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ENTROPY OF SELF-HOMEOMORPHISMS OF STATISTICAL PSEUDO-METRIC SPACES

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A pseudo-Menger space is a set X together with a function $\theta: X \times X \rightarrow \mathscr{D}$, the set of distribution functions, satisfying certain natural axioms similar to those of a pseudo-metric space. Let $T: X \rightarrow X$ be a bijection and let θ_T denote the topology generated by $\{T^i U(p, \varepsilon, \lambda): i \in Z, p \in X, \varepsilon > 0, \lambda > 0\}$ where $U(p, \varepsilon, \lambda) = \{q: \theta(p, q)(\varepsilon) > 1 - \lambda\}$. Assume that θ_T is compact. Let $h(T, \theta)$ denote the topological entropy of T with respect to the θ_T topology. The purpose of this note is to show that if one is given a sequence $\{\theta_n\}$ of pseudo-Menger structures on X satisfying $\theta_n(p, q) \geqq \theta(p, q)$ and $\theta_n(p, q) \rightarrow \theta(p, q)$ in distribution for all $p, q \in X$ then $h(T, \theta_n) \rightarrow h(T, \theta)$. A counterexample is then given to show that, in general, the condition $\theta_n(p, q) \geqq \theta(p, q)$ cannot be removed.

1. The investigation of statistical metric spaces was undertaken by Karl Menger [5] in 1942. Essentially these are spaces in which the "distance" between any two points is given by a probability distribution function. Our purpose is to investigate the behavior of the topological entropy of a self-homeomorphism of a compact Menger space under perturbations of these distribution functions. We proceed to give precise definitions.

2. Preliminaries. Let I denote the closed unit interval, Q^+ the positive rationals, Z^+ the positive integers, and $\mathscr{D}$ the set of all left-continuous monotone increasing functions $F: \boldsymbol{R} \rightarrow I$ satisfying $F(0) = 0$ and $\sup F(x) = 1$. Let H be the function defined by: $H(t) = 0$ for $t \leqq 0$ and $H(t) = 1$ for $t > 0$.

Throughout our discussion, X will be a fixed set. Let $\mathscr{F}$ denote the collection of all functions $\theta: X \times X \rightarrow \mathscr{D}$. For convenience we shall often write θ_{pq} in place of $\theta(p, q)$. A *statistical pseudo-metric space* is an ordered pair (X, θ) where $\theta \in \mathscr{F}$ satisfies

(a) $\theta_{pq} = \theta_{qp}$ for all $p, q \in X$.

(b) $\theta_{pq}(a + b) = 1$ whenever $\theta_{pr}(a) = \theta_{rq}(b) = 1$.

(c) $\theta_{pp} = H$ for all $p \in X$.

If, in addition, θ satisfies

(d) $\theta_{pq} = H$ only if $p = q$

then (X, θ) is a *statistical metric space*.

Let $\mathscr{S}$ denote the collection of all θ for which (X, θ) is a statistical pseudo-metric space.

A *triangular norm* is a function $\Delta: I \times I \rightarrow I$ which is associative, commutative, nondecreasing in each variable and satisfies $\Delta(y, 1) = y$ for each $y \in I$. A continuous *Menger space* [*pseudo-Menger space*] is a statistical metric space [statistical pseudo-metric space] (X, θ) for which there exists a continuous triangular norm Δ satisfying:

(e) $\theta_{pr}(a + b) \geqq \Delta(\theta_{pq}(a), \theta_{qr}(b))$ for all $p, q, r \in X$ and all $a, b \in \boldsymbol{R}$.

Let $\mathscr{M}$ denote the set of all θ for which (X, θ) is a continuous pseudo-Menger space.

If $\theta \in \mathscr{T}$ then one defines a topology on X, $\tau(\theta)$, in the following manner: If $p \in X$, $\varepsilon, \lambda > 0$, let $U(p, \varepsilon, \lambda, \theta) = \{q \in X: \theta_{pq}(\varepsilon) > 1 - \lambda\}$. These sets are a subbasis for $\tau(\theta)$. It was proven in [9] that if (X, θ) is a continuous Menger space then $\tau(\theta)$ is metrizable.

