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FINITELY GENERATED F-ALGEBRAS WITH APPLICATIONS TO STEIN MANIFOLDS

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Let A be a commutative n-generated F-algebra with identity, M_A the spectrum of A. Conditions are given which insure that M_A can be mapped homeomorphically into C^n and that A is topologically isomorphic to the algebra of all holomorphic functions on a polynomially convex open subset of C^n. A characterization is given of those Stein manifolds X for which $\mathcal{O}(X)$ is finitely generated.

1. Introduction. We seek conditions on a commutative F-algebra A with identity that force A to be topologically isomorphic to an algebra of all functions analytic on a polynomially convex open subset of C^n. Several conditions for the case A is singly-generated have been given by Arens in [1], Brooks in [4], and Carpenter [6]. Birtel in [3] characterized the algebra of all entire functions on C.

Our results are for the case A is n-generated. We first show that, under various conditions, the spectrum of A can be mapped homeomorphically into C^n (see Theorems 3.4 and 3.6). Theorem 3.7 shows that if the spectrum of A is a topological $2n$-manifold without boundary, then A is an algebra of all analytic functions on an open, polynomially convex subset of C^n. In Theorem 3.8 we obtain an F-algebra characterization of the algebra of all entire functions on C^n.

In § 4, we apply the results of § 3 to Stein manifolds. We obtain a characterization of those Stein manifolds whose algebras of analytic functions are finitely generated.

2. Preliminaries. An F-algebra A is a complete topological algebra over the complex numbers in which the topology is given by a countable family $\{\|\cdot\|_k : k = 1, 2, \cdots\}$ of algebraic seminorms. It can be assumed that the seminorms are increasing; i.e., $\|f\|_k \leqq \|f\|_{k+1}$ for each positive integer k and each f in A. All algebras in this paper are assumed to be commutative and contain units.

For each seminorm $\|\cdot\|_k$, we can obtain a Banach algebra A_k by letting A_k be the completion of the quotient algebra $A/(\ker \|\cdot\|_k)$ with respect to the norm induced on $A/(\ker \|\cdot\|_k)$ by $\|\cdot\|_k$. It follows that A is the inverse limit of the Banach algebras A_k, denoted $A = \lim \operatorname{inv} A_k$. For details of this construction the reader is referred to [10].

The spectrum of A, denoted by M_A, is the space of all continuous homomorphisms of A onto the complex numbers with the Gelfand topology (the relative weak * topology). For each positive integer k, the spectrum M_k of A_k is a compact Hausdorff space and is imbedded homeomorphically in M_A by the dual map π_k^* of π_k, where π_k is the natural projection of A into A_k. It follows that $M_A = \bigcup_{k=1}^{\infty} M_k$ and any compact set in M_A is a subset of M_k for some k. Hence, M_A is a hemicompact Hausdorff space.

For each f in A, we define $\hat{f}: M_A \rightarrow C$ by $\hat{f}(\phi) = \phi(f)$ for each $\phi \in M_A$. This mapping is called the Gelfand transform of f, and the mapping $f \rightarrow \hat{f}$ is a homomorphism of A onto a separating subalgebra $\hat{A}$ of $C(M_A)$. We define seminorms on the algebra $\hat{A}$ by letting $|\hat{f}|_k = \sup\{|\hat{f}(x)|; x \in M_k\}$ for each f in A. The topology on $\hat{A}$ defined by these seminorms is the compact-open topology.

If each of the seminorms for an F-algebra A satisfies $\|f^2\|_k = \|f\|_k^2$ for each f in A, then A is called a uniform algebra. If A is a uniform algebra, then $\|f\|_k = |\hat{f}|_k$ for each f and each k. Hence, for uniform algebras, the mapping $f \rightarrow \hat{f}$ is a topological isomorphism of A onto a complete subalgebra of $C(M_A)$. Therefore, a uniform F-algebra is a complete, hence closed, subalgebra of $C(X)$ where X is a hemicompact Hausdorff space.

The spectrum M_A also satisfies "convexity" conditions which have been particularly prominent in the study of algebras of analytic functions on complex manifolds. In particular if E is compact in M_A then the $\hat{A}$-hull of E is

$$\hat{E} = \{\phi \in M_A: |\hat{f}(\phi)| \leqq \|\hat{f}\|_E \text{ for all } \hat{f} \in \hat{A}\} .$$

If $\hat{E} = E$ then E is said to be $\hat{A}$-convex. Since each M_k is the spectrum of the Banach algebra A_k it follows easily that M_k is $\hat{A}$-convex. Furthermore, this implies that $\hat{E}$ is compact in M_A for any compact set E; in this case M_A is said to be $\hat{A}$-convex. For a discussion of convexity in function algebras see Rickart [12].

