PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

TOPOLOGIES ON THE TORSION-THEORETIC SPECTRUM OF A NONCOMMUTATIVE RING

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Let R-sp be the collection of all prime torsion theories on the category of left R-modules over an associative ring R. Three topologies — the order topology, the finitary order topology, and the reverse order topology (in the case that R is left noetherian) — are defined on R-sp and each is shown to exhibit some properties of the Zariski topology on the spectrum of a commutative ring. A fourth topology — the Gillman topology — is defined on R-sp when R is left noetherian and is used to characterize the separation of the reverse order topology.

1. Background and notation. Throughout the following R will always denote an associative ring with unit element 1. Unless the contrary is specifically stated, all modules and morphisms will be taken from the category R-mod of unitary left R-modules. Homomorphisms will be written as acting on the side opposite scalar multiplication, i.e., on the right. The injective hull of a module M will be denoted by $E(M)$.

The term "torsion theory" will always be used to mean hereditary torsion theory in the sense of [2]. In this section we summarize the information about torsion theories which we will need. The reader is referred to [2, 4, 6, 10] for further elucidation and for proofs.

A torsion theory τ can be completely characterized by any of the following data, each of which uniquely determines all of the others:

(i) The class $\mathscr{T}_\tau$ of torsion modules. This class is closed under taking submodules, factor modules, direct sums, and extensions (i.e., if $0 \to M' \to M \to M'' \to 0$ is an exact sequence with $M', M'' \in \mathscr{T}_\tau$, then $M \in \mathscr{T}_\tau$).

(ii) The class $\mathscr{F}_\tau$ of torsion-free modules. This class is closed under taking submodules, injective hulls, direct products, and extensions.

(iii) The set $\mathscr{L}_\tau$ of left ideals I of R satisfying $R/I \in \mathscr{T}_\tau$. This set is an idempotent filter, i.e., if $I \in \mathscr{L}_\tau$ then so does every left ideal of R properly containing I and so does $(I: r) = \{r' \in R \mid r'r \in I\}$ for every $r \in R$. Furthermore, $\mathscr{L}_\tau$ is closed under taking finite intersections and, if $I \in \mathscr{L}_\tau$ and $(H: r) \in \mathscr{L}_\tau$ for every $r \in I$ then $H \in \mathscr{L}_\tau$.

(iv) The class $\mathscr{E}_\tau$ of absolutely pure modules. These are elements N of $\mathscr{F}_\tau$ satisfying the condition that if N is a submodule of $M \in \mathscr{F}_\tau$ then $M/N \in \mathscr{F}_\tau$. The full subcategory of R-mod defined by $\mathscr{E}_\tau$ is abelian.

(v) The functor $T_\tau(\_)$: R-mod $\to$ R-mod which assigns to each module M the (unique) submodule $T_\tau(M)$ of M satisfying $T_\tau(M) \in \mathscr{T}_\tau$ and $M/T_\tau(M) \in \mathscr{F}_\tau$.

(vi) The functor $Q_\tau(\_)$: R-mod $\to \mathscr{E}_\tau$ which is the left adjoint of the inclusion functor.

For any module M, the module $Q_\tau(M)$ is called the *localization* of M with respect to the torsion theory τ. The endomorphism ring of $Q_\tau(R)$ is called the *quotient ring* of R with respect to τ and will be denoted by R_τ. As left R-modules, $Q_\tau(R)$ and R_τ are isomorphic. Furthermore, every module $Q_\tau(M)$ is canonically a left R_τ-module. For each left R-module M we have a canonical R-homomorphism $\hat{\tau}_M: M \to Q_\tau(M)$.

The class of all hereditary torsion theories on R-mod will be denoted by R-tors. This class can be partially ordered by setting $\tau \leqq \tau'$ if and only if $\mathscr{T}_\tau \subseteqq \mathscr{T}_{\tau'}$. If $\{\tau_i \mid i \in \Omega\}$ is a family of torsion theories then we denote the largest torsion theory less than or equal to all of them by $\bigwedge_{i \in \Omega} \tau_i$. Such a theory always exists and is defined by

$$\mathscr{T}_{\wedge \tau_i} = \bigcap \mathscr{T}_{\tau_i} .$$

Similarly we denote by $\bigvee_{i \in \Omega} \tau_i$ the smallest torsion theory greater than or equal to all of the τ_i. This theory always exists and is defined by

$$\mathscr{F}_{\vee \tau_i} = \bigcap \mathscr{F}_{\tau_i} .$$

The class R-tors has a minimal element ξ, defined by $\mathscr{T}_\xi = \{0\}$, and a maximal element χ, defined by $\mathscr{F}_\chi = \{0\}$. A torsion theory τ which is not equal to χ is called *proper*; a torsion theory τ which is not equal to ξ is called *nontrivial*. The collection of all proper torsion theories on R-mod will be denoted by R-prop.

