $\mathfrak{U}^G \cap \mathfrak{H} = \mathfrak{Z}^{R^iG}$ for some i and so $\mathfrak{F}^{R^iG} \in \widetilde{\mathscr{F}}$, as $\mathfrak{U}^G \subseteqq \mathfrak{F}^{R^iG}$, so that $\mathfrak{F}^{R^iG} \not\subseteqq \mathfrak{H}$.

Case 3. $\mathfrak{U}^G \cap \mathfrak{H} = 1$.

In this case, the four-group $\mathfrak{U}^G$ acts faithfully on $\mathfrak{H}\mathfrak{P}/\mathfrak{H}$, where $\mathfrak{P}$ is a $S_{2'}$-subgroup of $\mathfrak{M}$, and so there is a prime $p \geqq 5$ such that $\mathfrak{U}^G$ does not centralize $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}$, where $|\mathfrak{Q}| = p$. Let $\boldsymbol{C}_{\mathfrak{U}^G}(\mathfrak{H}\mathfrak{Q}/\mathfrak{H}) = \langle X \rangle$. Thus, $\langle X \rangle = \mathfrak{Z}^{R^iG}$ for some i. Set $G' = R^iG$. Thus, $\mathfrak{F}^{G'}$ normalizes $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}$, $\mathfrak{F}^{G'}$ does not centralize $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}$, and $\mathfrak{Z}^{G'}$ centralizes $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}$. Let $\mathfrak{Y} = \boldsymbol{C}_{\mathfrak{F}^{G'}}(\mathfrak{H}\mathfrak{Q}/\mathfrak{H})$, and let $\bar{\mathfrak{F}} = \mathfrak{F}/\mathfrak{Z}$, $\bar{\mathfrak{F}}_1 = \boldsymbol{C}_{\bar{\mathfrak{F}}}(\mathfrak{Y})$. We have $\mathfrak{F}^{G'} = \mathfrak{Y} \times \langle F \rangle$, and we may assume that F inverts $\mathfrak{Q}$. Thus, $\bar{\mathfrak{F}}_1$ is a free $F_2\langle F \rangle$-module, and since $p \geqq 5$, $|\bar{\mathfrak{F}}_1| \geqq 16$. Let $\mathfrak{F}_2 = \boldsymbol{C}_{\bar{\mathfrak{F}}_1}(\mathfrak{Z}^{G'})$, so that $|\mathfrak{F}_1 : \mathfrak{F}_2| \leqq 2$. Choose $F' \in \mathfrak{F}_2 - \boldsymbol{C}(F)$. Then $F' \in \boldsymbol{C}(\mathfrak{Z}^{G'}) = \mathfrak{M}^{G'}$, and $|\mathfrak{Y} : \boldsymbol{C}_{\mathfrak{Y}}(F')| \leqq 2$. Since $\mathfrak{Z}^{G'} \not\subseteqq \mathfrak{F}$, we have $[\mathfrak{F}^{G'}, F'] \not\subseteqq \mathfrak{Z}^{G'}$. Since $|\mathfrak{F}^{G'} : \boldsymbol{C}(F') \cap \mathfrak{F}^{G'}| \leqq 4$, it follows that $|\mathfrak{F}| = 2^5$. But in this case, $\mathfrak{T}/\mathfrak{H}$ is cyclic, against the fact that $\mathfrak{U}^G$ acts faithfully on $\mathfrak{H}\mathfrak{P}/\mathfrak{H}$. Putting the pieces together, we see that we have shown that

$$\widetilde{\mathscr{F}} \neq \emptyset .$$

Choose G in $\mathfrak{G}$ such that $\mathfrak{F}^G = \mathfrak{F}^* \in \widetilde{\mathscr{F}}$. Let $\mathfrak{Z}^* = \mathfrak{Z}^G = \langle Z^* \rangle$, $Z^* = Z^G$. Let $\mathfrak{F}_1 = C_{\mathfrak{F}}(Z^*)$, so that $|\mathfrak{F} : \mathfrak{F}_1| \leqq 2$. Let $\mathfrak{F}_0^* = \mathfrak{F}^* \cap \mathfrak{H}$. We proceed to show that $|\mathfrak{F}| = 2^5$.

Case 1. $|\mathfrak{F}^* : \mathfrak{F}_0^*| \geqq 4$.

Since $\mathfrak{F}\mathfrak{P}/\mathfrak{Z}$ is a Frobenius group, it follows that $\mathfrak{F}/\mathfrak{Z}$ is a free $F_2\mathfrak{F}^*/\mathfrak{F}_0^*$-module. On the other hand, $[\mathfrak{F}_1, X, Y] = 1$ for all $X, Y \in \mathfrak{F}^*$, and so we get $|\mathfrak{F}/\mathfrak{Z}| = 2^4$. This is impossible, since in this case $\mathfrak{T}/\mathfrak{H}$ is cyclic.

