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A COUNTEREXAMPLE TO A CONJECTURE ON AN INTEGRAL CONDITION FOR DETERMINING PEAK POINTS (COUNTEREXAMPLE CONCERNING PEAK POINTS)

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Let X be a compact plane set. Denote by $R(X)$ the uniform algebra generated by the rational functions with poles off X and by $H(X)$ the space of functions harmonic in a neighborhood of X endowed with the sup norm. A point $p \in \partial X$ is a peak point for $R(X)$ if there exists a function $f \in R(X)$ such that $f(p) = 1$ and $|f(x)| < 1$ if $x \neq p$. Moreover, p is a peak point for $H(X)$ (consider Re f) and hence, by a theorem of Keldysh, p is a regular point for the Dirichlet problem. Conditions which determine whether or not a point is a peak point for $R(X)$ are thus of interest in harmonic analysis. Melnikov has given a necessary and sufficient condition that p be a peak point for $R(X)$ in terms of analytic capacity, γ; namely p is a peak point for $R(X)$ if and only if

$$\sum_{n=0}^{\infty} 2^n \gamma(A_{np} \backslash X) = \infty \, . \quad A_{np} = \left\{z: \frac{1}{2^{n+1}} \leqq |z - p| \leqq \frac{1}{2^n}\right\} .$$

Analytic capacity is generally difficult to compute, so it is desirable to obtain more computable types of conditions. Let $X^c = C \backslash X$ and

$$I = \{t \in [0, 1]: z \in X^c \text{ and } |z| = t\} .$$

In this note the following conjecture, which can be found in Zalcman's Springer Lecture Notes and which is true for certain sets X, is shown to be false in general:

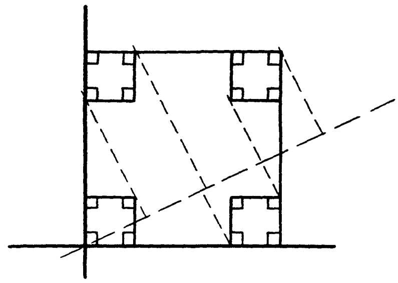
Conjecture. **If $\int_I t^{-1} dt = \infty$ then 0 is a peak point for $R(X)$.**

Our counterexample uses Melnikov's theorem and the following lemma:

LEMMA. *Given $0 < a < b$ and $\log b/a < 2\pi$ there exists a set K_{ab} such that $K_{ab} \subset \{z: a \leqq |z| \leqq b\}$, $\gamma(K_{ab}) = 0$ and $\{t: z \in K_{ab}$ and $|z| = t\} =$ $[a, b]$.*

The author is indebted to the referee for the following proof.

Garnett in [3] showed that the "Cantor corner square" set K constructed by removing all but the four corner squares of length 1/4 from the unit square, then removing all but the sixteen corner squares of length 1/16 from



these four squares, etc., has zero analytic capacity while the projection on the line $y = x/2$ is full, i.e., is the same as the projection of the unit square on that line. Thus given $0 < a < b$ and $\log b/a < 2\pi$ there exists after a suitable rotation, expansion and translation a compact plane set L_{ab} such that $\gamma(L_{ab}) = 0$, the projection on the x-axis of L_{ab} is $[\log a, \log b]$ and $L_{ab} \subset \{z: \log a \leqq x \leqq \log b, 0 \leqq y < 2\pi\}$. Let $K_{ab} = \{e^z: z \in L_{ab}\}$. Let W be a small neighborhood of L_{ab} such that the exponential map is 1-1 on W and $V = \{e^z: z \in w\}$. If g is bounded and analytic on $V \backslash K_{ab}$ then $g(e^z)$ is bounded and analytic on $W \backslash L_{ab}$. Since $\gamma(L_{ab}) = 0$, $g(e^z)$ extends analytically to L_{ab} so g extends analytically to K_{ab}. Thus $\gamma(K_{ab}) = 0$. The other properties required of K_{ab} obviously hold.

To construct our counterexample we choose open sets $U_n \subset A_{n0}$ such that $K_{6/5 \cdot 1/2^{n+1}, 5/6 \cdot 1/2^n} \subset U_n$, and such that $\gamma(U_n) < 1/4^n$. Let $X = \Delta(0, 1) \backslash (\bigcup_{n=0}^{\infty} U_n)$. Then

$$\sum_{n=0}^{\infty} 2^n \gamma(A_{n0} \backslash X) = \sum_{n=0}^{\infty} 2^n \gamma(U_n) < \infty$$

so 0 is not a peak point for $R(X)$ by Melnikov's theorem. On the other hand,

$$\int_I t^{-1} dt \geqq \sum_{n=0}^{\infty} \int_{6/5 \cdot 1/2^{n+1}}^{5/6 \cdot 1/2^n} t^{-1} dt = \sum_{n=0}^{\infty} \mathrm{Ln} \frac{\frac{5}{6} \cdot \frac{1}{2^n}}{\frac{6}{5} \cdot \frac{1}{2^{n+1}}} = \sum_{n=0}^{\infty} \mathrm{Ln} \frac{50}{60} = \infty .$$

If we choose U_n such that $\gamma(U_n) \leqq 1/(2^n)^{2n}$ then 0 supports bounded point derivations of all orders for $R(X)$, see [4], so 0 is indeed far from being a peak point for $R(X)$. The question remains whether 0 might be a regular point for the Dirichlet problem. We note that this could only happen if the set of representing measures M_x for $R(K)$ at $x \in X^0$ is not norm compact, see [1].

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