If $\mathscr{A}$ is any family of modules then we denote by $\xi(\mathscr{A})$ the smallest torsion theory in which every $M \in \mathscr{A}$ is torsion and by $\chi(\mathscr{A})$ the largest torsion theory in which every $M \in \mathscr{A}$ is torsion free. Then $\mathscr{F}_{\xi(\mathscr{A})} = \{N \mid \mathrm{Hom}_R(M, E(N)) = 0$ for all $M \in \mathscr{A}\}$ and $\mathscr{T}_{\chi(\mathscr{A})} = \{M \mid \mathrm{Hom}_R(M, E(N)) = 0$ for all $N \in \mathscr{A}\}$. Furthermore, for any $\tau \in R$-tors, we have $\tau = \bigvee \{\xi(R/I) \mid I \in \mathscr{L}_\tau\}$.

LEMMA 1.1. *Let I, I' be left ideals of R. Then*

$$\xi(R/I) \vee \xi(R/I') = \xi(R/[I \cap I']) .$$

Proof. Let $\tau = \xi(R/I) \vee \xi(R/I')$. Then $N \in \mathscr{F}_\tau$ if and only if $\mathrm{Hom}_R(R/I, E(N)) = \mathrm{Hom}_R(R/I', E(N)) = 0$. Clearly this holds if $N \in \mathscr{F}_{\xi(R/[I \cap I'])}$. Conversely, assume that $N \in \mathscr{F}_\tau$ and $\alpha \in \mathrm{Hom}_R(R/[I \cap I'], E(N))$. Then we have a canonical monomorphism

$\theta: R/[I \cap I'] \to R/I \oplus R/I'$. Since $E(N)$ is injective, there then exists a homomorphism $\beta: R/I \oplus R/I' \to E(N)$ with $\alpha = \theta\beta$. Since $N \in \mathscr{F}_\tau$ we must have $\beta = 0$ whence $\alpha = 0$, proving $N \in \mathscr{F}_{\xi(R/[I \cap I'])}$.

For any module M, we define the *wide support* of M by $\operatorname{wsupp}(M) = \{\tau \in R\text{-prop} \mid M \notin \mathscr{T}_\tau\} = \{\tau \in R\text{-prop} \mid Q_\tau(M) \neq 0\}$. The following lemma follows directly from this definition.

LEMMA 1.2. *For a module M,*
(1) $M = \Sigma M_i$ *implies that* $\operatorname{wsupp}(M) = \bigcup \operatorname{wsupp}(M_i)$.
(2) $N \subseteq M$ *implies that* $\operatorname{wsupp}(M) = \operatorname{wsupp}(N) \cup \operatorname{wsupp}(M/N)$.

2. The order topologies on R-prop.

We define functions

$$R\text{-tors} \underset{c}{\overset{d}{\rightleftarrows}} \text{subsets of } R\text{-prop}$$

as follows

$$c: \tau \longmapsto \{\tau' \in R\text{-prop} \mid \tau \leqq \tau'\}$$
$$d: U \longmapsto \wedge U .$$

Then we clearly have $dc(\tau) = \tau$ for all $\tau \in R$-tors (making the convention that $\wedge \varnothing = \chi$).

LEMMA 2.1. *If* $\{\tau_i \mid i \in \Omega\} \subseteq R$-tors *then*
(1) $\tau_{i_1} \leqq \tau_{i_2}$ *implies that* $c(\tau_{i_1}) \supseteq c(\tau_{i_2})$.
(2) $c(\tau_{i_1} \wedge \tau_{i_2}) \supseteq c(\tau_{i_1}) \cup c(\tau_{i_2})$.
(3) $c(\vee \tau_i) = \bigcap c(\tau_i)$.

Proof. (1) follows directly from the definition. By (1), we have $c(\tau_{i_j}) \subseteq c(\tau_{i_1} \wedge \tau_{i_2})$ for $j = 1, 2$ which implies (2). As for (3), if $\tau \in \bigcap c(\tau_i)$ then $\tau \geqq \tau_i$ for all $i \in \Omega$ and so, by definition, $\tau \geqq \bigvee \tau_i$, which is to say that $\tau \in c(\vee \tau_i)$. The reverse inclusion is trivial.

The proof of the following proposition is based on [1].