Case 2. $|\mathfrak{F}^* : \mathfrak{F}_0^*| = 2$, and $\mathfrak{Z}^* \not\subseteq \mathfrak{F}$. Let $\mathfrak{F}^* = \mathfrak{F}_0^* \times \langle F \rangle$. We can then choose $F_1 \in \mathfrak{F}_1$ such that $[F, F_1] \neq 1$. Since $\mathfrak{F}_0^* \cap C(F_1)$ is of index at most 2 in $\mathfrak{F}_0^*$, we have

$$F_1 \in \mathfrak{M}^G , \quad [\mathfrak{F}^G, F_1] \not\subseteq \mathfrak{Z}^G , \quad |\mathfrak{F}^G : \mathfrak{F}^G \cap C(F_1)| \leqq 4 .$$

This forces $|\mathfrak{F}| = 2^5$, as $\mathfrak{F}^G/\mathfrak{Z}^G$ is a free $F_2\langle F_1 \rangle$-module.

Case 3. $|\mathfrak{F}^* : \mathfrak{F}_0^*| = 2$ and $\mathfrak{Z}^* \subseteq \mathfrak{F}$. Since $\mathfrak{F}/\mathfrak{Z}$ is a free $F_2\langle F \rangle$-module of order $\geqq 16$, there is $F_1 \in \mathfrak{F}$ such that $[F, F_1] \notin \mathfrak{Z}^G$. Hence, $|\mathfrak{F}^G : C(F_1) \cap \mathfrak{F}^G| \leqq 4$, and $F_1 \in \mathfrak{M}^G$, so once again, we have $|\mathfrak{F}| = 2^5$.

We thus have established the important equality:

$$|\mathfrak{F}| = 2^5 .$$

LEMMA 20.26. *If* $\mathfrak{Z} \subset \mathfrak{X} \subset \mathfrak{F}$ *and* $|\mathfrak{F} : \mathfrak{X}| = 2$, *then* $N(\mathfrak{X}) \subseteq \mathfrak{M}$.

Proof. Let $\mathfrak{L} = N(\mathfrak{X})$, $\mathfrak{T}_0 = \mathfrak{L} \cap \mathfrak{M}$. Let $\mathfrak{P}$ be a $S_{2'}$-subgroup of $\mathfrak{M}$. Since $\mathfrak{P}\mathfrak{F}/\mathfrak{Z}$ is a Frobenius group, it follows that $\mathfrak{T}_0$ is a 2-group. Clearly, $\mathfrak{H} = O_2(\mathfrak{M}) \subseteq \mathfrak{T}_0$, as $[\mathfrak{H}, \mathfrak{F}] \subseteq \mathfrak{Z}$. Since $\mathfrak{Z} = \Omega_1(Z(\mathfrak{H}))$, it follows that $\mathfrak{Z} = \Omega_1(Z(\mathfrak{T}_0))$, and so $N(\mathfrak{T}_0) \subseteq \mathfrak{M}$. Thus, $\mathfrak{T}_0$ is a S_2-subgroup of $\mathfrak{L}$. Suppose by way of contradiction that $\mathfrak{T}_0 \subset \mathfrak{L}$.

Let $\mathfrak{L}_0 = C(\mathfrak{X})$, so that $\mathfrak{L}_0 \lhd \mathfrak{L}$, and $\mathfrak{L}_0$ is a 2-group, since $C(\mathfrak{X}) \subseteq \mathfrak{M} \cap \mathfrak{L} = \mathfrak{T}_0$. Let $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{L}_0$. Since $\mathfrak{H} \subseteq \mathfrak{L}$, it follows that $\bar{\mathfrak{H}} = \mathfrak{H}\mathfrak{L}_0/\mathfrak{L}_0$ is elementary abelian of order 8 and stabilizes the chain $\mathfrak{X} \supset \mathfrak{Z} \supset 1$.

