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TTF CLASSES AND QUASI-GENERATORS

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Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_A\mathscr{M}$, the category of left A-modules. In this paper the property that the torsionfree class $\mathscr{F}$ be closed under homomorphic images is investigated. Particular attention is given to the case where the torsion class $\mathscr{T}$ is torsion-torsionfree (TTF). Applications to projective quasi-generators are given.

When $\mathscr{T}$ is a TTF class the question naturally arises as to when A_t, the $\mathscr{T}$-torsion submodule of A, is contained in a certain idempotent topologizing filter of right ideals of A. This condition is shown to be equivalent to the property that the torsionfree class $\mathscr{F}$ be closed under homomorphic images. Our results generalize results of Jans [6] and Bernhardt [2] characterizing the property that the torsion theory $(\mathscr{T}, \mathscr{F})$ is centrally splitting. Dropping the assumption that $\mathscr{T}$ is TTF, further investigation of the property that $\mathscr{F}$ is closed under homomorphic images yields information as to when $\mathscr{T}$ is TTF, generalizing a result due to Rutter [10]. Finally, our methods are applied to the TTF class $\mathscr{T} = \{{}_AX \mid P \otimes {}_AX = 0\}$ where P_A is a projective right A-module. The definition of P_A being a quasi-generator is given and characterizations are obtained.

Section 2 of this paper was taken from the author's doctoral dissertation, under the direction of Professor F. L. Sandomierski, at the University of Wisconsin. Section 1 provides a generalization of the material in §2 to arbitrary TTF classes. The author expresses his gratitude to Professor Sandomierski for his guidance and encouragement.

In this paper A will be an associative ring with unit and all modules will be unitary. The left (right) annihilator of I in X will be denoted by $l_X(I)$ $(r_X(I))$. It is easy to see that for a left A-module X and a two-sided ideal I of A, $r_X(I) \cong \mathrm{Hom}_A(A/I, X)$.

Dickson [4] has defined a torsion theory for ${}_A\mathscr{M}$ to be a pair $(\mathscr{T}, \mathscr{F})$ of classes of left A-modules satisfying

(1) $\mathscr{T} \cap \mathscr{F} = \{0\}$.

(2) $\mathscr{T}$ is closed under homomorphic images.

(3) $\mathscr{F}$ is closed under submodules.

(4) For each $X \in {}_A\mathscr{M}$ there exists a (unique) submodule $X_t \in \mathscr{T}$ such that $X/X_t \in \mathscr{F}$.

A class $\mathscr{T}(\mathscr{F})$ of left modules is called a torsion (torsionfree) class provided there is a (unique) class $\mathscr{F}(\mathscr{T})$ such that $(\mathscr{T}, \mathscr{F})$ is a torsion theory. A torsion theory $(\mathscr{T}, \mathscr{F})$ is said to be hereditary if the torsion class $\mathscr{T}$ is closed under submodules. For further information on torsion and torsionfree classes the reader is referred to [4].

Gabriel [5] has shown that for a ring A there is a one-to-one correspondence between the hereditary torsion classes of ${}_A\mathscr{M}$ and the idempotent topologizing filters F of left ideals of A given by

$$\mathscr{T} \longrightarrow F(\mathscr{T}) = \{{}_AI \subseteqq {}_AA \mid A/I \in \mathscr{T}\} .$$

The inverse correspondence is given by

$$F \longrightarrow \mathscr{T}(F) = \{X \in {}_A\mathscr{M} \mid l_A(x) \in F \text{ for all } x \in X\} .$$

