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DETERMINANTS OF PETRIE MATRICES

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The number of nonsingular square Petrie matrices is calculated by setting up a correspondence with spanning trees and then using Cayley's Theorem to count these trees. This work has applications in the 'excluded volume' problem in Polymer Science.

A Petrie matrix is a finite matrix whose elements are either zeros or ones such that the ones in each column occur consecutively. Such matrices have been studied in a molecular biological situation by Fulkerson and Gross [2] and in archaeology by Kendall [4 and 5] and Wilkinson [7]. The problem here treated has arisen from a combinatorial analysis [3] of the theory of the 'excluded volume' of a polymer chain [1], in which Petrie matrices play an important, though hitherto unrecognized, role.

The Fulkerson and Gross paper quotes the interesting fact that Petrie matrices are unimodular, that is all square sub-matrices have a determinant which is either -1, 0, or 1. The object of this paper is to investigate square Petrie matrices with the view of counting the number with nonzero determinant. We propose to do this by constructing a correspondence between n by n Petrie matrices and graphs on $n+1$ vertices with at most n edges, and then to show that precisely those graphs that are spanning trees correspond to nonsingular Petrie matrices. A well known theorem by Cayley can then be used to count up the number of distinct labelled nonrooted trees.

$$\begin{matrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix}$$

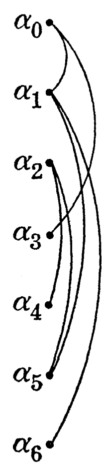

FIGURE 1

For an n by n Petrie matrix A, we construct a graph on $n+1$ points, which we label $\alpha_0 \cdots \alpha_n$. If a column of A has ones from row i to row j then we insert an edge between α_{i-1} and α_j. Figure 1 illustrates the construction of a tree from a Petrie matrix.

If the graph has m nonzero columns, there will be m edges in the graph. So we have constructed a mapping from the set of n by n Petrie matrices to the graphs on $n+1$ points with at most n edges. This mapping is onto, for given any such graph, we can trivially construct a Petrie matrix that maps onto it. If the graphs of two matrices are the same, then one is a column permutation of the other. So we can construct a one-to-one correspondence between the equivalence classes of Petrie matrices with respect to column permutations and the set of graphs.

THEOREM 1. *A square Petrie matrix is nonsingular if and only if its associated graph is a spanning tree. The determinant is either* $0, 1$ *or* -1.

Proof. If the graph is not a spanning tree then either there is a circuit, or the graph is disconnected. If there is no circuit and the graph is disconnected then there are less than n edges and hence at least one of the columns of A must be all zeros. If there is a circuit then we can construct a linear dependence between the columns. Consider a journey round a circuit. When going from α_i to α_j along an edge, add the corresponding column of A if $i > j$ and subtract if $j > i$. On completing the circuit the sum of these columns will be zero, since one has traversed each row as many times upwards as downwards.

If, on the other hand, the graph is a spanning tree, we prove by induction on n that the determinant of the Petrie matrix is plus or minus one. If $n = 1$ the result is true. Assume the hypothesis is true for all matrices of size $n - 1$. There is at least one edge incident on α_n. Consider those columns corresponding to edges incident on α_n. There is just one (X, say) with fewest ones, since otherwise there would be a circuit in the tree. We subtract this column X from the others, leaving a new Petrie matrix with a single one in the last row. The determinant of the matrix is the cofactor of the single one in the last row. The graph corresponding to this determinant of order $(n-1)$ is obtained from that of the original determinant of order n by elementary contraction of the edge corresponding to X, and thus is again a spanning tree.

THEOREM 2. *There are* $n!\,(n+1)^{n-1}$ *distinct nonsingular* n *by* n *Petrie matrices.*

Proof. By Cayley's theorem, there are $(n+1)^{n-1}$ distinct vertex-labelled trees on $n+1$ vertices. Each tree corresponds to a class of Petrie matrices. The columns are necessarily distinct and so each class contains $n!$ distinct matrices.

It is clear that there are $((n(n+1))/2+1)^n$ distinct Petrie matrices, so the proportion that are nonsingular tends to zero as n tends to infinity. This is interesting because for the general class of n by n (0, 1)-matrices, of which Petrie matrices form a sub-class, Komlós [6] has proved the opposite kind of asymptotic behavior: for them, the fraction of nonsingular ones tends to unity.

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