PROPOSITION 2.2. *If R is left noetherian and if $\tau \in R$-prop then $c(\tau)$ contains a maximal element of R-prop.*

Proof. Let $\mathscr{A}$ be the class of all proper ideals I of R satisfying the conditions
(1) $I = T_{\tau'}(R)$ for some $\tau' \in R$-prop;
(2) $R/I \in \mathscr{F}_\tau$.
Then $\mathscr{A}$ is nonempty since $T_\tau(R) \in \mathscr{A}$. Since R is left noetherian, $\mathscr{A}$ has a maximal element I_0. Let $\tau_0 = \chi(R/I_0)$. Then $\tau \leqq \tau_0$ since $R/I_0 \in \mathscr{F}_\tau$. On the other hand, if $\tau_0 \leqq \tau' \in R$-prop then $I_0 \subseteq T_{\tau'}(R)$ and

$\mathscr{F}_{\tau'} \subseteqq \mathscr{F}_{\tau_0} \subseteqq \mathscr{F}_{\tau}$ so that $R/T_{\tau'}(R) \in \mathscr{F}_{\tau}$. But this implies that $T_{\tau'}(R) \in \mathscr{A}$ and so $T_{\tau'}(R) \subseteqq I_0$, proving equality. Therefore $\tau_0 = \tau'$.

LEMMA 2.3. *If U_1 and U_2 are subsets of R-prop then*
(1) $U_1 \subseteqq U_2$ *implies that* $d(U_1) \geqq d(U_2)$.
(2) $d(U_1 \cup U_2) = d(U_1) \wedge d(U_2)$.

Proof. (1) follows directly from the definition. As for (2),

$$\begin{aligned}\mathscr{T}_{d(U_1 \cup U_2)} &= \{M \mid \operatorname{wsupp}(M) \cap (U_1 \cup U_2) = \varnothing\} \\ &= \{M \mid [\operatorname{wsupp}(M) \cap U_1] \cup [\operatorname{wsupp}(M) \cap U_2] = \varnothing\} \\ &= \{M \mid \operatorname{wsupp}(M) \cap U_1 = \varnothing\} \cap \{M \mid \operatorname{wsupp}(M) \cap U_2 = \varnothing\} \\ &= \mathscr{T}_{d(U_1)} \cap \mathscr{T}_{d(U_2)} = \mathscr{T}_{d(U_1) \wedge d(U_2)}\,.\end{aligned}$$

From Lemma 2.1(3) it is clear that the family $\{c(\tau) \mid \tau \in R\text{-tors}\}$ of subsets of R-prop is the base of a topology on R-prop, which we will call the *order topology*.

PROPOSITION 2.4. *For any $\tau \in R$-prop, $c(\tau)$ is quasicompact in the order topology. In particular, R-prop $= c(\xi)$ is quasicompact.*

Proof. If $\{c(\tau_i) \mid i \in \Omega\}$ is an open cover of $c(\tau)$, then $\tau \in c(\tau_k)$ for some $k \in \Omega$ whence $c(\tau) = c(\tau_k)$ by Lemma 2.1(1).

PROPOSITION 2.5. *For any $\tau \in R$-prop, the closure of $\{\tau\}$ in the order topology on R-prop is $\{\tau' \in R\text{-prop} \mid \tau' \leqq \tau\}$.*

Proof. By definition, τ' belongs to the closure of $\{\tau\}$ if and only if every open neighborhood of τ' intersects $\{\tau\}$. This clearly happens when $\tau' \leqq \tau$. Conversely, if $\tau' \nleqq \tau$ then there exists an $M \in \mathscr{T}_{\tau'} \backslash \mathscr{T}_{\tau}$. Then $\tau' \in c(\xi(M))$, $\tau \notin c(\xi(M))$, which shows that $c(\xi(M))$ is an open neighborhood of τ' not containing τ.

By Lemma 1.1, the family $\{c(\xi(R/I)) \mid I \text{ a left ideal of } R\}$ of subsets of R-prop also forms the base of a topology on R-prop. This topology is coarser than the order topology; we call it the *finitary order topology* on R-prop.

3. Prime torsion theories. The notion of a prime element of R-tors was first defined by Goldman [4] and has since been considered by several authors [7, 8, 11]. Of the equivalent definitions available in the literature, we will use the one from [7].

A left ideal I of a ring R is *critical* if and only if, for every left ideal H of R properly containing I, $R/H \in \mathscr{T}_{\chi(R/I)}$. It is easily shown that if I is critical, it is meet-irreducible. Furthermore, if R is commutative then I is critical if and only if it is prime. We therefore define a torsion theory $\tau \in R$-tors to be *prime* if and only if $\tau = \chi(R/I)$ for some critical left ideal I of R. The family of all prime elements of R-tors is called the *left spectrum* of R and will be denoted by R-sp. If $\tau \in R$-sp then the family of all critical left ideals I of R with $\tau = \chi(R/I)$ will be denoted by crit(τ).

EXAMPLE 3.1. Maximal left ideals of R are trivially critical. Therefore, if M is a simple left R-module, $\chi(M) \in R$-sp.

LEMMA 3.2. *Let* $\tau \in R$-sp *and* $\tau_1, \tau_2 \in R$-tors. *Then*

(1) $\tau = \tau_1 \wedge \tau_2$ *implies that* $\tau = \tau_1$ *or* $\tau = \tau_2$.

(2) $\tau \geqq \tau_1 \wedge \tau_2$ *implies that* $\tau \geqq \tau_1$ *or* $\tau \geqq \tau_2$.