Jans [6] has called a torsion class $\mathscr{T}$ which is also a torsionfree class for some torsion class $\mathscr{C}$, a torsion-torsionfree (TTF) class. In this case $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$ are called the torsion theories associated with $\mathscr{T}$. In [6] it is shown that $\mathscr{T}$ is a TTF class if and only if $F(\mathscr{T})$ contains a unique minimal left ideal T. Furthermore, T is an idempotent two-sided ideal, $T = A_c$ (the $\mathscr{C}$-torsion submodule of A), and there is a one-to-one correspondence between the TTF classes $\mathscr{T}$ of ${}_A\mathscr{M}$ and the idempotent two-sided ideals of A given by

$$\mathscr{T} \longrightarrow T .$$

The inverse correspondence is given by

$$T \longrightarrow \{X \in {}_A\mathscr{M} \mid TX = 0\} .$$

1. TTF classes. Let $\mathscr{T} \subseteqq {}_A\mathscr{M}$ be a TTF class with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$. Let T be the minimal, idempotent, two-sided ideal in $F(\mathscr{T})$. One easily checks that

$$\mathscr{T} = \{{}_AX \mid TX = 0\} ,$$

$$\mathscr{F} = \{{}_AX \mid \operatorname{Hom}(A/T, X) = 0\} ,$$

and

$$\mathscr{C} = \{{}_AX \mid A/T \otimes {}_AX = 0\} .$$

Note that the $\mathscr{T}$-torsion submodule of ${}_AX$ is $X_t = r_X(T)$ while the $\mathscr{C}$-torsion submodule is $X_c = TX$. Furthermore, let

$$\mathscr{H} = \{{}_AX \mid A/T \otimes {}_AM = 0 \text{ for every submodule } M \text{ of } X\} .$$

LEMMA 1.1. *$\mathscr{H}$ is a hereditary torsion class and* $F(\mathscr{H}) = \{{}_AI \subseteqq {}_AA \mid (I\colon a) + T = A$ *for all* $a \in A\}$ *where* $(I\colon a) = \{x \in A \mid x\, a \in I\}$.

Proof. It is left to the reader to check that ${}_AX \in \mathscr{H}$ if and only if $x \in Tx$ for all $x \in X$. If ${}_AI \in F(\mathscr{H})$, then $A/I \in \mathscr{H}$. Hence $a + I \in T(a + I)$ for all $a \in A$. Thus given $a \in A$ there exists $t \in T$ such that $ta - a = (t - 1)a \in I$. Therefore, $t - 1 \in (I\colon a)$, which implies that $1 \in (I\colon a) + T$.

Conversely, for a left ideal I suppose that $(I\colon a) + T = A$ for all $a \in A$. Hence given $a \in A$, we have that $1 = x + t$ where $x \in (I\colon a)$ and $t \in T$. Thus $a + I = x(a + I) + t(a + I)$, and so $a + I \in T(a + I)$.

Lemma 1.1 can also be deduced from [9, Lemma 1.1].

LEMMA 1.2. *For the torsion class $\mathscr{C}$ the following statements are equivalent.*

(1) $(A/T)_A$ *is flat.*

(2) $\mathscr{C}$ *is hereditary (i.e.,* $\mathscr{C} = \mathscr{H}$).

(3) $\bigcap_{i=1}^n l_A(t_i) + T = A$ *for all* $t_i \in T$.

(4) $l_A(t) + T = A$ *for all* $t \in T$.

Furthermore, if $\mathscr{C}$ *is hereditary, then* $F(\mathscr{C}) = \{{}_AI \subseteqq {}_AA \mid I + T = A\}$.

Proof. (1) $\rightarrow$ (2) Take ${}_AX \in \mathscr{C}$. Since $(A/T)_A$ is flat M/TM may be viewed as a submodule of X/TX. Thus $M/TM = 0$ as $X/TX = 0$.

(2) $\rightarrow$ (3) Since $T \in \mathscr{C}$ we have that $l_A(t) \in F(\mathscr{C})$ for all $t \in T$. Thus $\bigcap_{i=1}^n l_A(t_i) \in F(\mathscr{C})$ for all $t_i \in T$ as $F(\mathscr{C})$ is closed under finite intersections. Hence $\bigcap_{i=1}^n l_A(t_i) + T = A$ for all $t_i \in T$ by Lemma 1.1.