Let $\mathfrak{Y} = \mathfrak{Z}^{\mathfrak{L}}$ so that $\mathfrak{Z} \subset \mathfrak{Y} \subseteq \mathfrak{X}$. Since $\mathfrak{Z} \subseteq Z(\mathfrak{T}_0)$, $\mathfrak{Y}$ is 2-reducible in $\mathfrak{L}$. Let $\mathfrak{D}$ be a $S_{2'}$-subgroup of $\mathfrak{L}$. Thus, $|\mathfrak{D}| \mid 15$ and $\mathfrak{D}$ acts faithfully on $\mathfrak{Y}$. If $5 \mid |\mathfrak{D}|$, then $\mathfrak{Y} = \mathfrak{X}$ is an irreducible $F_2\mathfrak{L}$-module, and so $\bar{\mathfrak{H}}$ acts faithfully on a cyclic group of order 5 or 15. This is false, since $\bar{\mathfrak{H}}$ is elementary of order 8. So $|\mathfrak{D}| = 3$, and $|\mathfrak{Y}| = 4$. Let $\mathfrak{D} = \langle D \rangle$ and let $\mathfrak{H}_1 = C_{\mathfrak{H}}(\mathfrak{Y})$. Thus, $\bar{\mathfrak{H}}_1 = \mathfrak{H}_1\mathfrak{L}_0/\mathfrak{L}_0$ is a four-group and $\bar{\mathfrak{H}}_1$ stabilizes the chains $\mathfrak{X} \supset \mathfrak{Y} \supset 1$, $\mathfrak{X} \supset \mathfrak{Z} \supset 1$. Hence, $\mathfrak{H}_1^D$ stabilizes the chains $\mathfrak{X} \supset \mathfrak{Y} \supset 1$, $\mathfrak{X} \supset \mathfrak{Z}^D \supset 1$. Since $\mathfrak{H}_1^D/\mathfrak{H} \cap \mathfrak{H}_1^D$ contains a four-group, it follows that S_2-subgroups of $\mathfrak{M}/\mathfrak{H}$ are non cyclic. This contradiction completes the proof.

We next note that $|\mathfrak{M}|_{2'} = 5$. Namely $|\mathfrak{F}/\mathfrak{Z}| = 2^4$, and so $|\mathfrak{M}|_{2'} = 5$ or 15. If $|\mathfrak{M}|_{2'} = 15$, then $|N_{\mathfrak{M}}(\Omega_1(\mathfrak{T}))| = 3|\mathfrak{T}|$, whence $N_{\mathfrak{M}}(\mathfrak{V}) \supset \mathfrak{T}$, which is false. So

$$|\mathfrak{M}|_{2'} = 5 .$$

Again, choose G in $\mathfrak{G}$ such that $\mathfrak{F}^G \in \tilde{\mathscr{F}}$. Let $\mathfrak{F}^* = \mathfrak{F}^G$, $\mathfrak{F}_1^* = \mathfrak{F}^* \cap \mathfrak{H}$, $\mathfrak{F}_2^* = \mathfrak{F}^* \cap \mathfrak{H}_0$, where $\mathfrak{H}_0 = C(\mathfrak{F})$. Also, let $\mathfrak{Z}^* = \mathfrak{Z}^G$. Thus,

$$|\mathfrak{F}^* : \mathfrak{F}_1^*| = 2 , \qquad \mathfrak{F}^* = \mathfrak{F}_1^* \times \langle F \rangle ,$$

F inverts a subgroup $\mathfrak{P}$ of $\mathfrak{M}$ of order 5,

$$\mathfrak{Z}^* \subset \mathfrak{F}_1^* .$$

Now $|\mathfrak{H}/\mathfrak{H}_0| = 2^4$ and if $\mathfrak{H}_1/\mathfrak{H}_0 = C_{\mathfrak{H}/\mathfrak{H}_0}(F)$, then $\mathfrak{H}_1/\mathfrak{H}_0$ is a four-group. Since $\mathfrak{F}^*$ is abelian, it follows that

$$\mathfrak{F}_1^* \subseteq \mathfrak{H}_1 , \qquad |\mathfrak{F}_1^* : \mathfrak{F}_2^*| \leqq 2^2 .$$

It is important to show that

$$\mathfrak{Z}^* \subseteq \mathfrak{F}_2^* .$$

Suppose false.

Case 1. $|\mathfrak{F}_1^* : \mathfrak{F}_2^*| = 4$. Here $\mathfrak{F}_1^* = \langle \mathfrak{F}_2^*, \mathfrak{Z}^*, X \rangle$ for some $X \in \mathfrak{F}_1^*$. Let $\tilde{\mathfrak{F}} = C_{\mathfrak{F}}(X)$, $\tilde{\mathfrak{F}}_1 = C_{\mathfrak{F}}(\mathfrak{Z}^*)$. Then

$$[\tilde{\mathfrak{F}}, \mathfrak{Z}^*] = \mathfrak{Z} , \qquad [\tilde{\mathfrak{F}}_1, X] = \mathfrak{Z} .$$

The second equality shows that $Z \in \mathfrak{F}^G \cap \mathfrak{F} \subseteq \mathfrak{F}_1^*$, and then the first equality shows that $\tilde{\mathfrak{F}} \subseteq N(\mathfrak{F}_1^*)$. This, however, violates Lemma 20.26.

