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THE HADAMARD PRODUCT OF A AND A^*

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Coefficient-wise multiplication was introduced by Hadamard and has been studied for certain square matrices by I. Schur and later authors. For $A \in M_n(C)$, the n by n complex matrices, this paper examines the Hadamard product of A and A^*. Upper estimates are given for the largest characteristic root of this necessarily Hermitian product, and three conditions on A sufficient for the product to be positive definite are presented.

1. Preliminaries. If $A = (a_{ij})$ and $B = (b_{ij})$ are elements of $M_n(C)$, the *Hadamard product* [see 4, 5, 6] of A and B is the matrix $A \circ B = (a_{ij}b_{ij}) \in M_n(C)$. Let Σ_n denote the class of Hermitian positive definite elements of $M_n(C)$. I. Schur [7] showed that Σ_n is closed under Hadamard multiplication and this fact was further investigated in [5]. Fiedler [1] provided the result that $A \in \Sigma_n$ implies $A \circ A^{-1} \geqq I$.

Whereas the usual product of A and A^* is Hermitian and positive semidefinite, the Hadamard product $A \circ A^* = f(A)$ is necessarily Hermitian but not necessarily positive semidefinite. We first develop several facts, some of which are of interest by themselves, with which to study $f(A)$. Theorem 1, for instance, generalizes Schur's result.

NOTATION 1. We shall adopt the following additional notational conveniences. For $A \in M_n(C)$, $H(A) = (A + A^*)/2$, the *Hermitian part* and $S(A) = (A - A^*)/2$, the *skew-Hermitian part* of A, and let Π_n denote the class of $A \in M_n(C)$ for which $H(A) \in \Sigma_n$. Also let $F(A) = \{x^*Ax \mid x \in C^n, x^*x = 1\}$, the *field of values* and $F_{\text{ang}}(A) = \{x^*Ax \mid 0 \neq x \in C^n\}$, the *angular field of values* of A. Starting with the upper right and proceeding counterclockwise, number the interiors of the *quadrants* of the complex plane Q_1, Q_2, Q_3, Q_4. If S and S_0 are two sets in the complex plane their *sum* $S + S_0 = \{x + x_0 \mid x \in S, x_0 \in S_0\}$ and their *product* $SS_0 = \{xx_0 \mid x \in S, x_0 \in S_0\}$ and denote the closure of S with respect to the Euclidean norm by $\bar{S}$. Now it is clear that $A \in \Pi_n$ if and only if $F_{\text{ang}}(A) \subset \text{interior } (\bar{Q}_1 \cup \bar{Q}_4)$. Denote by $\sigma(A)$ the set of all characteristic roots of $A \in M_n(C)$, and for Hermitian A, B let $A > B$ mean $A - B \in \Sigma_n$. $X^{(m)}$ will denote the mth Hadamard power of $X \in M_n(C)$ and $J \in M_n(C)$ will be the *Hadamard identity*, the matrix of all ones. D will always be a diagonal matrix. It is well known that $\sigma(A) \subseteqq F(A) \subseteqq F_{\text{ang}}(A)$ and the latter is a positive convex cone. Both F and F_{ang} are subadditive as set-valued functions of a matrix argument.

THEOREM 1. *If* $H \in \Sigma_n$, $A \in M_n(C)$, *then* $F_{\text{ang}}(H \circ A) \subseteqq F_{\text{ang}}(A)$.

Proof. Since $H \in \Sigma_n$ we may write $H = B^*B$ where B is nonsingular. The i, j-entry of $H \circ A$ is then $\sum_{k=1}^n \bar{b}_{ki}b_{kj}a_{ij}$ so that we have

$$\begin{aligned} x^*(H \circ A)x &= \sum_{i,j,k=1}^n \bar{b}_{ki}b_{kj}a_{ij}\bar{x}_i x_j \\ &= \sum_{k=1}^n y_k^* A y_k \quad \text{where} \quad y_k^* = (\bar{b}_{kl}\bar{x}_l, \cdots, \bar{b}_{kn}\bar{x}_n) \ . \end{aligned}$$

Since $F_{\text{ang}}(A)$ is a positive convex cone and since B is nonsingular, the latter sum is in $F_{\text{ang}}(A)$ when $x \neq 0$. We then conclude $x^*(H \circ A)x \in F_{\text{ang}}(A)$ which completes the proof.

COROLLARY 1. *If* $A, B \in M_n(C)$ *and* $F_{\text{ang}}(A) \subseteqq Q_1$, *then*

$$F_{\text{ang}}(A \circ B) \subseteqq F_{\text{ang}}(B) + iF_{\text{ang}}(B) \ .$$

Proof. $F_{\text{ang}}(A) \subseteqq Q_1$ if and only if $H(A) \in \Sigma_n$ and $1/iS(A) = K \in \Sigma_n$. Now $A \circ B = H(A) \circ B + iK \circ B$ so that

$$F_{\text{ang}}(A \circ B) \subseteqq F_{\text{ang}}(H(A) \circ B) + iF_{\text{ang}}(K \circ B)$$

because of the subadditivity of F_{ang}. By Theorem 1 it then follows that $F_{\text{ang}}(A \circ B) \subseteqq F_{\text{ang}}(B) + iF_{\text{ang}}(B)$ as the corollary asserts.