(3) $\rightarrow$ (4) Trivial.

(4) $\rightarrow$ (1) For $t \in T$, by assumption we have that $1 = a + t'$ where $a \in l_A(t)$ and $t' \in T$. So $t = at + t't = t't \in Tt$. That $(A/T)_A$ is flat now follows by [8, Lemma 4.1].

If $\mathscr{C}$ is hereditary then ${}_AI \in F(\mathscr{C})$ if and only if $T(A/I) = A/I$; i.e., if and only if $I + T = A$.

Since the minimal idempotent ideal T in $F(\mathscr{T})$ is two-sided let

$$\mathscr{H}' = \{X_A \mid M \otimes {}_AA/T = 0 \text{ for every submodule } M \text{ of } X\}\,,$$

a hereditary torsion class in $\mathscr{M}_A$. Let I be a right ideal of A contained in $F(\mathscr{H}')$. Then $I + T = A$. If $x \in A_t = r_A(T)$, then $Ax = Ix + Tx = Ix \subseteqq I$. That is, A_t is contained in every right ideal in the filter $F(\mathscr{H}')$. Hence the question arises as to when A_t is itself in $F(\mathscr{H}')$?

THEOREM 1.3. *Let* $\mathscr{T} \subseteqq {}_A\mathscr{M}$ *be a TTF class with associated torsion theories* $(\mathscr{T}, \mathscr{F})$ *and* $(\mathscr{C}, \mathscr{T})$. *Using the above notation the following statements are equivalent.*

(1) $A_t \in F(\mathscr{H}')$.

(2) $A = A_t + A_c$.

(3) $X = X_t + X_c$ *for all* $X \in {}_A\mathscr{M}$.

(4) ${}_A(A/A_c)$ *is projective.*

(5) *$\mathscr{F}$ is closed under homomorphic images.*

(6) $\mathscr{F} = \mathscr{H}$.

(7) $\mathscr{F} \subseteqq \mathscr{C}$.

Proof. (1) $\to$ (2) This is immediate from Lemma 1.1.

(2) $\leftrightarrow$ (3) If $A = A_t + A_c$, then for every ${}_AX$ we have ${}_AX = A_tX + A_cX \subseteqq X_t + X_c$. The converse is trivial.

(2) $\to$ (4) This is Lemma 4.10 of [8].

(4) $\to$ (5) Let ${}_AX \in \mathscr{F}$ and let $\alpha\colon {}_AX \to {}_AY$ be an epimorphism. Since ${}_A(A/T)$ is projective we have an epimorphism $\alpha^*\colon \mathrm{Hom}_A(A/T, X) \to \mathrm{Hom}_A(A/T, Y)$. Thus $\mathrm{Hom}_A(A/T, Y) = 0$; hence $Y \in \mathscr{F}$.

(5) $\to$ (6) For ${}_AX \in \mathscr{H}$ consider $X_t \subseteqq X$. Now $TX_t = 0$; but also $TX_t = X_t$ since $X \in \mathscr{H}$. Thus $X \in \mathscr{F}$ as $X_t = 0$.

Conversely, let ${}_AX \in \mathscr{F}$. For $x \in X$, $Ax \in \mathscr{F}$ since $\mathscr{F}$ is closed under submodules. Thus $Ax/Tx \in \mathscr{F}$ by assumption. But $Ax/Tx \in \mathscr{T}$ since $T(Ax/Tx) = 0$. Hence $Tx = Ax$ for all $x \in X$, which implies that $X \in \mathscr{H}$.

(6) $\to$ (7) Trivial.

(7) $\to$ (2) By assumption $A/A_t \in \mathscr{C}$. Thus $A/A_t = T(A/A_t) = T + A_t/A_t$; i.e., $A_t + A_c = A$.

(2) $\to$ (1) For all $t \in T$, $r_A(t) + T = A$ since $A_t \subseteqq r_A(t)$. Thus by Lemma 1.2 $F(\mathscr{H}') = \{I_A \subseteqq A_A \mid I + T = A\}$. Hence $A_t \in F(\mathscr{H}')$.