For a hemicompact complex analytic manifold X, the algebra $\mathcal{O}(X)$ of analytic functions is a uniform F-algebra (compact-open topology). It is possible that the spectrum of $\mathcal{O}(X)$ is also a complex manifold which is $\mathcal{O}(X)$ convex (more commonly called holomorphically convex); such a manifold is called a Stein manifold. A particular example is C^n itself. In this case, since analytic polynomials are dense in $\mathcal{O}(C^n)$, the phrase polynomial convexity is used rather than holomorphic convexity. We will say that an arbitrary subset of C^n is polynomially convex if it is the hemicompact union of polynomially convex compact subsets of C^n.

3. Finitely generated F-algebras.

An F-algebra is n-generated provided there exist elements $f_1, \cdots, f_n$ in A such that A is the closure of the polynomials in $f_1, \cdots, f_n$. In order that a family $\{f_1, \cdots, f_n\}$

generate A it is necessary and sufficient that for each k the family $\{\pi_k f_1, \cdots, \pi_k f_n\}$ generate A_k.

In this section we will give conditions on the spectrum of A which will insure that an n-generated algebra A is topologically isomorphic to the algebra of all functions holomorphic on some polynomially convex open subset U of C^n (denoted $\mathscr{O}(U)$). We will first consider the problem of mapping the spectrum of A homeomorphically into C^n.

DEFINITION 3.1. Suppose A is an n-generated F-algebra with spectrum M_A. A map $F: M_A \to C^n$ is called a *spectrum map* provided there exist generators $f_1, \cdots, f_n$ for A such that $F(\phi) = (\hat{f}_1(\phi), \cdots, \hat{f}_n(\phi))$ for each ϕ in M_A.

It is easy to show that any spectrum map is one-to-one and continuous. In [5] examples are given where (i) one spectrum map is a homeomorphism (onto its image), while another is not, and (ii) no spectrum map is a homeomorphism. Since M_k is the spectrum of a finitely generated Banach algebra, $F \mid M_k$ is a homeomorphism and $F(M_k)$ is a polynomially convex compact set in C^n (see [9]). The following theorem, which is Theorem 1.3 of [5] for the case A is an F-algebra, gives a necessary and sufficient condition in order that a spectrum map be a homeomorphism.

THEOREM 3.2. *Suppose A is an n-generated F-algebra with spectrum M_A and $M_A = \bigcup_{k=1}^{\infty} M_k$. A spectrum map $F: M_A \to C^n$ is a homeomorphism if and only if for each compact $E \subseteqq F(M_A)$ there exists a positive integer k such that $E \subseteqq F(M_k)$.*

COROLLARY 3.3. *If F is a homeomorphism then $F(M_A)$ is polynomially convex.*

Proof. Since $F(M_A) = \bigcup_{k=1}^{\infty} F(M_k)$, F is a homeomorphism, and $F(M_k)$ is polynomially convex, it follows that $F(M_A)$ is the hemicompact union of polynomially convex compact subsets of C^n; hence, $F(M_A)$ is polynomially convex.

We will now give a sufficient condition in order that a spectrum map be a homeomorphism and have a closed image in C^n.

THEOREM 3.4. *Suppose A is an n-generated F-algebra with spectrum M_A and $F: M_A \to C^n$ is a spectrum map defined by generators $f_1, \cdots, f_n$. If for each $c > 0$ the set $K_c = \{\phi \in M_A: |\phi(f_i)| \leqq c$ for $i = 1, \cdots, n\}$ is compact in M_A, then F is a homeomorphism and $F(M_A)$ is closed in C^n.*

Proof. Let E be a compact subset of $F(M_A)$. Then there exists

$c > 0$ such that $F^{-1}(E) \subseteqq K_c$. Since F is continuous and K_c is compact, $F^{-1}(E)$ is compact; hence, $E \subseteqq F(M_k)$ for some k. Theorem 3.2 now implies that F is a homeomorphism.

Suppose λ is a limit point of $F(M_A)$. Then there exists a sequence $\{F(\phi_n)\}$ converging to λ. There exists $c > 0$ such that $\{\phi_1, \phi_2, \cdots\}$ is a subset of K_c. Hence, some subsequence $\{\phi_{k_n}\}$ converges to $\phi \in K_c$. Since F is continuous, $F(\phi) = \lambda$ and $\lambda \in F(M_A)$. Thus $F(M_A)$ is closed in C^n.