We shall be concerned with studying a bijective map $T: X \to X$. If τ is a topology on X then τ^T will denote the topology generated by $\{T^i A: A \in \tau, i \in Z\}$. We will say that a map $\theta \in \mathscr{F}$ is *T-admissible* if $\tau(\theta)^T$ is compact and $\theta(T^i x, T^i y) = H$ $(i \in Z)$ whenever $\theta_{xy} = H$.

Let $\mathscr{F}_0$, $\mathscr{S}_0$, $\mathscr{M}_0$ denote the T-admissible maps belonging to $\mathscr{F}$, $\mathscr{S}$, $\mathscr{M}$ respectively. Of course $\mathscr{M}_0 \subset \mathscr{S}_0 \subset \mathscr{F}_0$. We define a partial order on $\mathscr{F}_0$ as follows: If $\theta, \Psi \in \mathscr{F}_0$ then $\theta \leqq \Psi$ if and only if $\theta_{pq}(t) \leqq \Psi_{pq}(t)$ for all $t \in \boldsymbol{R}$ and $p, q \in X$. If $\{\theta^n: n \in Z^+\}$ and Ψ belong to $\mathscr{T}$ then $\theta^n \xrightarrow{D} \Psi$ will mean that for every $(p, q) \in X \times X$, $\theta^n_{pq}(y) \to \Psi_{pq}(y)$ for each $y \in X$ at which Ψ_{pq} is continuous.

For definitions and properties of measure theoretic entropy the reader is referred to [2] or [6]. If Σ is a σ-algebra of subsets of X, μ an invariant measure on Σ, $T\Sigma = \Sigma \pmod 0$ and Γ a sub-σ-algebra of Σ then $h_\mu(T, \Gamma)$ will denote the measure theoretic entropy of $(X, \bigvee_{-\infty}^{\infty} T^i \Gamma, \mu, T)$.

For definitions and properties of topological entropy we refer the reader to [1] and [3]. If τ is a compact topology on X for which $T^{-1}(\tau) \subset \tau$ then $h(T, \tau)$ will denote the topological entropy of (X, τ, T). In case $\theta \in \mathscr{M}_0$ we shall write $h(T, \theta)$ in place of $h(T, \tau(\theta)^T)$.

If τ is a topology on X then $\sigma(\tau)$ will denote the Borel σ-algebra generated by τ. If A and B are subsets of X let $A + B$ represent $(A \cup B) \sim (A \cap B)$.

3. A convergence theorem. If $\theta \in \mathscr{M}_0$ one can define the following relation on X: $x \sim y$ if and only if $\theta_{xy} = H$. This is an equivalence relation on X due to condition (b) of $\mathscr{S}$ above. Note that θ induces the structure of a continuous Menger space on $\tilde{X}$. Let $\pi: X \to X/\sim$ be the projection map. Since $X/\sim$ is metrizable, the topology on X is induced by a pseudo-metric. Consequently π is a continuous open and closed surjection. Let $\tilde{T}$ be the self-homeomorphism of $X/\sim$ defined by $\tilde{T}\pi = \pi T$.

LEMMA 1. *If* $\theta \in \mathscr{M}_0$ *then*
$h(T) = h(\tilde{T})$ *and*
$h(T) = \sup\{h_\mu(T)$: μ *is a T-invariant regular probability measure on the Borel sets of* $X\}$.

Proof. π induces a Boolean isomorphism $\hat{\pi}$ between the Borel σ-algebra of X and that of $X/\sim$ satisfying $\tilde{T}\hat{\pi} = \hat{\pi}T$. Since $X/\sim$ is compact Hausdorff, we can apply the theorem of Goodwyn-Dinaburg-Goodman [3].

LEMMA 2. *Let T be a bijective map of a set X onto itself and let* $\theta \in \mathscr{M}_0$. *Suppose* $\{\tau_i\}_1^\infty$ *is a sequence of sub-topologies of* $\tau = \tau(\theta)$. *Denote* $\sigma(\tau^T)$ *by* Σ *and* $\sigma(\tau_i)$ *by* Σ_i. *In addition, assume that:*

(*) *Given any regular T-invariant probability measure* μ *on* Σ, $A \in \tau$ *and* $\xi > 0$ *then there is a positive integer N such that for each* $i \geqq N$ *there exists a set* $A_i \in \Sigma_i$ *satisfying* $\mu(A + A_i) < \xi$.

Then $h(T, \tau_i^T) \rightarrow h(T, \tau^T)$.