REMARK. The equivalence of (2), (3), and (7) was recently shown by Kurata in [7]. He also shows that (5) $\to$ (7).

A torsion theory $(\mathscr{T}, \mathscr{F})$ is centrally splitting if $\mathscr{T}$ is a TTF class (with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$) and $X = X_t \oplus X_c$ for all $X \in {}_A\mathscr{M}$. The following corollary characterizing centrally splitting torsion theories is due to Jans [6] and Bernhardt [2].

COROLLARY 1.4. *Let $\mathscr{T} \subseteqq {}_A\mathscr{M}$ be a TTF class with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$. The following statements are equivalent.*

(1) $A = A_t \oplus A_c$ *(ring direct sum).*

(2) $X = X_t \oplus X_c$ *for all* $X \in {}_A\mathscr{M}$.

(3) $\mathscr{F} = \mathscr{C}$.

(4) *$\mathscr{T}$ is stable and A_c is a direct summand of A.*

(5) *$\mathscr{F}$ is closed under homomorphic images and A_t is a direct summand of A.*

(6) *A_c is a ring direct summand of A.*

Proof. $\mathscr{T}$ being stable (closed under injective hulls) is equivalent to $\mathscr{C}$ being hereditary (see e.g. [2]). Using Lemma 1.2 and Theorem 1.3 conditions (1), (2), (3), (4), and (6) are easily seen to be equivalent to the condition that ${}_A(A/A_c)$ is projective and $(A/A_c)_A$ is flat. That (1) is equivalent to (5) is easy using Theorem 1.3.

An example of a TTF class $\mathscr{T}$ for which $\mathscr{F}$ is closed under homomorphic images, but $(\mathscr{T}, \mathscr{F})$ is not centrally splitting is given in §2. See Example 2.5 (i).

In the remainder of this section we investigate the condition that the torsionfree class of a hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ be closed under homomorphic images, dropping the condition that $\mathscr{T}$ be a TTF class. Assuming $\mathscr{F}$ is closed under homomorphic images, $\mathscr{F}$ is now a torsion class; i.e., a TTF class. $(\mathscr{T}, \mathscr{F})$ is one torsion theory associated with $\mathscr{F}$. Let $(\mathscr{F}, \mathscr{L})$ be the other. The $\mathscr{T}$-torsion submodule of A, A_t, is in $F(\mathscr{F})$ since $A/A_t \in \mathscr{F}$. Furthermore, if ${}_AI \in F(\mathscr{F})$, then $A_t \cap I \in F(\mathscr{F})$, which means that $A/A_t \cap I \in \mathscr{F}$. Thus $A_t/A_t \cap I \in \mathscr{F}$. Since $A_t/A_t \cap I$ is also in $\mathscr{T}$ we have that $A_t/A_t \cap I = 0$; i.e., $A_t \subseteqq I$. Hence A_t is the minimal element of the filter $F(\mathscr{F})$. As before we have that

$$\mathscr{F} = \{{}_AX \mid A_tX = 0\} ,$$

$$\mathscr{L} = \{{}_AX \mid \mathrm{Hom}_A(A/A_t, X) = 0\} ,$$

and

$$\mathscr{T} = \{{}_AX \mid A/A_t \otimes {}_AX = 0\} .$$

(By Theorem 1.3 if $\mathscr{T}$ is a TTF class then $\mathscr{L}$ is the torsionfree class associated with $\mathscr{H}$.) Since $\mathscr{T}$ is hereditary, $(A/A_t)_A$ is flat by Lemma 1.2. Furthermore, $F(\mathscr{T}) = \{{}_AI \subseteqq {}_AA \mid I + A_t = A\}$.