Suppose A is an n-generated uniform F-algebra with spectrum $M_A = \bigcup_{k=1}^{\infty} M_k$ and $F: M_A \to C^n$ is a spectrum map defined by the generators $f_1, \cdots, f_n$. Let $\operatorname{Hol} F(M_A)$ denote the algebra of all functions h such that h is holomorphic in some open set U_h containing $F(M_A)$. Let the topology on $\operatorname{Hol} F(M_A)$ be generated by the family of seminorms $\{\|\cdot\|_{F(M_k)}: k = 1, 2, \cdots\}$. The map $F^*: \operatorname{Hol} F(M_A) \to \hat{A}$ is defined by $F^*(h) = h \circ F$ for each h in $\operatorname{Hol} F(M_A)$. The function calculus for F-algebras (see [2]) implies that $h \circ F$ is in $\hat{A}$. The following lemma is immediate.

LEMMA 3.5. *F^* is an isometric isomorphism of* $\operatorname{Hol} F(M_A)$ *onto a dense subalgebra of $\hat{A}$.*

THEOREM 3.6. *F is a homeomorphism if and only if the topology on* $\operatorname{Hol} F(M_A)$ *is equivalent to the compact open topology.*

Proof. If F is a homeomorphism, then any compact subset of $F(M_A)$ is a subset of $F(M_k)$ for some k; hence the topology on $\operatorname{Hol} F(M_A)$ is the compact open topology.

Suppose that the topology on $\operatorname{Hol} F(M_A)$ is the compact open topology. Let E be a compact subset of $F(M_A)$. If for each k, $E \not\subseteqq F(M_k)$ then there exists a sequence $\{\lambda_k\}$ such that $\lambda_k \in E \backslash F(M_k)$. Since $F(M_k)$ is polynomially convex there exists a polynomial P_k of n-variables such that $P_k(\lambda_k) \geqq 1$ and $\|P_k\|_{F(M_k)} < 1/k$. Then the sequence $\{P_k\}$ is a sequence in $\operatorname{Hol} F(M_A)$ which converges to 0. Since the topology on $\operatorname{Hol} F(M_A)$ is equivalent to the compact open topology the sequence $\{P_k\}$ converges uniformly to 0 on the set E. But $P_k(\lambda_k) \geqq 1$ for each k, hence $\{P_k\}$ cannot converge uniformly to 0 on E. This is a contradiction, therefore, it must be that $E \subseteqq F(M_k)$ for some k. Consequently, by Theorem 3.2, F is a homeomorphism.

We will now give an F-algebra characterization of the algebra of all functions holomorphic on a polynomially convex open subset of C^n.

THEOREM 3.7. *An n-generated, uniform F-algebra is topologically* *isomorphic to the algebra of all functions holomorphic on a polynomially convex open subset of* C^n *if and only if the spectrum of* A *is a topological* $2n$*-dimensional manifold (without boundary).*

Proof. If A is topologically isomorphic to $\mathscr{O}(U)$ where U is a polynomially convex open subset of C^n, then the spectrum of A is homeomorphic to the spectrum of $\mathscr{O}(U)$ which is U. Hence M_A is a topological $2n$-manifold.

Suppose A is generated by $f_1, \cdots, f_n$ and F is the associated spectrum map. Since M_A is a $2n$-manifold, for each point ϕ of M_A there exists an open set V_ϕ in M_A and a map h_ϕ: $V_\phi \to R^{2n}$ such that $\phi \in V_\phi$, $\bar{V}_\phi$ is compact, and h_ϕ is a homeomorphism of V_ϕ onto an open set in R^{2n}. Since $\bar{V}_\phi$ is compact, $F \mid \bar{V}_\phi$ is a homeomorphism, hence $F \mid V_\phi$ is a homeomorphism of V_ϕ into C^n. The Brouwer domain invariance theorem (see [7], page 358) implies that $F(V_\phi)$ is an open subset of C^n. It now follows that F is a homeomorphism and that $U = F(M_A)$ is an open subset of C^n. U is polynomially convex since $F(M_k)$ is polynomially convex for each k and U is the hemicompact union of these sets. Since A is a uniform algebra, A is topologically isomorphic to $\hat{A}$. Lemma 3.5 implies that $\mathrm{Hol}\, F(M_A)$ is topologically isomorphic to a dense subalgebra of $\hat{A}$. Since F is a homeomorphism, the topology on $\mathrm{Hol}\, F(M_A)$ is equivalent to the compact open topology (Theorem 3.6). But $F(M_A)$ is open, so $\mathrm{Hol}\, F(M_A)$ is topologically isomorphic to $\mathscr{O}(U)$. Since $\mathscr{O}(U)$ is complete, it follows that $\hat{A}$ is topologically isomorphic to $\mathscr{O}(U)$; hence A is topologically isomorphic to $\mathscr{O}(U)$.