Proof. We shall begin by assuming $h(T, \tau^T) < \infty$. Let $\varepsilon > 0$ be given. Lemma 1 allows us to choose a regular T-invariant probability measure μ on Σ such that $h_\mu(T, \Sigma) \geqq h(T, \tau^T) - \varepsilon/3$. Let $P = \{P^1, \cdots, P^s\} \subset \Sigma$ be a finite partition of X such that $h_\mu(T, P) \geqq h_\mu(T, \Sigma) - \varepsilon/3$. Choose $\delta > 0$ such that if $Q \subset \Sigma$ is a finite partition of X into s sets then $|h_\mu(T, P) - h_\mu(T, Q)| < \varepsilon/3$ provided $\sum_1^s \mu(P^i + Q^i) < 3\delta$. Since μ is regular, there exist sets $A^i \in \tau^T$ $(1 \leqq i \leqq s)$ satisfying $\mu(P^i + A^i) < \delta/3s$. Choose $K > 0$ and $B^1, \cdots, B^s \in \bigvee_{-K}^{K} T^i\tau$ such that $\mu(A^i + B^i) < \delta/3s$. This can be achieved since finite intersections of $\{T^iA: A \in \tau, i \in Z\}$ constitute a basis for τ^T. Applying condition (*) of the hypothesis, there exists an integer $N > 0$ such that for each $j \geqq N$ one can find sets $C^1, \cdots, C^s \in \bigvee_{i=-K}^{K} T^i\Sigma_j$ satisfying $\mu(B^i + C^i) < \delta/3s$ $(1 \leqq i \leqq s)$. Applying the triangle inequality one obtains $\mu(P^i + C^i) < \delta/s$. Now since $\Sigma\mu(C^i) \leqq \Sigma(\mu(P^i) + \delta/s) = 1 + \delta$ one can construct a partition $\bar{C} = \{\bar{C}^1, \cdots, \bar{C}^s\} \subset \bigvee_{i=-K}^{K} T^i\Sigma_j$ of X satisfying $\sum_{i=1}^s \mu(P^i + \bar{C}^i) \leqq s(\delta/s) + 2\delta = 3\delta$. Therefore, $|h_\mu(T, \bar{C}) - h_\mu(T, P)| < \varepsilon/3$. Consequently, for all $j \geqq N$, $h(T, \tau_j^T) \geqq h_\mu(T, \sigma(\tau_j)) \geqq h_\mu(T, \bar{C}) \geqq h_\mu(T, P) - \varepsilon/3 \geqq h_\mu(T, \Sigma) - 2/3\,\varepsilon \geqq h(T, \Sigma) - \varepsilon$. It remains only to note that since $\tau_j \subset \tau$, $h(T, \tau_j^T) \leqq h(T, \tau^T)$.

In case $h(T, \tau^T) = \infty$ a similar argument may be used.

LEMMA 3. *If* $\theta \in \mathscr{M}_0$ *and* $p \in X$ *then* $\{U(p, \varepsilon, \lambda, \theta): \varepsilon > 0, \lambda > 0\}$ *is a local basis for* $\tau(\theta)$ *at p.*

Proof. Since $\hat{\pi}$ is an isomorphism of the Borel σ-algebra of X

with that of $\tilde{X}$ such that $\hat{\pi}(U(p, \varepsilon, \lambda)) = \tilde{U}(\pi(p), \varepsilon, \lambda)$, it suffices to prove the lemma under the additional hypothesis that (X, θ) is a Menger space.

It was proven in Theorem 7.2 of [8] that $\{U(p, \varepsilon, \lambda): p \in X, \varepsilon > 0, \lambda > 0\}$ is a basis for $\tau(\theta)$. Let $z \in U(p, \varepsilon, \lambda)$ be given. We must show that there exist $\bar{\varepsilon} > 0$, $\bar{\lambda} > 0$ such that $U(z, \bar{\varepsilon}, \bar{\lambda}) \subset U(p, \varepsilon, \lambda)$. Suppose that for each positive integer j there exists $y_j \in U(p, 1/j, 1/j) \sim U(p, \varepsilon, \lambda)$. By compactness there exists a $y \in X$ and a subsequence $\{y_{j_n}\}$ such that $y_{j_n} \to y$. Now Theorem 8.1 of [8] yields $\liminf_{n\to\infty} \theta(y_{j_n}, z) = \theta(y, z)$. But $\theta(y_{j_n}, z)(1/m) > 1 - 1/m$ for $j_n \geqq m$. Hence $\theta(y, z) = H$ which implies $y = z$ contradicting the fact that $y \in X \sim U(p, \varepsilon, \lambda)$.