Case 2. $|\mathfrak{F}_1^* : \mathfrak{F}_2^*| = 2$.

Let $\mathfrak{F}_0 = C_{\mathfrak{F}}(\mathfrak{Z}^*) = C_{\mathfrak{F}}(\mathfrak{F}_1^*)$. Thus, $|\mathfrak{F} : \mathfrak{F}_0| = 2$, and so $[\mathfrak{F}_0, F] \neq 1$. By Lemma 20.13, $[\mathfrak{F}_0, F] \subseteq \mathfrak{Z}^*$, against $[\mathfrak{F}_0, F] \subseteq \mathfrak{F}$, $\mathfrak{Z}^* \not\subseteq \mathfrak{F}$.

So $\mathfrak{Z}^* \subseteq \mathfrak{F}_2^* \subseteq C(\mathfrak{F})$.

We now proceed to show that

$$\mathfrak{Z}^* \subseteq \mathfrak{F} , \quad |\mathfrak{F}_1^* : \mathfrak{F}_2^*| = 2 , \quad \mathfrak{Z} \subseteq \mathfrak{F}_2^* .$$

Namely, if $\mathfrak{F}_1^* = \mathfrak{F}_2^*$, then Lemma 20.13 gives $[\mathfrak{F}, F] \subseteq \mathfrak{Z}^*$, against

$|[\mathfrak{F}, F]| = 2^2$. So $\mathfrak{F}_1^* \supset \mathfrak{F}_2^*$. Since $\mathfrak{Z}^* \subseteq C(\mathfrak{F})$, we have $\mathfrak{F} \subseteq \mathfrak{M}^G$, and since $[\mathfrak{F}_1^*, \mathfrak{F}] = \mathfrak{Z}$, we get $\mathfrak{Z} \subset \mathfrak{F}^*$, whence $\mathfrak{Z} \subseteq \mathfrak{F}_2^*$. Thus, $[\mathfrak{F}, \mathfrak{F}^*] \subseteq \mathfrak{F} \cap \mathfrak{F}^*$, and $|[\mathfrak{F}, \mathfrak{F}^*]| = 2^3$. Hence, $[\mathfrak{F}, \mathfrak{F}^*] = \mathfrak{F}_2^*$, and so $\mathfrak{Z}^* \subset \mathfrak{F}$. Since $|\mathfrak{F}_1^*| = 2^4$, it follows that $|\mathfrak{F}_1^* : \mathfrak{F}_2^*| = 2$.

Set $\mathfrak{D} = \langle \mathfrak{P}, F \rangle$, a dihedral group of order 10. Now $\mathfrak{D}$ acts on $\mathfrak{H}_0 = C(\mathfrak{F})$. Since $\mathfrak{Z}^* \subseteq \mathfrak{F}$, we have $\mathfrak{H}_0 \subseteq \mathfrak{M}^G$. Thus, $[\mathfrak{H}_0, F] \subseteq \mathfrak{H}_0 \cap \mathfrak{F}^* = \mathfrak{F}_2^* \subset \mathfrak{F}$. So F centralizes $\mathfrak{H}_0/\mathfrak{F}$, whence $\mathfrak{P}$ centralizes $\mathfrak{H}_0/\mathfrak{F}$, and so

$$\mathfrak{H}_0 = \mathfrak{H}^0 \times \mathfrak{F}_0 ,$$

where

$$\mathfrak{H}^0 = C_{\mathfrak{H}_0}(\mathfrak{P}) , \quad \mathfrak{F}_0 = [\mathfrak{F}, \mathfrak{P}] , \quad \mathfrak{F} = \mathfrak{F}_0 \times \mathfrak{Z} .$$

Let $\mathfrak{W} = \langle \mathfrak{F}, \mathfrak{F}^* \rangle$, so that $\mathfrak{W}' \subseteq \mathfrak{F} \cap \mathfrak{F}^* \subseteq Z(\mathfrak{W})$, $|\mathfrak{W}| = 2^7$. Thus,

$$\mathfrak{F}^* = \langle \mathfrak{F} \cap \mathfrak{F}^*, U^*, V^* \rangle , \quad \mathfrak{F}_1^* = \langle \mathfrak{F} \cap \mathfrak{F}^*, U^* \rangle , \quad V^* = F ,$$
$$\mathfrak{F} = \langle \mathfrak{F} \cap \mathfrak{F}^*, U, V \rangle , \quad \mathfrak{F} \cap O_2(\mathfrak{M}^G) = \langle \mathfrak{F} \cap \mathfrak{F}^*, U \rangle ,$$

and V inverts a subgroup $\mathfrak{P}^*$ of $\mathfrak{M}^G$ of order 5.