THEOREM 3.8. *Suppose* A *is a uniform* F*-algebra generated by* $f_1, \cdots, f_n$, M_A *is a topological* $2n$*-manifold, and for each* $c > 0$ *the set* $K_c = \{\phi \in M_A \colon |\phi(f_i)| \leqq c \text{ for } i = 1, \cdots, n\}$ *is compact in* M_A. *Then* A *is topologically isomorphic to* $\mathscr{O}(C^n)$.

Proof. From the proof of Theorem 3.7 we know that $U = F(M_A)$ is open in C^n and A is topologically isomorphic to $\mathscr{O}(U)$. Theorem 3.4 implies that U is closed in C^n. Since C^n is connected, $U = C^n$ and A is therefore topologically isomorphic to $\mathscr{O}(C^n)$.

4. Stein manifolds. If X is a Stein manifold, then $\mathscr{O}(X)$ is a uniform F-algebra with the compact open topology. It follows from the proper imbedding theorem (Hörmander [9], Theorem 5.3.9) that Stein manifolds are just closed submanifolds of C^n. Hence from Rickart [12] Theorem 1.3, it follows that $X = M_{\mathscr{O}(X)}$. So, if $\mathscr{O}(X)$ is n-generated then a spectrum map F can be viewed as an analytic map of X into C^n. In this section we give a more complete description of the image $F(X)$ when F is a homeomorphism and finally give a charac-

terization of those Stain manifolds for which $\mathscr{O}(X)$ is finitely-generated. For properties of analytic maps, see Narasimhan [11] and Herve [8].

THEOREM 4.1. *If X is a Stein manifold and $\mathscr{O}(X)$ is generated by $f_1, \cdots, f_n$ then*

(i) $n \geqq \dim X$;

(ii) *if F, the associated spectrum map, is a homeomorphism then $F(X)$ is a closed polynomially convex submanifold of an open set in C^n and F is bianalytic;*

(iii) *if $n = \dim X$ then $F(X)$ is a polynomially convex open set in C^n.*

Proof. Part (i) follows from the fact that an injective analytic map can not decrease dimension.

To prove (ii), first note that since F is injective, each $x \in X$ has a fundamental system of neighborhoods whose images have germs at $F(x)$ which are the germs of analytic varieties. But since F is a homeomorphism, the germ of the images of the neighborhood at $F(x)$ are the same as the germ of $F(X)$ at $F(x)$ so $F(X)$ is an analytic subvariety of some open set in C^n. To show that $F(X)$ is a submanifold it suffices to show that $F^{-1}: F(X) \to X$ is analytic.

$\mathscr{O}(X)$ provides local coordinate system for X, so to show F^{-1} is analytic it suffices to show $g \circ F^{-1} \in \mathscr{O}(F(X))$ for $g \in \mathscr{O}(X)$. But since F is a homeomorphism $\mathrm{Hol}\, F(X)$ has the compact open topology and $g \circ F^{-1}$ is in the closure of $\mathrm{Hol}\, F(X)$ in $C(F(X))$. Furthermore, $\mathrm{Hol}\, F(X) \subset \mathscr{O}(F(X))$ and $\mathscr{O}(F(X))$ is closed in $C(F(X))$ so $g \circ F^{-1} \in \mathscr{O}(X)$.

If X is given as a hemicompact union of $\{M_k\}$ where $M_k = \hat{M}_k$ then $F(M_k)$ is polynomially convex; since F is a homeomorphism $F(X)$ is the hemicompact union of $\{F(M_k)\}$ hence is also polynomially convex.

Part (iii) follows immediately from Theorem 3.7.

THEOREM 4.2. *If X is a Stein manifold then $\mathscr{O}(X)$ is finitely generated if and only if X can be imbedded as a closed polynomially convex submanifold of C^N for some N.*

Proof. The proper imbedding theorem states that there is an analytic, regular, proper map $G: X \to C^n$. If $\mathscr{O}(X)$ is finitely generated and $F: X \to C^m$ is any spectrum map then $G \times F$ is a spectrum map. Furthermore, it is regular and proper, hence is an imbedding. That the image is polynomially convex follows from 4.1. (ii).

The converse follows immediately from the following lemma.

LEMMA 4.3. *If Ω is a domain of holomorphy in C^n and X is a polynomially convex submanifold of Ω then $\mathcal{O}(X)$ is generated by* $\{z_1|_X, \cdots, z_n|_X\}$.

Proof. Since Ω is a domain of holomorphy $\mathcal{O}(X)$ is precisely the restriction of $\mathcal{O}(\Omega)$ to X. Furthermore, each function in $\mathcal{O}(\Omega)$ can be approximated on polynomially convex compact subsets of X by analytic polynomials (Hörmander [9], Theorem 2.7.7).

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