If in fact $(A/A_t)_A$ is projective, then $A = l_A(A_t) + A_t$ by [8, Lemma 4.10]. Thus $l_A(A_t) \in F(\mathscr{T})$. Furthermore, $l_A(A_t)$ is the minimal element of $F(\mathscr{T})$ by an argument similar to the one used prior to Theorem 1.3. Hence $\mathscr{T}$ is a TTF class.

THEOREM 1.5. *For a hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ the following statements are equivalent.*

(1) *$\mathscr{F}$ is closed under homomorphic images and $(A/A_t)_A$ has a projective cover.*

(2) *$\mathscr{F}$ is closed under homomorphic images and A_t is finitely generated as a right A-module.*

(3) *$\mathscr{T}$ is a TTF class where $A_c = l_A(A_t)$ and $(A/A_t)_A$ is projective.*

Proof. By our above argument (3) will follow from (1) or (2) if we can show that the flat module $(A/A_t)_A$ is projective. In (1) this follows since $(A/A_t)_A$ has a projective cover [8, Lemma 1.2]. In (2) this follows by [3, corollary to Proposition 2.2].

By Theorem 1.3 (1) and (2) will follow from (3) if we show that $A = A_t + A_c$. Since $(A/A_t)_A$ is projective, $A = A_t + l_A(A_t)$ by [8, Lemma 4.10]. Thus $A = A_t + A_c$ by assumption.

COROLLARY 1.6. (Rutter [10, Proposition 2].) *Let A be a semi-perfect ring and $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory in ${}_A\mathscr{M}$. If $\mathscr{F}$ is closed under homomorphic images, then $\mathscr{T}$ is a TTF class where $A_c = l_A(A_t)$.*

COROLLARY 1.7. *Let A be a right Noetherian ring and $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory in ${}_A\mathscr{M}$. If $\mathscr{F}$ is closed under homomorphic images, then $\mathscr{T}$ is a TTF class where $A_c = l_A(A_t)$.*

2. **Projective quasi-generators.** In this section P_A will be a projective right A-module with trace ideal $T = \sum_{f \in \mathrm{Hom}(P,A)} \mathrm{im}\, f$ and A-endomorphism ring B. For further information on projective modules and the idempotent trace ideal the reader is referred to [13]. Let $\mathscr{T} = \{{}_AX \mid P \otimes_A X = 0\}$, a hereditary torsion class in ${}_A\mathscr{M}$. It is easily seen that $\mathscr{T} = \{{}_AX \mid TX = 0\}$ and $F(\mathscr{T}) = \{{}_AI \subseteqq {}_AA \mid T \subseteqq I\}$. Hence $\mathscr{T}$ is a TTF class, and the notation of § 1 applies.

Sandomierski [12] has defined an A-module ${}_AX(X_A)$ to be T-accessible if $TX = X(XT = X)$. Note that X_A is T-accessible if and only if X_A is a homomorphic image of a direct sum of copies of P_A [12]. Define an A-module X to be strongly T-accessible if every submodule of X is T-accessible. The class of T-accessible (strongly T-accessible) left A-modules is our class $\mathscr{C}(\mathscr{H})$. From Lemma 1.1 we see that ${}_AX$ is strongly T-accessible if and only if $x \in Tx$ for all $x \in X$.

We shall call P_A a quasi-generator if every T-accessible right A-module is strongly T-accessible. That is, if every submodule of a homomorphic image of a direct sum of copies of P_A is itself a homomorphic image of a direct sum of copies of P_A. This definition is dual to the definition of self-cogenerator given in [11].

THEOREM 2.1. *For P_A projective with trace ideal T the following statements are equivalent.*

(1) *P_A is a quasi-generator.*
(2) *P_A is strongly T-accessible.*
(3) *T_A is strongly T-accessible.*
(4) *${}_A(A/T)$ is flat.*
(5) *$r_A(p) + T = A$ for all $p \in P$.*
(6) *$r_A(t) + T = A$ for all $t \in T$.*

Proof. By definition P_A is a quasi-generator if and only if the torsion class $\mathscr{C}' = \{X_A \mid X \otimes_A A/T = 0\}$ is hereditary. Thus the equivalence of (1), (4), and (6) follows by Lemma 1.2. As $\mathscr{H}' = \{X_A \mid M \otimes {}_A A/T = 0$ for every submodule M of $X\}$ is the class of strongly T-accessible right A-modules, the equivalence of (2) and (5), and the equivalence of (3) and (6) follows from Lemma 1.1. That (1) implies (2) is by definition.