LEMMA 4. *If* $\theta \in \mathscr{M}_0$, $\Gamma \in \mathscr{S}_0$, *and* $\theta \geqq \Gamma$ *then* $\tau(\theta) \subset \tau(\Gamma)$.

Proof. This follows at once from Lemma 3 together with the observation that $U(p, \varepsilon, \lambda, \theta) \supset U(p, \varepsilon, \lambda, \Gamma)$.

LEMMA 5. *Given* $\theta \in \mathscr{M}_0$ *and a countable dense subset* $\{y_i\}_1^\infty$ *of* $(X, \tau(\theta))$ *then* $\{U(y_i, \varepsilon, \lambda): i \in Z^+, \varepsilon, \lambda \in Q^+\}$ *is a countable basis for* $\tau(\theta)$.

Proof. This follows from an argument similar to the one given in the proof of Lemma 3.

LEMMA 6. *Suppose* $F_n, F \in \mathscr{D}$ *and* $F_n \geqq F$ *for all* n. *Let* $F_n(x+)$ *denote the right limit of* F_n *at* x. *If* $F_n \xrightarrow{D} F$ *then* $F_n(x+) \to F(x+)$ *for all* x.

Proof. Let $x \in \boldsymbol{R}$ and $\varepsilon > 0$. Choose y to be a continuity point of F such that $y > x$ and $F(y) - F(x+) < \varepsilon$. Since $F(x+) \leqq F_n(x+) \leqq F_n(y)$ we have $F(x+) \leqq \liminf_{n\to\infty} F_n(x+) \leqq \limsup_{n\to\infty} F_n(x+) \leqq F(y)$. The desired conclusion now easily follows.

THEOREM. *Given* $\{\theta^i \in \mathscr{M}_0: i \in Z^+\}$ *and* $\Psi \in \mathscr{M}_0$ *satisfying* $\theta^i \geqq \Psi$ *for each* i *and* $\theta^i \xrightarrow{D} \Psi$ *then* $h(T, \theta^i) \to h(T, \Psi)$.

Proof. Let $A \in \tau(\Psi)$ and $\varepsilon > 0$ be given and let μ be a T-invariant probability measure on $\sigma(\tau(\Psi)^T)$. Since $(X, \tau(\Psi))$ is separable, Lemma 5 implies the existence of sequences $\{p_i \in A\}$, $\{\varepsilon_i > 0\}$ and $\{\lambda_i > 0\}$ satisfying $\bigcup_{i=1}^\infty U(p_i, \varepsilon_i, \lambda_i, \Psi) = A$. Since μ is finite, there exists a positive integer N_1 such that $\mu(A \sim \bigcup_{i=1}^{N_1} U(p_i, \varepsilon_i, \lambda_i, \Psi)) < \varepsilon/4$. Using the fact that $\bigcup_{j=1}^\infty U(p_i, \varepsilon_i, \lambda_i(1 - 2^{-j}), \Psi) = U(p_i, \varepsilon_i, \lambda_i, \Psi)$ we can select $0 < \bar{\lambda}_i < \lambda_i$ for $1 \leqq i \leqq N_1$ such that $\mu(A \sim \bigcup_{i=1}^{N_1} U(p_i, \varepsilon_i, \bar{\lambda}_i, \Psi)) < \varepsilon/3$. Furthermore, since Ψ_{pq} is left continuous we may choose $0 <$