Since

$$[\mathfrak{F}_1^*, \mathfrak{F} \cap O_2(\mathfrak{M}^G)] \subseteq \mathfrak{Z} \cap \mathfrak{Z}^* = 1 ,$$

we have

$$[U, U^*] = 1 .$$

Thus,

$$[U^*, V] = Z , \quad [U, V^*] = Z^* , \quad \text{and } [V, V^*] = T ,$$

where

$$\mathfrak{F} \cap \mathfrak{F}^* = \langle Z, Z^*, T \rangle .$$

So the isomorphism class of $\mathfrak{W}$ is determined.

Set

$$\mathfrak{K} = C(\mathfrak{W}) = C(\mathfrak{F}) \cap C(\mathfrak{F}^G) .$$

Since $\mathfrak{H}/\mathfrak{H}_0$ is an irreducible module for $\mathfrak{M}$, and since $\mathfrak{F}_2^* \subset \mathfrak{F}_1^*$, it follows that

$$\mathfrak{H} = \mathfrak{H}_0 \cdot \Omega_1(\mathfrak{H}) .$$

Thus, there is an involution I of $\mathfrak{H}$ such that

$$[Z^*, I] = Z .$$

Thus, I normalizes $\langle \mathfrak{Z}, \mathfrak{Z}^* \rangle$, since I normalizes every subgroup of $\mathfrak{F}$

which contains $\mathfrak{Z}$. By symmetry, there is an involution J in $\mathfrak{H}^G$ such that

$$[Z, J] = Z^* .$$

Thus, $\langle K\rangle = \mathfrak{F} \cap \mathfrak{F}^* \cap C(I) \cap C(J)$ is of order 2. It follows that $N(\mathfrak{F} \cap \mathfrak{F}^*)$ contains an element S of order 3 such that

$$K^S = K , \quad Z^S = Z^* , \quad Z^{*S} = Z \cdot Z^* .$$

With this information, we can show that $\mathfrak{M}/\mathfrak{H}$ is a Frobenius group of order 20. Namely, $\mathfrak{H}$ contains an element H such that $H \in C(\langle\mathfrak{Z}, \mathfrak{Z}^*\rangle)$, $K^H = KZ$. Let $\tilde{H} = H^S$. Then $\tilde{H} \in C(\langle\mathfrak{Z}, \mathfrak{Z}^*\rangle)$, $K^{\tilde{H}} = KZ^*$. Thus, $\langle\mathfrak{H}, F\rangle$ does not map onto $A_{\mathfrak{M}}(\mathfrak{F} \cap \mathfrak{F}^*)$, and so $\langle\mathfrak{H}, F\rangle$ is not a S_2-subgroup of $\mathfrak{M}$. So

$$\mathfrak{M}/\mathfrak{H} \text{ is a Frobenius group of order } 20 ,$$

$$\mathfrak{F} \cap \mathfrak{F}^* \lhd \mathfrak{T} .$$

The fact that $\mathfrak{F} \cap \mathfrak{F}^*$ is normal in T follows from the fact that $\mathfrak{F} \cap \mathfrak{F}^* \lhd \langle\mathfrak{H}, F\rangle$, and $\mathfrak{T}$ is the only S_2-subgroup of $\mathfrak{M}$ which contains $\langle\mathfrak{H}, F\rangle$. Since $S \in C(K)$, it follows that

$$Z \text{ and } K \text{ are not } \mathfrak{G}\text{-conjugate} .$$

Let $N_{\mathfrak{T}}(\mathfrak{P}) = \tilde{\mathfrak{T}}$. Thus, $\tilde{\mathfrak{T}} \supset \mathfrak{H}^0$, and $\tilde{\mathfrak{T}}/\mathfrak{H}^0$ is cyclic of order 4. Since $\tilde{H}$ centralizes $\langle Z, Z^*\rangle$, it follows that $\tilde{\mathfrak{T}} \subseteq C(\mathfrak{Z}^*) = \mathfrak{M}^G$. Let $\tilde{\mathfrak{H}} = C_{\mathfrak{H}}(\mathfrak{Z}^*)$, so that

$$|\mathfrak{T}: \tilde{\mathfrak{T}} \cdot \tilde{\mathfrak{H}}| = 2 , \qquad \tilde{\mathfrak{T}} \cdot \tilde{\mathfrak{H}} = C_{\mathfrak{T}}(\mathfrak{Z}^*) ,$$

$$\mathfrak{T} = \tilde{\mathfrak{T}} \cdot \tilde{\mathfrak{H}}\langle I\rangle .$$