COROLLARY 2. *If* $A, B \in M_n(C)$ *and* $F_{\text{ang}}(A) \subseteqq Q_1$ *and* $F_{\text{ang}}(B^*) \subseteqq Q_1$, *then* $A \circ B \in \Pi_n$.

Proof. Since $F_{\text{ang}}(B^*) \subseteqq Q_1$, $F_{\text{ang}}(B) \subseteqq Q_4$ and since $F_{\text{ang}}(A) \subseteqq Q_1$, we have by Corollary 1 that $F_{\text{ang}}(A \circ B) \subseteqq F_{\text{ang}}(B) + iF_{\text{ang}}(B) \subseteqq Q_4 + iQ_4 = Q_4 + Q_1 \subseteqq$ interior $(\bar{Q}_1 \cup \bar{Q}_4)$. That $F_{\text{ang}}(A \circ B) \subseteqq$ interior $(\bar{Q}_1 \cup \bar{Q}_4)$ means $A \circ B \in \Pi_n$ and completes the proof.

REMARK. $A \circ B \in \Pi_n$ if and only if $H(A) \circ H(B) + S(A) \circ S(B) > 0$ and thus $f(A) \in \Sigma_n$ if and only if $H(A)^{(2)} > S(A)^{(2)}$.

Proof. An easy computation shows that $H(A \circ B) = H(A) \circ H(B) + S(A) \circ S(B)$ so that the first part of the remark follows. The second portion then follows by taking $B = A^*$ and thus $S(B) = -S(A)$.

THEOREM 2. *Suppose* $A, D \in M_n(C)$ *and* D *is a nonsingular diagonal matrix. Then* $f(A) \in \Sigma_n$ *if and only* if $f(DA) \in \Sigma_n$.

Proof. Since Σ_n is closed under congruence, the statement of the theorem follows from the observation that $f(DA) = DA \circ A^*D^* = D(A \circ A^*)D^* = Df(A)D^*$.

2. The largest eigenvalue of $A \circ A^*$. Since $f(A)$ is Hermitian, $\sigma(f(A))$ is real. Employing a result of [4] we next estimate the largest member of $\sigma(f(A))$ which is necessarily nonnegative.

NOTATION 2. Denote the numerical radius of $A \in M_n(C)$ by $r(A) = \max_{t \in F(A)} |t|$. If $\sigma(A)$ is real, let $\lambda_M(A) = \max_{\lambda \in \sigma(A)} \lambda$ and $\lambda_m(A) = \min_{\lambda \in \sigma(A)} \lambda$. In case A is Hermitian, $r(A) = \max \{\lambda_M(A), |\lambda_m(A)|\}$.

LEMMA 1. [4]. *If $A, N \in M_n(C)$ and N is normal, then*

$$r(N \circ A) \leqq r(N)r(A) .$$

THEOREM 3. *For $A \in M_n(C)$, we have*

$$r(A \circ A^*) \leqq r(H(A))^2 + r(S(A))^2 .$$

Proof. Since $f(A) = H(A)^{(2)} - S(A)^{(2)}$, it follows that $r(f(A)) = r(H(A)^{(2)} - S(A)^{(2)}) \leqq r(H(A)^{(2)}) + r(-S(A)^{(2)}) \leqq r(H(A))^2 + r(S(A))^2$. The latter inequality is from the lemma and completes the proof.

COROLLARY 3. *For $A \in M_n(C)$,*

$$\lambda_M(A \circ A^*) \leqq \lambda_M(H(A)^2) - \lambda_m(S(A)^2) .$$

Proof. Since $\lambda_M(f(A)) \leqq r(f(A))$, $r(H(A))^2 = \lambda_M(H(A)^2)$, and

$$r(S(A))^2 = -\lambda_m(S(A)^2) ,$$

this follows directly from Theorem 3.

EXAMPLE. The estimates of Theorem 3 and Corollary 3 are sharp. Equality may be attained even for nonHermitian matrices. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$; then $F(A)$ is the unit closed circular disk and thus $r(A) = r(H(A)) = r(S(A)) = 1$. Also $f(A) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ so that $r(f(A)) = \lambda_M(f(A)) = 2 = r(H(A))^2 + r(S(A))^2 = \lambda_M(H(A)^2) - \lambda_m(S(A)^2)$.

Although we will not do so here, estimates for $\lambda_m(A \circ A^*)$ may straightforwardly be obtained from the results of the next section.

