SHARPENED POLYNOMIAL APPROXIMATION

ALAN G. LAW AND ANN L. MCKERRACHER

In 1951, W. Wolibner showed that a real continuous function on a closed interval can be uniformly approximated by a polynomial which interpolates at prescribed points and which has a uniform norm agreeing with the function's. This fit can be sharpened to include matching of some relative extrema as well. The paper characterizes functions that permit Simultaneous Approximation and Interpolation which is Norm-Preserving and Extrema-Matching over the entire interval except, perhaps, for a subset of arbitrarily-small diameter.

1. **Introduction.** The classical Weierstrass Theorem of (uniform) polynomial approximation to real, continuous functions has been generalized in many ways [2, 3]; one type of extension concerns approximations in various normed linear spaces [3]. However, if the element to be approximated remains in the space of continuous functions on a closed and finite interval, and if the uniform approximation is by polynomial, then the generalizations involve "closeness" of approximation. As an example of this second type, the Walsh result of 1935 [2] shows existence of a uniformly-good polynomial approximation to a function that also interpolates the function at prescribed points. Deutsch and Morris [3] have pointed out that W. Wolibner [4] extended this last result in 1951; he, in essence, proved existence of a polynomial which (uniformly) approximates a continuous function, and which interpolates the function, and whose (uniform) norm is equal to that of the function on the prescribed interval. Theorem 1 shows that such a fit can easily be sharpened to include matching of relative extrema too; that is, finitely-many relative extrema of the function are relative extrema of the approximating polynomial (where a minimum corresponds to a minimum and a maximum to a maximum, of course).

If f has an infinite number of relative extrema, no polynomial can match at all of them—otherwise, the derivative of the polynomial would have an infinite number of zeros. However, some continuous functions have approximating polynomials that match all extreme points except those over a closed set of arbitrarily-small diameter; Theorem 2 shows that a function has such approximating polynomials when, and only when, the points where the function achieves its extrema constitute a (countable) convergent sequence.

Throughout the discussion, the space considered is the set $C[a, b]$ of continuous, real-valued functions endowed with the max norm,

$\| \ \|$, on the closed and finite interval $[a, b]$.

2. **Polynomial approximations which match extrema.**
Wolibner's theorem actually shows that a finite number of points in
the plane, for which the ordinates of any two adjacent points are
distinct, can be interpolated by a polynomial that preserves mono-
tonicity of the piecewise-linear function, ϕ, connecting the points. It
is clear in the proof of this theorem that the polynomial constructed
not only is monotonically increasing or decreasing where ϕ is, but
is a uniform approximation to ϕ as well. Now, any member of
$C[a, b]$ can be approximated arbitrarily closely by the piecewise linear
function which connects certain points on the graph (a sufficient
number with equally-spaced abscissae, for instance). If to such a set
of points there are adjoined, if necessary, (i) a point $(x_0, f(x_0))$ where
$|f(x_0)| = \| f \|$, and (ii) intermediate points so that any two adjacent
points in the final set will have distinct ordinates, then Wolibner's
development for the corresponding polygonal function gives:

THEOREM 1. *Suppose f is a member of $C[a, b]$ and that $t_i(1 \leq$
$i \leq m)$ are m given values in $[a, b]$.   Let $s_j(1 \leq j \leq k)$ be k points
at which f has relative extrema.   Then, for any positive ε, there exists
a polynomial p such that:*
    (i) $\| f - p \| < \varepsilon$,
    (ii) $p(t_i) = f(t_i)$ *for* $1 \leq i \leq m$,
    (iii) $\| p \| = \| f \|$, *and*
    (iv) $p(s_j) = f(s_j)$ *and p has a relative extremum at each s_j for*
$1 \leq j \leq k$ *(a maximum of p corresponds to a maximum of f, and a
minimum to a minimum).*

Thus, any f in $C[a, b]$ has a polynomial that simultaneously
approximates, interpolates, preserves norm and, further, matches
extrema at a finite number of points.

DEFINITION. Suppose f is a member of $C[a, b]$, and let X be a
given subset of $[a, b]$. f is said to have property S on X if for any
$\varepsilon > 0$ and for any prescribed finite subset, T, of $[a, b]$ there exists a
polynomial p such that:
    (i) $\| f - p \| < \varepsilon$,
    (ii) $p(t) = f(t)$ for $t \in T$,
    (iii) $\| p \| = \| f \|$, and
    (iv) at each point of X where f has a relative extremum, p
interpolates f and p has a relative extremum (of the same type)
there.