Proof. (1) Assume that $\tau = \tau_1 \wedge \tau_2$ where $\tau_1 > \tau$ and $\tau_2 > \tau$. If $I \in \text{crit}(\tau)$ then R/I belongs to neither $\mathscr{F}_{\tau_1}$ nor $\mathscr{F}_{\tau_2}$ and so we have nonzero modules $W_j/I = T_{\tau_j}(R/I)$, $j = 1, 2$. On the other hand, $(W_1/I) \cap (W_2/I) = T_\tau(R/I) = 0$. This contradicts the fact that I is meet irreducible.

(2) For $j = 1, 2$, let $\tau'_j = \tau \vee \tau_j$. If $\tau \geqq \tau_1 \wedge \tau_2$ then $\tau = \tau'_1 \wedge \tau'_2$ and so, by (1), $\tau = \tau'_j$ for $j = 1$ or $j = 2$. This implies that $\tau \geqq \tau_j$ for that j.

For each ordinal t, define $[R\text{-sp}]_t$ by

(1) $[R\text{-sp}]_0 = \{\tau \in R\text{-sp} \mid \tau \text{ is maximal}\}$.

(2) $[R\text{-sp}]_t = \{\tau \in R\text{-sp} \mid \tau < \tau' \in R\text{-sp} \Rightarrow \tau' \in \bigcup_{s<t} [R\text{-sp}]_s\}$. If there exists an ordinal t with $[R\text{-sp}]_t = R$-sp then we say that t is the *Krull-Krause dimension* of R-sp and that R-sp has Krull-Krause dimension. A proof analogous to that of [5, Proposition 1.2] then establishes

PROPOSITION 3.3. *The following conditions are equivalent for a ring* R:

(1) R-sp *has Krull-Krause dimension.*

(2) R-sp *satisfies the maximum condition.*

Alternatively, for each ordinal t define the torsion theory τ_t as follows:

(1) $\tau_0 = \xi$.

(2) If t is not a limit ordinal, $\tau_t = \xi(\{M \mid Q_{\tau_{t-1}}(M) \text{ is of finite length}\})$.

(3) If t is a limit ordinal, $\tau_t = \bigvee \{\tau_s \mid s < t\}$. If there exists an ordinal t with $\chi = \tau_t$ then we say that t is the *Krull-Gabriel dimension*

of R-tors and that R-tors has Krull-Gabriel dimension. It is then easy to establish the following result [9, Corollaire 2.4]:

PROPOSITION 3.4. *If* R-tors *has Krull-Gabriel dimension then* R-sp *satisfies the minimum condition.*

In particular we have

COROLLARY 3.5. *If* R *is left noetherian then* R-sp *satisfies the minimum condition.*

For a left R-module M we define the *assassin* $\operatorname{ass}(M)$ of M to be the family of all $\tau \in R$-sp for which there exists an $m \in M$ with $(0: m) \in \operatorname{crit}(\tau)$.

PROPOSITION 3.6.
(1) *If* $M = \bigcup M_i$ *then* $\operatorname{ass}(M) = \bigcup \operatorname{ass}(M_i)$.
(2) *If* $I \in \operatorname{crit}(\tau)$ *then for all* $0 \neq {}_RN \subseteq R/I, \operatorname{ass}(N) = \{\tau\}$.
(3) *If* $N \subseteq M$ *then* $\operatorname{ass}(N) \subseteq \operatorname{ass}(M) \subseteq \operatorname{ass}(N) \cup \operatorname{ass}(M/N)$.
(4) *If* $M = \bigoplus M_i$ *then* $\operatorname{ass}(M) = \bigcup \operatorname{ass}(M_i)$.
(5) *If* N *is a large submodule of* M *then* $\operatorname{ass}(N) = \operatorname{ass}(M)$.

Proof. Parts (1)–(4) follow from [11, Proposition 3.1]. As for part (5), $\operatorname{ass}(N) \subseteq \operatorname{ass}(M)$ by (3). Conversely, assume that $\tau \in \operatorname{ass}(M)$. Then there exists an $r \in R$ with $0 \neq rm \in N$, where $(0: m) \in \operatorname{crit}(\tau)$. Furthermore, $(0: rm) = ((0: m): r)$. Since $(0: m) \in \operatorname{crit}(\tau)$, we have $(0: rm) \in \operatorname{crit}(\tau)$ by [6, Proposition 2.8] and so $\tau \in \operatorname{ass}(N)$.

PROPOSITION 3.7. *The following conditions are equivalent*:
(1) $M \neq 0$ *implies that* $\operatorname{ass}(M) \neq \varnothing$.
(2) *If* I *is a proper left ideal of* R *then there exists an* $r \in R$ *with* $(I: r)$ *critical.*

Proof. (1) $\Rightarrow$ (2): Let I be a proper left ideal of R. Then by (1) there exists a $\tau \in \operatorname{ass}(R/I)$ and so there exists an $r \in R$ with $(I: r) = (0: r + I) \in \operatorname{crit}(\tau)$.