Next, we show that

$$\mathfrak{N} = N(\tilde{\mathfrak{T}} \cdot \tilde{\mathfrak{H}}) , \qquad |\mathfrak{N}| = 3\,|\mathfrak{T}| .$$

In any case, $|\langle\mathfrak{T}, S\rangle| = 3\,|\mathfrak{T}|$, and $\langle\mathfrak{T}, S\rangle \subseteq \mathfrak{N}$. If $|\mathfrak{N}| = 15 \cdot |\mathfrak{T}|$, then $|N_{\mathfrak{N}}(\Omega_1(\mathfrak{T}))| = 3\,|\mathfrak{T}|$. This is false, since $I \in \Omega_1(\mathfrak{T})$, and I inverts an element of $\langle\mathfrak{T}, S\rangle$ of order 3.

Let $\mathfrak{C} = C(\mathfrak{F} \cap \mathfrak{F}^*)$, so that $\mathfrak{C} \lhd \mathfrak{N}$, $\mathfrak{C} = (\mathfrak{C} \cap \mathfrak{H})\langle F\rangle$, $\mathfrak{N}/\mathfrak{C} \cong \Sigma_4$.

We next show that

$$\mathfrak{M} \text{ has no normal subgroup of order } 4 .$$

Suppose false, and $\mathfrak{E} \lhd \mathfrak{M}$, $|\mathfrak{E}| = 4$. By Theorem 20.2, $\mathfrak{E} = \langle E\rangle$ is cyclic. It is then straightforward to check that $V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{E}); \mathfrak{T}) \lhd \langle\mathfrak{M}, \mathfrak{N}\rangle$, which is false. In particular, we have

$$\mathfrak{Z} = Z(\mathfrak{M}) = Z(\mathfrak{H}) .$$

We now tackle $\mathfrak{H}^0$. We will show that $|\mathfrak{H}^0| \leqq 4$. Suppose false. Now $\mathfrak{H}_0 = \mathfrak{H}^0 \times \mathfrak{F}_0 \lhd \mathfrak{M}$, and so $\boldsymbol{D}(\mathfrak{H}^0) \lhd \mathfrak{M}$. If $|\boldsymbol{D}(\mathfrak{H}^0)| \geqq 4$, then $\mathfrak{M}$ has a normal subgroup of order 4. This is false, so $\boldsymbol{D}(\mathfrak{H}^0) \subseteq \mathfrak{Z}$. Since $\mathfrak{H}^0 \subseteq \boldsymbol{C}(\mathfrak{Z}^*)$, it follows that

$$[\mathfrak{H}^0, \mathfrak{F}_1^*] \subseteq \mathfrak{F}^* \cap \mathfrak{H}_0 = \mathfrak{F}^* \cap \mathfrak{F} \subseteq \mathfrak{F} .$$

Since $\mathfrak{P}$ acts irreducibly on $\mathfrak{H}/\mathfrak{H}_0$, it follows that $[\mathfrak{H}^0, \mathfrak{H}] \subseteq \mathfrak{F}$. Set $\mathfrak{Y} = \mathfrak{H}^0/\mathfrak{Z}$, $\bar{\mathfrak{F}} = \mathfrak{F}/\mathfrak{Z}$, $\bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{H}_0$. If $Y \in \mathfrak{Y}$, $H \in \bar{\mathfrak{H}}$, so that $Y = Y_0\mathfrak{Z}$, $H = \mathfrak{H}_0 H_0$, set

$$\varphi_Y(H) = [Y_0, H_0]\mathfrak{Z} .$$

So $\varphi_Y \in \operatorname{Hom}(\bar{\mathfrak{H}}, \bar{\mathfrak{F}})$, and φ_Y commutes with $\mathfrak{P}$. If $\varphi_Y = 1$, then $[\mathfrak{H}, Y_0] \subseteq \mathfrak{Z}$, and so $\langle Y_0, \mathfrak{Z}\rangle \lhd \mathfrak{H}$, whence $Y_0 \in \mathfrak{Z}$, $Y = 1$. Since F normalizes $\mathfrak{H}^0$, we have $[\mathfrak{H}^0, F] \subseteq \mathfrak{H}^0 \cap \mathfrak{F}^* = \mathfrak{Z}$, and so φ_Y commutes with F. So $\{\varphi_Y \mid Y \in \mathfrak{Y}\} \cong \mathfrak{Y}$, and $\mathfrak{Y}$ may be identified with a subgroup of $\operatorname{Hom}(\bar{\mathfrak{H}}, \bar{\mathfrak{F}})$ which commutes with the action of $\langle \mathfrak{P}, F\rangle$, a dihedral group of order 10. Hence, $|\mathfrak{Y}| = 4$, $|\mathfrak{H}^0| = 8$.