(2) $\to$ (1) Let X_A be T-accessible. Then X_A is a homomorphic image of P_A^I, a direct sum of copies of P_A. Now $P_A^I \in \mathscr{H}'$ as $P_A \in \mathscr{H}'$ by assumption and $\mathscr{H}'$ is closed under direct sums. Thus $X \in \mathscr{H}'$ as $\mathscr{H}'$ is closed under homomorphic images.

Ware [13] has called a projective module regular if every homomorphic image is flat.

COROLLARY 2.2. *Let P_A be a regular module. Then P_A is a quasi-generator. Hence over a regular ring, every projective is a quasi-generator.*

Proof. Let $x \in P_A$ and consider the exact sequence

$$0 \longrightarrow xA \longrightarrow P \longrightarrow P/xA \longrightarrow 0 .$$

Since P/xA is flat there exists a map $\theta: P_A \to xA$ such that $\theta(x) = x$ [13, Lemma 2.2]. Letting T be the trace ideal of P_A, we have that $x = \sum_{i=1}^n p_i t_i$ where $p_i \in P$, $t_i \in T$, $i = 1, 2, \cdots, n$. Hence $x = \theta(x) = \theta(\sum_{i=1}^n p_i t_i) = \sum_{i=1}^n \theta(p_i) t_i \in xAT = xT$. Thus P_A is strongly T-accessible, hence a quasi-generator by Theorem 2.1.

The remainder of the corollary follows from the fact that over a regular ring every projective is regular [13, Example 1, page 238].

In our present framework the question that we asked prior to Theorem 1.3 becomes the following: When is $r_A(T) = r_A(P)$ strongly T-accessible as a right A-module?

THEOREM 2.3. *Let P_A be projective with trace ideal T and $B = \operatorname{End}(P_A)$. The follwoing statements are equivalent.*

(1) *$r_A(P)$ is strongly T-accessible as a right A-module.*

(2) *$r_A(P) + T = A$.*

(3) *${}_A(A/T)$ is projective.*

(4) *$P_{\bar{A}}$ is a generator where $\bar{A} = A/r_A(P)$.*

(5) *P_A is a quasi-generator and ${}_AT$ is finitely generated.*

(6) *P_A is a quasi-generator and ${}_BP$ is finitely generated.*

Proof. The equivalence of (1), (2), and (3) follows from Theorem 1.3. That (3) is equivalent to (5) follows since P_A is a quasi-generator if and only if ${}_A(A/T)$ is flat.

(2) $\leftrightarrow$ (4) $P_{\bar{A}}$ is projective with trace ideal $\bar{T} = T + r_A(T)/r_A(T)$.

Thus $r_A(P) + T = A$ if and only if $\bar{T} = \bar{A}$; i.e., if and only if $P_{\bar{A}}$ is a generator.

(4) $\to$ (6) Since $B = \operatorname{End}(P_{\bar{A}})$, ${}_BP$ is finitely generated. Thus (6) follows by (3).

(6) $\to$ (1) Since ${}_BP$ is finitely generated we may write $P = Bp_1 + \cdots + Bp_n$ where $p_1, \cdots, p_n \in P$. Since P_A is a quasi-generator $r_A(p_i) \in F(\mathscr{H}')$ for $i = 1, \cdots, n$. Thus $r_A(T) = r_A(P) = \bigcap_{i=1}^n r_A(p_i) \in F(\mathscr{H}')$ since $F(\mathscr{H}')$ is closed under finite intersections.