(2) $\Rightarrow$ (1): Let $M \neq 0$ and pick $0 \neq m \in M$. Then by (2) there exists an $r \in R$ with $(0: rm) = ((0: m): r)$ critical. If $\tau = \chi(Rm)$ then $\tau \in \operatorname{ass}(M)$.

This condition is satisfied if R is left noetherian. In fact, we have the slightly stronger result.

PPOPOSITION 3.8. *If* M *is a nonzero noetherian module then* ass (M) *is a nonempty finite set.*

Proof. By [11, Proposition 3.3] and [4, Theorem 6.14].

COROLLARY 3.9. *If R is left noetherian then R-*sp *is a dense subset of R-*prop *in the order topology.*

Proof. Let $\tau \in R$-prop and let M be an injective cogenerator of $\mathscr{F}_\tau$. If $0 \neq m \in M$ then Rm is noetherian. Let $\tau' \in \text{ass}(Rm) \subseteq \text{ass}(M)$. Then $\mathscr{F}_{\tau'} \subseteq \mathscr{F}_\tau$ and so $\tau' \in c(\tau) \cap R\text{-}sp$.

PROPOSITION 3.10. *If R is left noetherian then the homomorphism $\psi: M \to \Pi\{Q_\tau(M) \mid \tau \in \text{ass}(M)\}$ defined by $m \mapsto < m\hat{\tau}_M >$ is a monomorphism.*

Proof. If $0 \neq K = \ker(\psi)$ then by Propositions 3.8 and 3.6(3) there exists a $\tau \in \text{ass}(K) \subseteq \text{ass}(M)$. If $k \in K$ with $(0: k) \in \text{crit}(\tau)$ then $Rk \in \mathscr{F}_\tau$ so $k\hat{\tau}_M \neq 0$, contradicting the fact that $K = \bigcap \{\ker(\hat{\tau}_M) \mid \tau \in \text{ass}(M)\}$.

For a module M we define the *support* of M by $\text{supp}(M) = \text{wsupp}(M) \cap R\text{-sp}$. We then have the following result analogous to Lemma 1.2, again directly from the definition.

LEMMA 3.11. *For a module M,*
(1) $M = \Sigma M_i$ *implies that* $\text{supp}(M) = \bigcup \text{supp}(M_i)$.
(2) $N \subseteq M$ *implies that* $\text{supp}(M) = \text{supp}(N) \cup \text{supp}(M/N)$.

It is clear that, for any module M, $\text{ass}(M) \subseteq \text{supp}(M)$. Therefore, as a corollary to Proposition 3.8 we have

PROPOSITION 3.12. *If R is left noetherian then $M = 0$ if and only if* $\text{supp}(M) = \varnothing$.

4. The order topologies induced on R-sp. The [finitary] order topology defined in §2 induces a topology on R-sp, a basis for which is the family of sets $c'(\tau) = c(\tau) \cap R$-sp, for each $\tau \in R$-tors [resp. $\tau = \xi(R/I)$].

LEMMA 4.1. *If $\tau_1, \tau_2 \in R$-*sp *then $c'(\tau_1 \wedge \tau_2) = c'(\tau_1) \cup c'(\tau_2)$.*

Proof. That $c'(\tau_1) \cup c'(\tau_2) \subseteq c'(\tau_1 \wedge \tau_2)$ follows from Lemma 2.1(2).

Conversely, if $\tau \in c'(\tau_1 \wedge \tau_2)$ then $\tau \in c'(\tau_1) \cup c'(\tau_2)$ by Lemma 3.2(2).

LEMMA 4.2. *If* $\tau \in R$-tors *and* $M \in \mathscr{F}_\tau$ *then* $\mathrm{ass}(M) \subseteq c'(\tau)$.

Proof. If $\tau' \in \mathrm{ass}(M)$ then there exists an $0 \neq m \in M$ with $\tau' = \chi(Rm)$. Since $M \in \mathscr{F}_\tau$, $Rm \in \mathscr{F}_\tau$ and so $\tau \leqq \tau'$.

If $U \subseteq R$-sp, then in general $d(U) \notin R$-sp.

PROPOSITION 4.3. *Let* $\tau \in R$-tors *satisfy*
(*) *Every* $0 \neq M \in \mathscr{F}_\tau$ *has a nonzero noetherian submodule.*
Then $dc'(\tau) = \tau$.