$\bar{\varepsilon}_i < \varepsilon_i$ for $1 \leqq i \leqq N_1$ such that $\mu(A \sim \bigcup_{i=1}^{N_1} U(p_i, \bar{\varepsilon}_i, \bar{\lambda}_i, \Psi)) < \varepsilon/2$. Applying Lemma 6 and Egoroff's theorem, there exists a set $G \subset A$ such that $\mu(A \sim G) < \varepsilon/2$ and $\theta^n_{p_iq}(\bar{\varepsilon}_i +) \to \Psi_{p_iq}(\bar{\varepsilon}_i +)$ uniformly in $q \in G$ as $n \to \infty$ for $1 \leqq i \leqq N_1$. As a consequence of the uniform convergence of the functions $\theta^n_{p_iq}(\bar{\varepsilon}_i +)$ on G there exists a positive integer N_2 such that:

$$U(p_i, \bar{\varepsilon}_i, \bar{\lambda}_i, \Psi) \cap G \subset \{q: \theta^n_{p_iq}(\bar{\varepsilon}_i +) > 1 - \bar{\lambda}_i\} \cap G$$
$$\subset \{q: \Psi_{p_iq}(\bar{\varepsilon}_i +) > 1 - \lambda_i\} \cap G \subset U(p_i, \varepsilon_i, \lambda_i, \Psi) \subset A$$

for $1 \leqq i \leqq N_1$ and all $n \geqq N_2$. Then for $n \geqq N_2$, $\mu(A + \bigcup_{i=1}^{N_1} U(p_i, \varepsilon_i, \lambda_i, \theta^n)) < \varepsilon$. Lemma 4 implies that the $\{\tau(\theta^i)\}$ are subtopologies of $\tau(\Psi)$. An application of Lemma 2 completes the proof.

As a consequence of the theorem we obtain:

COROLLARY. *Let $\{d, d_n: n \in Z^+\}$ be pseudo-metrics on the space X and $T: X \to X$ be a bijection satisfying*

(a) $d_n(x, y) \leqq d(x, y)$ *for all* $x, y \in X$.

(b) $d_n(x, y) \to d(x, y)$ *as* $n \to \infty$.

(c) *T is a self-homeomorphism of (X, d) and of (X, d_n) for each n.*

(d) *(X, d) is compact.*

Let T_n denote the self-homeomorphism T of the topological space (X, d_n).

Then $h(T_n) \to h(T)$.

Proof. The proof follows from Theorem 5.1 of [8] where it is shown that every metric space is a continuous Menger space. Since T_n is a homeomorphism it is easy to see that $d_n(T^ix, T^iy) = 0$ whenever $d_n(x, y) = 0$. Now the theorem may be applied.

We present two examples to illustrate the use of the theorem and to show that the theorem is no longer valid if one removes the condition that $\theta_n \geqq \theta$.

EXAMPLE A. Let $Y = \{0, 1/n: n \in Z^+\}$ be endowed with the induced topology of I, $X = Y^Z$ and $T: X \to X$ be the shift defined by $T(\{y_i\}) = \{y_{i+1}\}$. The space X is metrized by:

$$d(\{y_i\}, \{z_i\}) = \sum_{-\infty}^{\infty} \frac{|y_i - z_i|}{2^{|i|}}.$$

Subtopologies of (X, d) are determined by pseudo-metrics d_n defined by:

$$d_n(\{y_j\}, \{z_j\}) = \sum_{j=-\infty}^{\infty} \frac{|y_j\omega_n(y_j) - z_j\omega_n(z_j)|}{2^{|j|}}$$

where ω_n is the characteristic function of $[1/n, 1]$. Let T_n denote the self-homeomorphism T of (X, d_n). Then $d_n \leqq d$, $d_n \to d$, $h(T_n) = \ln n$ and $h(T) = \infty$.

EXAMPLE B. Let T be the shift on X as above. For $n \geqq 1$ let u_n be the pseudo-metric on X given by:

$$u_n(\{y_j\}, \{z_j\}) = \sum_{j=-\infty}^{\infty} \frac{|t_n(y_j) - t_n(z_j)|}{2^{|j|}}$$

where

$$t_n(s) = \begin{cases} \frac{1}{n} & \text{if } s \geqq \frac{1}{n} \\ s & \text{otherwise} . \end{cases}$$

Then $u_n \to u$ as $n \to \infty$, where $u(x, y) = 0$ for all $x, y \in X$. Now $h(T, u_n) = \infty$ but $h(T, u) = 0$.

We conclude with the following speculation:

Conjecture. Given $\{\theta^i \in \mathscr{M}_0: i \in Z^+\}$ and $\Psi \in \mathscr{M}_0$ satisfying $\theta^i \xrightarrow{D} \Psi$ then

$$\liminf_{i \to \infty} h(T, \theta^i) \geqq h(T, \Psi) .$$

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