Since $|\mathfrak{T}| = |\mathfrak{T}: \mathfrak{H}| \cdot |\mathfrak{H}| = 4\,|\mathfrak{H}| = 2^6 \cdot |\mathfrak{H}_0| = 2^{10} \cdot |\mathfrak{H}^0|$, we find that $|\mathfrak{T}| = 2^{13}$. Consider $\tilde{T} = \langle \mathfrak{H}^0, T_0\rangle$, where $F \in \langle \mathfrak{H}^0, T_0^2\rangle$. Since $\mathfrak{T} \cap \mathfrak{H}^G \supseteq \langle \mathfrak{Z}, F\rangle$, and $\tilde{\mathfrak{T}}/\mathfrak{T} \cap \mathfrak{H}^G$ is cyclic of order at most 4, while $\tilde{\mathfrak{T}}/\langle \mathfrak{Z}, F\rangle$ is a dihedral group of order 8, it follows that $\tilde{\mathfrak{T}} \cap \mathfrak{H}^G$ is of index at most 2 in $\tilde{\mathfrak{T}}$. This implies that $\tilde{\mathfrak{T}} \cap \mathfrak{H}^G$ contains an element $\tilde{T}$ such that $[\tilde{T}, \mathfrak{F}_1^*] = 1$, $[\tilde{T}, F] = \mathfrak{Z}^*$. Since $\boldsymbol{C}_{\tilde{\mathfrak{T}}}(\mathfrak{F}_2^*) = \langle \mathfrak{H}^0, F\rangle$, we have $\tilde{T} \in \langle \mathfrak{H}^0, F\rangle$. Since $F \in \boldsymbol{C}(\mathfrak{F}^*)$, we may choose $\tilde{T} \in \mathfrak{H}^0$. But then $\boldsymbol{C}_{\mathfrak{H}}(\tilde{T}) \supset \mathfrak{H}_0$, and since $\mathfrak{P}$ acts irreducibly on $\mathfrak{H}/\mathfrak{H}_0$, it follows that $\mathfrak{H} = \mathfrak{H}_0 \cdot \boldsymbol{C}_{\mathfrak{H}}(\tilde{T})$. As we have already shown this forces $\tilde{T} \in \mathfrak{Z}$, against $[\tilde{T}, F] = Z^*$. So

$$|\mathfrak{H}^0| = 2^h , \qquad h = 1 \text{ or } 2 ,$$
$$|\mathfrak{T}| = 2^{10+h} .$$

Suppose $h = 1$. In this case, $\mathfrak{H} = [\mathfrak{H}, \mathfrak{P}]$ is of order 2^9. (It is precisely at this point that I made my mistake. I thought I could show that $\mathfrak{H}$ was extra special.) We argue that $\mathfrak{H}$ is not extra special. Suppose indeed that $\mathfrak{H}$ is extra special. In this case, $\mathfrak{H} \cap \mathfrak{M}^G$ is of index 2 in $\mathfrak{M}^G$, and $\mathfrak{H} \cap \mathfrak{H}^G$ is of index at most 8 in $\mathfrak{H}^G$, whence $(\mathfrak{H} \cap \mathfrak{H}^G)' = \mathfrak{Z} = \mathfrak{Z}^*$, the desired contradiction. So $\mathfrak{H}$ is not extra special. This implies that $\mathfrak{H}' = \mathfrak{F}$. Since $\mathfrak{H} = \langle \mathfrak{F}, \mathfrak{F}_1^{*\mathfrak{P}}\rangle$, and since $\tilde{\mathfrak{T}}$ is of type (2, 4) (the type of $\tilde{\mathfrak{T}}$ is uniquely determined since $F \in \tilde{\mathfrak{T}}$), it is straightforward to show that the isomorphism class of $\mathfrak{M}$ is uniquely determined, and so $G \cong {}^2F_4(2)'$, by a result of Parrott [A characterization of the Tits' simple group, to appear].

Suppose $h = 2$, so that $|\mathfrak{H}^0| = 4$, $|\mathfrak{T}| = 2^{12}$.

We argue that $\mathfrak{H}^0$ is cyclic. Suppose false. Then $\mathfrak{H}^0 = \langle Z, Y\rangle$ for some involution $\mathfrak{Y}$.