3. Conditions sufficient for $A \circ A^* \in \Sigma_n$. We next study three rather different sufficient conditions (Theorems 4, 5, and 6) for the Hermitian matrix $f(A)$ to be positive definite.

NOTATION 3. If $X \in M_n(C)$ denote the union of the *Gersgorin circles* [3] obtained from the rows of X by $G_r(X)$ and the union of the Gersgorin circles obtained from the columns of X by $G_c(X)$.

Let $G(X) = G_r(X) \cap G_c(X)$. Then $\sigma(X) \subseteqq G(X)$, [3], and $0 \notin G_r(X)$ is the assumption of *row diagonal dominance* while $0 \notin G_c(X)$ is *column diagonal dominance*. We shall call a matrix $T = (t_{ij}) \in M_n(C)$ *combinatorially triangular* if for all pairs $i \neq j$ either of $t_{\ j}$ or t_{ji} is 0.

THEOREM 4. *If $A \in M_n(C)$ and there is a diagonal matrix $D \in M_n(C)$ such that $F(DA) \subseteqq Q_1$, then $f(A) \in \Sigma_n$.*

Proof. If there is such a D, then it must be nonsingular and by Theorem 2 it suffices to prove the statement of this theorem for $D = I$. By letting $B = A^*$, the hypothesis of Corollary 2 is satisfied in our case and we may conclude $f(A) = A \circ A^* \in \Pi_n$. But since $f(A)$ is Hermitian it is then in Σ_n which completes the proof.

REMARK. It is an easy observation that $f(e^{i\theta}A) = f(A)$. By Theorem 3 this means that if $F_{\text{ang}}(A) \subseteqq Q$, where Q is any rotation of Q_1, then $f(A) \in \Sigma_n$.

LEMMA 2. *If $0 \notin G_r(A) \cup G_c(A)$, then $0 \notin G(f(A))$.*

Proof. Since $f(A)$ is Hermitian, $G(f(A)) = G_r(f(A)) = G_c(f(A))$. Since $0 \notin G_r(A) \cup G_c(A)$, $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ and $|a_{ii}| > \sum_{j\neq i} |a_{ji}|$, for all $i = 1, \cdots, n$. Thus

$$a_{ii}\overline{a_{ii}} = |a_{ii}|^2 > \Big(\sum_{j\neq i} |a_{ij}|\Big)\Big(\sum_{j\neq i} |a_{ji}|\Big) \geqq \sum_{j\neq i} |a_{ij}|\,|a_{ji}| = \sum_{j\neq i} |a_{ij}\overline{a_{ji}}|$$

which means that $0 \notin G(f(A))$.

LEMMA 3. *If $0 \notin G_r(A)$, there is a positive diagonal matrix D such that $0 \notin G_r(DA) \cup G_c(DA)$.*

Proof. Since D diagonal and invertible and $0 \notin G_r(A)$ imply $0 \notin G_r(DA)$, it suffices to show that under the assumption a D may be found such that $0 \notin G_c(DA)$. This may be done by an M-matrix argument [2]. Without loss of generality we may assume A is real with positive diagonal entries and nonpositive off-diagonal entries. Our assumption, $0 \notin G_r(A)$, then implies that A and thus A^* are M-matrices. By [2, Theorem 4.3] this implies the existence of a positive diagonal D such that $0 \notin G_r(A^*D) = G_c(DA)$. For this D, then, $0 \notin G_r(DA) \cup G_c(DA)$ as desired.

THEOREM 5. *If $A \in M_n(C)$ and there is a diagonal matrix $D \in M_n(C)$ such that $0 \notin G(DA)$, then $f(A) \in \Sigma_n$.*

Proof. Again by Theorem 2 it suffices to prove the weaker statement that $0 \notin G(A)$ implies $f(A) \in \Sigma_n$, and since $f(A) = f(A^*)$ we may assume without loss of generality that $0 \notin G_r(A)$. Then by Lemma 3, there is a positive diagonal matrix D such that $0 \notin G_r(DA) \cup G_c(DA)$. According to Lemma 2 this implies $0 \notin G(f(DA))$. Since $f(DA)$ is Hermitian with nonnegative diagonal entries, $0 \notin G(f(DA))$ implies $G(f(DA)) \subseteqq$ interior $(\bar{Q}_1 \cup \bar{Q}_4)$ and that all eigenvalues of $f(DA)$ are positive. This means that $f(DA) \in \Sigma_n$ and by Theorem 2 that $f(A) \in \Sigma_n$ which completes the proof.

THEOREM 6. *If* $A = (a_{ij}) \in M_n(C)$ *is combinatorially triangular and* $a_{ii} \neq 0$, $i = 1, \cdots, n$, *then* $f(A) \in \Sigma_n$.

Proof. Under the hypothesis $a_{ij}\overline{a_{ji}}$ is 0 if $i \neq j$ and positive if $i = j$. This means $f(A)$ is a positive diagonal matrix and, therefore, a member of Σ_n.

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