Proof. Clearly $dc'(\tau) \geqq \tau$. Conversely assume that $M \in \mathscr{T}_{dc'(\tau)} \backslash \mathscr{T}_\tau$. Then $0 \neq M/T_\tau(M)$ and so there exists a $\tau' \in \mathrm{ass}(M/T_\tau(M))$ by (*) and Proposition 3.8. Furthermore, $M \notin \mathscr{T}_{\tau'}$ since otherwise we would have $M/T_\tau(M) \in \mathscr{T}_{\tau'}$ which contradicts the definition of τ'. Therefore $\tau' \in \mathrm{supp}(M)$. On the other hand, $\mathscr{F}_{\tau'} \subseteq \mathscr{F}_\tau$ by construction and so $\tau' \in c'(\tau)$ whence $\tau' \notin \mathrm{supp}(M)$ by the choice of M. From this contradiction we deduce that $dc'(\tau) \leqq \tau$ and so we have equality.

We have thus seen that, particularly for the case of a left noetherian ring R, the order topology on R-sp exhibits various "nice" features of the Zariski topology on the spectrum $\mathrm{spec}(R)$ of a commutative ring R. It is the finitary order topology, however, which reduces to the Zariski topology in the case that R is commutative.

PROPOSITION 4.4. *If* R *is commutative then* R-sp *with the finitary order topology is homeomorphic to* spec (R) *with the Zariski topology.*

Proof. Define the function $h\colon \mathrm{spec}(R) \to R$-sp by $P \mapsto \chi(R/P)$. Since the critical left ideals of a commutative ring R are precisely the prime ideals of R [7] the function h is clearly a surjection. Furthermore, by [4, Proposition 5.2],

$$T_{\chi(R/P)}(M) = \{m \in M \mid r^n m = 0 \text{ for some } r \in R\backslash P \text{ and some integer } n\}$$

which shows that h is a bijection.

If I is an ideal of R and $V(I) = \{P \in \mathrm{spec}(R) \mid I \subseteq P\}$ is a subset of $\mathrm{spec}(R)$ closed in the Zariski topology, then $h(V(I)) = \{\chi(R/P) \in R\text{-sp} \mid \chi(R/P) \leqq \chi(R/I)\}$ which is closed in the finitary order topology on R-sp by Proposition 2.5. Conversely, inverse images under h of subsets of R-sp closed in the finitary order topology are clearly closed in $\mathrm{spec}(R)$. Therefore, h is a homeomorphism.

5. The reverse order topology on R-sp.

PROPOSITION 5.1. *If R is left noetherian then $c'd$ is a closure operator on R-sp.*

Proof. Clearly $c'd(\varnothing) = \varnothing$. By definition, $c'd(U) \supseteqq U$ for every subset U of R-sp. In particular, if $U \subseteqq R$-sp then $c'd(U) \subseteqq c'dc'd(U)$. Conversely, if $\tau \in c'dc'd(U)$ then $dc'd(U) \leqq \tau$ and so by Proposition 4.3, $d(U) \leqq \tau$ which implies that $\tau \in c'd(U)$. Therefore $c'dc'd(U) = c'd(U)$. Finally, by Lemmas 2.3 and 4.1, $c'd(U_1 \cup U_2) = c'(d(U_1) \wedge d(U_2)) = c'd(U_1) \cup c'd(U_2)$.

Proposition 5.1 shows that, for a left noetherian ring R, we have another topology on R-sp which is opposite to the order topology in the sense that the open sets are precisely the sets of the form $R\text{-sp}\backslash c'(\tau)$ for some $\tau \in R$-tors. We call this topology the *reverse order topology* on R-tors.

LEMMA 5.2. *Let R be a left noetherian ring. Then for any module M,* supp (M) *is open in the reverse order topology on* R-sp.

Proof. Let $\tau = \bigwedge \{\tau' \mid M \in \mathscr{T}_{\tau'}\}$. Then $M \in \mathscr{T}_\tau$ so $\tau \notin \operatorname{supp}(M)$. If $\tau'' \in R\text{-sp}\backslash c'(\tau)$ then $M \notin \mathscr{T}_{\tau''}$ so $\tau'' \in \operatorname{supp}(M)$. The converse is trivial. Hence $\operatorname{supp}(M) = R\text{-sp}\backslash c'(\tau)$ is open.

We now develop another method for characterizing the reverse order topology on *R-sp*. To this end define a function

$$\text{subsets of } R\text{-sp} \xrightarrow{\ e\ } R\text{-tors}$$

by $e\colon U \mapsto \chi(\{N \mid \operatorname{ass}(N) \subseteqq U\})$. Then

LEMMA 5.3. *If U_1 and U_2 are subsets of* R-sp *then*
(1) *$U_1 \subseteqq U_2$ implies that $e(U_1) \geqq e(U_2)$.*
(2) *$e(U_1 \cup U_2) = e(U_1) \wedge e(U_2)$.*