There are 15 cosets of $\mathfrak{H}_0$ in $\mathfrak{H} - \mathfrak{H}_0$. Of these, the 5 cosets $\mathfrak{H}_0 \cdot F_1^{P^i}$ contain involutions, where $\mathfrak{P} = \langle P \rangle$, $0 \leqq i \leqq 4$, and $F_1 \in \mathfrak{F}_1^* - \mathfrak{F}_2^*$. As $N_{\mathfrak{M}}(\mathfrak{P})$ is transitive on the remaining 10 cosets, either all of them or none of them contain involutions. If all contain involutions, then $\mathfrak{H}/\mathfrak{F}$ is elementary of order 2^5, so $\mathfrak{H}_1 = [\mathfrak{H}, \mathfrak{P}]$ is of order 2^9. If every coset of $\mathfrak{F}$ in $\mathfrak{H}_1$ contains involutions, then $\mathfrak{H}_1$ is forced to be extra special. This is false, and so precisely 5 cosets of $\mathfrak{F}$ in $\mathfrak{H}_1$ contain involutions, (note that $F_1 \in \mathfrak{H}_1$) and if $\{\mathfrak{F}R_j \mid 1 \leqq j \leqq 10\}$ are the remaining cosets, then none of them contain involutions, while $\mathfrak{F}R_j Y$ contains involutions for all j. We may assume that $(R_j Y)^2 = 1$. Since Y is an involution, it follows that R_j has order 4 and is inverted by Y. This implies that Y inverts $\mathfrak{H}_1/\mathfrak{Z}$, a homocyclic group of exponent 4 and order 2^8. This is false, since $F_1 \in \mathfrak{H}_1$, and so the only cosets of $\mathfrak{H}_0$ in $\mathfrak{H} - \mathfrak{H}_0$ which contain involutions are the $\mathfrak{H}_0 F_1^{P^i}$.

Now $[F_1, Y] = Z^*$, and so $\mathfrak{F}YF_1$ contains no involutions. Thus, all involutions of $\mathfrak{H}_0 F_1$ are contained in $\mathfrak{F}F_1$, and there are thus 16 involutions in $\mathfrak{H}_0 F_1$, namely, $\tilde{\tilde{\mathfrak{F}}}F_1$, where $\tilde{\tilde{\mathfrak{F}}} = C_{\mathfrak{F}}(F_1)$. Now $\tilde{\tilde{\mathfrak{F}}} = \langle \mathfrak{F} \cap \mathfrak{F}^* \rangle \times \langle \tilde{F} \rangle$, and so every involution of $\mathfrak{H}_0 F_1$ is either conjugate to an element of $\mathfrak{F}$ or is in $(\mathfrak{F} \cap \mathfrak{F}^*)\tilde{\tilde{\mathfrak{F}}}F_1$.

Let $\mathfrak{H}^1 = C_{\mathfrak{H}}(Z^*)$, and choose $H \in \mathfrak{H} - \mathfrak{H}^1$. If $F_1^H \in (\mathfrak{F} \cap \mathfrak{F}^*)F_1$ then $H \in N(\mathfrak{F}_1^*) \subseteqq N(\mathfrak{F}^*)$. This is false, and so $F_1^H \in (\mathfrak{F} \cap \mathfrak{F}^*)\tilde{\tilde{\mathfrak{F}}}F_1$, whence $((\mathfrak{F} \cap \mathfrak{F}^*)\tilde{\tilde{\mathfrak{F}}}F_1)^H = (\mathfrak{F} \cap \mathfrak{F}^*)F_1$. Thus, all involutions of $\mathfrak{H} - \mathfrak{H}_0$ are fused to elements of $\mathfrak{F}$.

Since $\mathfrak{M}$ has 3 orbits on $\mathfrak{F}^\#$, with representatives $\{Z, Z^*, K\}$, it follows that every involution of $\mathfrak{H} - \mathfrak{H}_0$ is fused to precisely one of Z, K.

Now $Y \in C(Z^*)$, and $[F_1, Y] = Z^*$, whence $|C_{\mathfrak{Z}^*/\mathfrak{Z}^*}(Y)| \geqq 2^3$. So $Y \in \mathfrak{H}^G - \mathfrak{H}_0^G$, and so Y is fused to either Z or K. Since $\mathfrak{H}^0$ is a four-group, $\mathfrak{P}$ is a S_5-subgroup of $\mathfrak{G}$, and $N(\mathfrak{P}) \subseteqq \mathfrak{M}$. Thus, Y and Z are not fused in $\mathfrak{G}$. So Y and K are fused, whence $15 \mid |C(Y)|$. This is false, since $N(\mathfrak{P}) \subseteqq \mathfrak{M}$, and $\mathfrak{P}$ is a S_5-subgroup of $\mathfrak{G}$. This contradiction shows that $\mathfrak{H}^0$ is cyclic of order 4.

The exact determination of the isomorphism type of $\mathfrak{M}$ is now straightforward, if somewhat detailed. Thus, $G \cong {}^2F_4(2)$, by a result of Hearn.

The proof of the (augmented) Main Theorem is complete.

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