ON FINITE LEFT LOCALIZATIONS

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Of the several types of noncommutative quotient rings, finite left localizations have structure most like that of the original ring. This paper examines finite left localizations from two points of view: As rings of quotients with respect to hereditary torsion classes, and as endomorphism rings of finitely generated projective modules. In the first case, finite left localizations are shown to be the rings of quotients with respect to perfect TTF-classes. In the second, they are shown to be the double centralizers of finite projectors.

The first characterization (Corollary 1.2) shows that finite left localizations may be realized as endomorphism rings of finitely generated projective idempotent ideals. It follows that every left localization of a right perfect ring is finite.

Finally, the finite projectors of the second characterization (Corollary 2.2) are shown to have endomorphism rings whose quotient ring structure is very closely related to that of the original ring. For example, every ring of left quotients of the endomorphism ring is Morita equivalent to a ring of quotients of the original ring.

1. **Perfect TTF-classes.** For an associative ring R with identity let $_R\mathscr{M}$ denote the category of unital left R-modules. A (hereditary) torsion class $\mathscr{T} \subseteq {}_R\mathscr{M}$ is a nonvoid class of left R-modules closed under taking submodules, factor modules, extensions, and arbitrary direct sums. Corresponding to a torsion class $\mathscr{T}$ is a torsion-free class $\mathscr{F} = \{M \in {}_R\mathscr{M} \mid \operatorname{Hom}_R(N, M) = 0 \; \forall N \in \mathscr{T}\}$ and a topologizing filter of left ideals $f = \{I \text{ left ideals of } R \mid R/I \in \mathscr{T}\}$. A torsion class determines a localization functor $L_{\mathscr{T}}$ from $_R\mathscr{M}$ to the quotient category $_R\mathscr{M}/\mathscr{T}$ (see [13, § 2]) given by $L_{\mathscr{T}}(M) = \varinjlim_{I \in f} = \operatorname{Hom}_R(I, M/t(M))$ where $t(M)$ is the $\mathscr{T}$-torsion submodule of M. This localization functor is covariant, additive, and left exact, and there is a natural transformation $\sigma_M : M \to L_{\mathscr{T}}(M)$ whose kernel is $t(M)$. The object $Q = L_{\mathscr{T}}(R)$ is a ring, called the ring of left quotients of R with respect to $\mathscr{T}$, and the map σ_R is a ring homomorphism. A torsion class $\mathscr{T}$, or its ring of quotients Q, is called *faithful* if $t(R) = 0$. For more details, see [13] or [14].

Some special types of torsion classes will be of interest to us. Stenström [13, § 13] calls a topologizing filter f *perfect* in case $Q \cdot I = Q$ for every left ideal $I \in f$ (where Q is viewed as an R-module). We will call a torsion class $\mathscr{T}$ perfect if its associated filter is per-

fect. These torsion classes have been characterized in [13, Theorem 13.1] and [14, Theorem 3.2]. We will refer to these characterizations as needed.

Perfect torsion classes are important because they give the left localizations of a ring. A *left localization* of R is a ring Q and a ring homomorphism $\sigma: R \to Q$ such that σ is an epimorphism in the category of rings which gives Q the structure of a flat right R-module [7], [13]. Stenström [13, §13] shows that $\mathscr{T}$ is perfect if and only if $\sigma_R: R \to Q$ is a left localization of R and $\mathscr{T} = \{M \in {}_R\mathscr{M} \mid Q \otimes_R M = 0\}$. Silver [12] further calls a left localization *finite* if Q is finitely generated and projective as a right R-module.

The other type of useful torsion class is the TTF-class. $\mathscr{T}$ is a TTF-class if there is a (not necessarily hereditary) torsion class $\mathscr{C}$ for which $\mathscr{T}$ is the corresponding torsion-free class. These were characterized by Jans [8, Theorem 2.1] as those torsion classes whose associated filter contains a unique minimal idempotent ideal J.

Notice that if $\mathscr{T}$ is a TTF-class, the localization functor is just $L_{\mathscr{T}}(M) = \operatorname{Hom}_R(J, M/t(M))$ where J is the minimal element in the corresponding filter.

The following theorem characterizes finite left localizations in terms of these torsion classes.