Proof. (1) follows directly from the definition. As for (2),

$$\begin{aligned} e(U_1 \cup U_2) &= \chi(\{N \mid \operatorname{ass}(N) \subseteqq U_1 \cup U_2\}) \\ &= \chi(\{N \mid \operatorname{ass}(N) \subseteqq U_1\}) \wedge \chi(\{N \mid \operatorname{ass}(N) \subseteqq U_2\}) \\ &= e(U_1) \wedge e(U_2)\,. \end{aligned}$$

LEMMA 5.4. *If U is a subset of* R-sp *then $d(U) \geqq e(U)$.*

Proof. Let $M \in \mathscr{T}_{e(U)}$ and let $\tau \in U$. Then for any $I \in \operatorname{crit}(\tau)$, $\operatorname{ass}(E(R/I)) = \{\tau\}$ by Proposition 3.6. Therefore, $\operatorname{Hom}_R(M, E(R/I)) = 0$

and so $M \in \mathscr{T}_\tau$. Hence $M \in \mathscr{T}_{e(U)}$ implies that $M \in \bigcap \{\mathscr{T}_\tau \mid \tau \in U\} = \mathscr{T}_{d(U)}$.

We now prove a result analogous to Proposition 4.3.

PROPOSITION 5.5. *Let* $\tau \in R$-tors *satisfy*
(**) *Every* $0 \neq M \in \mathscr{T}_\tau$ *has a nonzero noetherian submodule.*
Then $ec'(\tau) = \tau$.

Proof. To prove that $ec'(\tau) = \tau$ it suffices to show that $M \in \mathscr{F}_\tau$ if and only if $\operatorname{ass}(M) \subseteqq c'(\tau)$. Assume that $\operatorname{ass}(M) \subseteqq c'(\tau)$ and that $M \notin \mathscr{F}_\tau$. Then $T_\tau(M) \neq 0$ and so by (**) and Proposition 3.8 there exists a $\tau' \in \operatorname{ass}(T_\tau(M)) \subseteqq \operatorname{ass}(M)$. Therefore, $\tau' \in c'(\tau)$ by assumption. But $T_\tau(M) \in \mathscr{T}_\tau$ implies that $\operatorname{ass}(T_\tau(M)) \cap c'(\tau) = \varnothing$ since $\tau \leqq dc'(\tau)$ and $\operatorname{ass}(M) \subseteqq \operatorname{supp}(M)$. This yields a contradiction which shows that we must have $M \in \mathscr{F}_\tau$. The reverse implication follows directly from Lemma 4.2.

PROPOSITION 5.6. *If* R *is left noetherian then*
(1) $dc' = ec' =$ *identity on* R-tors.
(2) $c'd = c'e$.

Proof. (1) follows directly from Propositions 5.5 and 4.3. As for (2), by definition we have $c'e(U) \supseteqq U$ for any subset U of R-sp and so, in particular, for any such U we have $c'e(U) \subseteqq c'ec'e(U)$. Conversely if $\tau \in c'ec'e(U)$ then $ec'e(U) \leqq \tau$. By part (1), $ec'e(U) = e(U)$ and so $e(U) \leqq \tau$ which implies that $\tau \in c'e(U)$. This proves that $c'e(U) = c'ec'e(U)$.

Now let $U \subseteqq R$-sp. Then $c'd(U) = c'dc'd(U) = c'ec'd(U)$ by part (1). Furthermore, $c'ec'd(U) \supseteqq c'e(U)$ since $c'd(U) \supseteqq U$. On the other hand, $c'e(U) = c'ec'e(U) = c'dc'e(U) \supseteqq c'd(U)$ by a similar argument and so we have $c'e(U) = c'd(U)$.

Thus we see that the reverse order topology also resembles the Zariski topology although it "goes the other way". In particular, the construction of the reverse order topology is formally the same as the classical "hull-kernel" construction of the Zariski topology.

6. The Gillman topology on R-sp.

If R is left noetherian we can define another function

$$\text{subsets of } R\text{-sp} \xrightarrow{\ g\ } R\text{-tors}$$

by $g\colon U \mapsto \bigvee \{d(U') \mid U \subseteqq U' \subseteqq R\text{-sp and } U' \text{ is open in the reverse order topology on } R\text{-sp}\}$.

LEMMA 6.1. *If R is left noetherian then for $U_1, U_2 \subseteqq R$-sp,*
(1) *$U_1 \subseteqq U_2$ implies that $g(U_1) \geqq g(U_2)$.*
(2) *$U_1 \subseteqq c'g(U_1)$.*

Proof. (1) follows directly from the definition. As for (2), if U' is a neighborhood of U_1 in the reverse order topology on R-sp then $d(U') = \bigwedge U'$ is the largest torsion theory less than or equal to every element of U'. In particular, if $\tau \in U$ then $\tau \geqq d(U')$. Since $\vee\, d(U')$ is the smallest torsion theory greater than or equal to all of the $d(U')$, $\tau \geqq \vee\, d(U') = g(U_1)$. Thus $\tau \in c'g(U_1)$ for all $\tau \in U_1$, proving (2).