REMARK. While $r_A(T) \in F(\mathscr{H}')$ implies that $F(\mathscr{H}')$ has a minimal element, it is possible for $F(\mathscr{H}')$ to have a minimal element without $r_A(T)$ being contained in $F(\mathscr{H}')$. Let A be a left perfect ring and let P_A be a faithful projective that is not a generator. $F(\mathscr{H}')$ has a minimal element L [1, Corollary 1.6], but $L \neq r_A(T)$. Otherwise $L = 0$, which implies that every right A-module is torsion; i.e., P_A is a generator.

COROLLARY 2.4. *Let P_A be projective with trace ideal T and $B = \operatorname{End}(P_A)$. The following statements are equivalent.*

(1) *P_A is a generator (progenerator).*

(2) *P_A is a (finitely generated) faithful quasi-generator and ${}_BP$ is finitely generated.*

(3) *P_A is (finitely generated) faithful and ${}_A(A/T)$ is projective.*

EXAMPLE 2.5. (i) Let A be the ring of all 2 by 2 upper triangular matrices over a field K. Let

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $P_A = eA$ is a projective quasi-generator. ${}_BP$ is finitely generated since ${}_BP = Be$ where $B = eAe$. However, P_A is not faithful, so P_A is not a generator.

Also, $T = AeA$ is a left direct summand; i.e., ${}_A(A/T)$ is projective. However, $(A/T)_A$ is not flat (see Corollary 1.4).

(ii) Let $A = \prod_{i \in I} K_i$ where $K_i = K$ a field and the index set I is countable infinite. Let $P_A = \bigoplus_{i \in I} K_i$. We may write $P_A = \bigoplus_{i \in I} e_iA$ where $e_i^2 = e_i \in A$ and $e_iA \cong K_i$ for $i \in I$. Then P_A is projective with trace ideal $T = \bigoplus_{i \in I} e_iA$. Since A is a regular ring ${}_A(A/T)$ is flat, and thus P_A is a quasi-generator. Furthermore, P_A is faithful, but ${}_BP$ is not finitely generated as ${}_AT$ is not finitely generated. Hence P_A is not a generator.

REMARK. By Corollary 2.4 (3) if A is a semiperfect ring, then a projective module P_A is a generator if and only if P_A is a faithful quasi-generator. Also, by Corollary 2.4 (2) if A is a regular ring, then a projective module P_A is a generator if and only if P_A is faithful and ${}_BP$ is finitely generated.

REFERENCES

1. J. Alin and E. Armendariz, *TTF-classes over perfect rings*, J. Austral. Math. Soc., **11** (1970), 499-503.

2. R. L. Bernhardt, *Splitting hereditary torsion theories over semiperfect rings*, Proc. Amer. Math. Soc., **22** (1969), 681-687.

3. S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc., **97** (1960), 457-473.

4. S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc., **121** (1966), 223-235.

5. P. Gabriel, *Des catégories abeliennes*, Bull. Soc. Math. France, **90** (1962), 323-448.

6. J. P. Jans, *Some aspects of torsion*, Pacific J. Math., **15** (1965), 1249-1259.

7. Y. Kurata, *On an n-fold torsion theory in the category ${}_RM$*, J. Algebra, **22** (1972), 559-572.

8. R. W. Miller, *Endomorphic rings of finitely generated projective modules*, Pacific J. Math., **47** (1973), 199-220.

9. K. Morita, *Flat modules, injective modules and quotient rings*, Math. Z., **120** (1971), 25-40.

10. E. Rutter, *Torsion theories over semiperfect rings*, Proc. Amer. Math. Soc., **34** (1972), 389-395.

11. F. L. Sandomierski, *Linearly compact modules and local Morita duality*, Proceedings of the Ring Theory Conference, Salt Lake City, Utah, March 1971.

12. ———, *Modules over the endomorphism ring of a finitely generated projective module*, Proc. Amer. Math. Soc., **31** (1972), 27-31.

13. R. Ware, *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc., **115** (1971), 233-256.

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