Wolibner's result is that any f has property S on the null set;

Theorem 1 can easily be used to show that f has property S on $[a, b]$ if, and only if, the set of points in $[a, b]$ at which f has relative extrema is finite. The next theorem characterizes those functions in $C[a, b]$ which have property S on sets $X = [a, b] - V$, where V is a closed set with arbitrarily-small diameter (that is, $d(V) = \sup_{x, y \in V} | x - y |$ is arbitrarily small).

THEOREM 2. *For any member, f, of $C[a, b]$, the following two assertions are equivalent:*

(a) *corresponding to any given $\varepsilon > 0$, there exists a closed subset V, with $d(V) < \varepsilon$, for which f has property S on $[a, b] - V$;*

(b) *the points of $[a, b]$ where f has relative extrema constitute a (countable) convergent sequence.*

Proof. Suppose, first, that the relative extrema of f occur only at $\{x_1, x_2, x_3, \cdots\}$ and let x_0, in $[a, b]$, denote the limit of this sequence. For $\varepsilon > 0$ the set $V = [x_0 - \varepsilon/3, x_0 + \varepsilon/3] \cap [a, b]$ has diameter less than ε and, by Theorem 1, f has property S on $[a, b] - V$.

For the converse, let E denote the set of points in $[a, b]$ at which f has relative extrema. If E is finite, there is nothing to prove; so, suppose that E is infinite. Let $\{V_n\}$, $n \geq 1$, be a sequence of closed subsets of $[a, b]$ such that f has property S on $[a, b] - V_n$, and such that $d(V_n) < 1/n$ for $n \geq 1$. Since E is infinite, $\{V_n\}$ has the finite intersection property; thus, $V = \bigcap_{n=1}^{\infty} V_n$ is nonempty. But $V \subset V_n$ implies that $d(V) = 0$ — consequently, $V = \{x_0\}$ for some x_0 in $[a, b]$. The relation

$$E = \{E \cap V\} \cup \left\{ E \cap \left([a, b] - \bigcap_{n=1}^{\infty} V_n\right)\right\}$$

$$= \{E \cap V\} \cup \bigcup_{n=1}^{\infty} \{E \cap ([a, b] - V_n)\}$$

shows that E is a countable union of finite sets and, hence, is countable. Finally, since each V_n contains both x_0 and infinitely-many elements of E, and because $d(V_n) \to 0$, x_0 must be the limit of the sequence E.

It is, perhaps, not surprising that Theorem 2 no longer holds if diameter is replaced by (Lebesgue) measure μ. Of course, (b) would still imply (a) since $\mu(V) = d(V)$; however, the following example shows that the converse would fail. For each $n = 1, 2, 3, \cdots$, let (a_n, b_n) be an open, middle-third interval removed from $[0, 1]$ in the construction of the Cantor ternary set K. Let $g(x)$ be defined as follows: $g(x) = 0$ for $x \in K$, $g(x) = x - a_n$ if $a_n < x < (a_n + b_n)/2$ and $g(x) = b_n - x$ if $b_n > x \geq (a_n + b_n)/2$, for $n \geq 1$. It is easy to check that g is in $C[0, 1]$, that it attains its relative minima on K and that it has a relative maximum at $(a_n + b_n)/2$ for each $n \geq 1$. Thus, the

corresponding set at which g has relative extrema is uncountable. On the other hand, for any $N \geqq 1$, g has only a finite number of extrema on $\bigcup_{n=1}^{N} (a_n, b_n)$; hence, with $V_N = [0, 1] - \bigcup_{n=1}^{N} (a_n, b_n)$, g has property S on $X = [0, 1] - V_N$ and $\mu(V_N)$ can be made arbitrarily small.

References

1. K. A. Bush, *Pathological functions, crinkly and wild*, J. Math. Anal. Appl., **35** (1971), 559-562.
2. P. J. Davis, *Interpolation and Approximation*, Blaisdell, New York, 1963.
3. F. Deutsch and P. D. Morris, *On simultaneous approximation and interpolation which preserves norm*, J. Approximation Theory, **2** (1969), 355-373.
4. W. Wolibner, *Sur un polynôme d'interpolation*, Colloq. Math., **2** (1951), 136-137.

UNIVERSITY OF SASKATCHEWAN, REGINA