LEMMA 6.2. *Let R be left noetherian and let $\tau \in R$-sp. Then for any $U \subseteqq R$-sp, $U \cap c'(\tau) \neq \varnothing$ implies $\tau \in c'g(U)$.*

Proof. To show that $\tau \in c'g(U)$ we have to show that for every open neighborhood U' of U, $d(U') \leqq \tau$. Let U' be such an open neighborhood and let $V' = R\text{-sp}\backslash U'$. Then $d(V') \wedge d(U') \leqq \tau$ so by Lemma 3.2 either $d(V') \leqq \tau$ or $d(U') \leqq \tau$. But $d(V') \leqq \tau$ implies that $\tau \in c'd(V') = V'$ (since V' is closed in the reverse orde topology) whence $c(\tau) \subseteqq V'$, contradicting the hypothesis that $U \cap c(\tau) \neq \varnothing$. Therefore, we must have $d(U') \leqq \tau$.

PROPOSITION 6.3. *If R is left noetherian then $c'g$ is a closure operator on R-sp.*

Proof. For any subset U of R-sp, $U \subseteqq c'g(U)$ by Lemma 6.1. Furthermore, it is clear that $c'g(\varnothing) = \varnothing$. Also, by Lemma 6.1, $c'g(U) \subseteqq c'gc'g(U)$. Conversely, $c'gc'g(U) \cap U \neq \varnothing$ and so, by Lemma 6.2, $gc'g(U) \in c'g(U)$, i.e., $gc'g(U) \geqq g(U)$. By Lemma 2.1 this implies that $c'g(U) \supseteqq c'gc'g(U)$ and so we have equality.

Finally we have to show that $c'g(U_1 \cup U_2) = c'g(U_1) \cup c'g(U_2)$. For $i = 1, 2$, $U_i \subseteqq U_1 \cup U_2$ and so $g(U_i) \geqq g(U_1 \cup U_2)$. Thus $g(U_1) \wedge g(U_2) \geqq g(U_1 \cup U_2)$. By Lemma 2.1 this implies that $c'g(U_1) \cup c'g(U_2) = c'(g(U_1) \wedge g(U_2)) \subseteqq c'g(U_1 \cup U_2)$. Conversely suppose that $\tau \in c'g(U_1 \cup U_2)$. To show that $\tau \in c'(g(U_1) \wedge g(U_2))$ it suffices to show that $\tau \geqq g(U_1)$ or $\tau \geqq g(U_2)$. Assume neither holds. Then $\tau \notin c'g(U_1) \cap c'g(U_2)$. In particular, there then exist open neighborhoods U_i' of U_i $(i = 1, 2)$ with $\tau \notin c'd(U_i')$. Therefore, $\tau \notin c'd(U_1') \cup c'd(U_2') = c'd(U_1' \cup U_2')$. But $U_1' \cup U_2'$ is an open neighborhood of $U_1 \cup U_2$ and so $c'd(U_1' \cup U_2')$ contains τ by hypothesis. From this contradiction we have $\tau \in c'(g(U_1) \wedge g(U_2)) = c'g(U_1) \cup c'g(U_2)$.

If R is left noetherian, the closure operator $c'g$ thus defines a topology on R-sp, which we will call the *Gillman topology* since the above construction is based on the construction in [3]. We now use the Gillman topology to characterize the reverse order topology.

PROPOSITION 6.4. *The following conditions are equivalent for a left noetherian ring R:*

(1) R-sp *is a T_2-space under the reverse order topology.*

(2) R-sp *is a T_1-space under the Gillman topology.*

Proof. $(1) \Rightarrow (2)$. To show that R-sp is a T_1-space under the Gillman topology we have to show that for every $\tau \in R$-sp, $\{\tau\} = c'g(\{\tau\})$. By Lemma 6.1 we know that $\{\tau\} \subseteqq c'g(\{\tau\})$. Conversely, assume that $\tau \neq \tau' \in c'g(\{\tau\})$. Then $\tau' \geqq g(\{\tau\})$ and so $\tau' \geqq d(U')$ for any open neighborhood U' of τ. This means that $\tau' \in c'd(U')$ for any open neighborhood U' of τ. Since $c'd(U')$ is the closure of U' in the reverse order topology, this means that there is no neighborhood of τ' which does not intersect a neighborhood of τ, contradicting the fact that R-sp is a T_2-space under the reverse order topology.

$(2) \Rightarrow (1)$: Let $\tau \neq \tau' \in R$-sp. Then $\tau' \notin c'g(\{\tau\})$ by (2) and so there exists some proper open neighborhood U' of τ with $\tau' \notin c'd(U')$. Then $R \backslash c'd(U')$ is an open neighborhoood of τ' not intersecting U'. This proves that R-*sp* is a T_2-space under the reverse order topology.

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Received December 12, 1972. This research was partially supported by the National Research Council of Canada.

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