THEOREM 1.1. *Let $\mathscr{T}$ be a hereditary torsion class for a ring R with corresponding ring of left quotients Q. The following statements are equivalent:*

(1) $\mathscr{T}$ *is a perfect TTF-class,*

(2) Q *is a finite left localization of R and* $\mathscr{T} = \{M \in {}_R\mathscr{M} \mid Q \otimes_R M = 0\}$,

(3) *the localization functor $L_{\mathscr{T}}$ is exact and commutes with both direct sums and direct products,*

(4) $\mathscr{T}$ *is a TTF-class for which the minimal element J of the corresponding filter satisfies:*

(a) ${}_R J$ *is finitely generated, and*

(b) ${}_R J$ *is projective relative to the class of epimorphisms* $\mathscr{E} = \{A \to B \to 0 \mid A, B \in \mathscr{T}\}$.

Proof. (1) $\Rightarrow$ (4). Since $\mathscr{T}$ is perfect, f has a cofinal set of finitely generated left ideals [14, Theorem 3.2(a)]. Further, since $\mathscr{T}$ is TTF, f has a minimal element. Thus this minimal element ${}_R J$ is finitely generated. Moreover, if $f: A \to B$ is any epimorphism, $L_{\mathscr{T}}(f): L_{\mathscr{T}}(A) \to L_{\mathscr{T}}(B)$ is onto. Since $L_{\mathscr{T}}(M) \cong \operatorname{Hom}_R(J, M)$ if $M \in \mathscr{T}$, ${}_R J$ is projective relative to $\mathscr{E}$.

(4) $\Rightarrow$ (3). Condition (a) shows that $L_{\mathscr{T}}$ commutes with direct sums, while $L_{\mathscr{T}}$ commutes with direct products since $\mathscr{T}$ is a TTF-

class. Finally, a straightforward check shows that condition (b) makes $L_{\mathcal{F}}$ right exact and thus exact.

(3) $\Rightarrow$ (2). Since $L_{\mathcal{F}}$ is exact and commutes with direct sums, $\mathcal{F}$ is a perfect torsion class [13, Theorem 13.1(c)]. Thus Q is a left localization of R and $\mathcal{F} = \{M \in {}_R\mathcal{M} \mid Q \otimes_R M = 0\}$. In particular, Q_R is flat. Moreover, since $L_{\mathcal{F}}$ is equivalent to $Q \otimes_R (\ )$ [13, Theorem 13.1(e)] and commutes with direct products, [4, Lemmas 1.1 and 1.2] show that Q_R is finitely generated and, in fact, finitely presented. Hence by [3], Q_R is projective and is a finite left localization of R.

(2) $\Rightarrow$ (1). Since Q_R is finitely generated and projective and $Q \cong \operatorname{End}_R(Q_R)$ by [9, Theorem 7.1], $\mathcal{F}$ is a TTF-class by [5, Proposition 1.4]. Finally, $\mathcal{F}$ is perfect by Stenström's characterization of left localizations.

REMARK. It would be tempting to replace condition (2) of the theorem by the condition that Q be a finite left localization of R. However, Example 2.4 below will show that this is impossible.

COROLLARY 1.2. *There is a one-to-one correspondence between the finite left localizations of R and the perfect* TTF-*classes, given by corresponding to Q the class $\{M \in {}_R\mathcal{M} \mid Q \otimes_R M = 0\}$.*

Proof. This follows from the above theorem, [13, Theorem 13.1(f)], and [5, Proposition 1.4].

The following corollary shows that if $\mathcal{F}$ is also faithful, we may strengthen condition (4(b)) above.

COROLLARY 1.3. *Let $\mathcal{F}$ be a faithful hereditary torsion class in ${}_R\mathcal{M}$. Then $\mathcal{F}$ is a perfect* TTF-*class if and only if the associated filter f has a minimal element J which is finitely generated and projective as a left ideal of R. Moreover, in this case $Q \cong \operatorname{End}_R({}_RJ)$.*

Proof. The condition on ${}_RJ$ is clearly sufficient. Conversely, since ${}_RR$ and ${}_RJ$ are both in $\mathcal{F}$, the usual epimorphism of a free module onto ${}_RJ$ is in $\mathcal{E}$. Since ${}_RJ$ is projective relative to $\mathcal{E}$, this map must split. Hence ${}_RJ$ is projective. The last statement follows since ${}_RJ \in \mathcal{F}$ and $R/J \in \mathcal{F}$, whence $L_{\mathcal{F}}(R) \cong \operatorname{Hom}_R(J, R) \cong \operatorname{Hom}_R(J, J)$.

REMARK. The last statement of the corollary extends Silver's remark on finite left localizations of Noetherian rings [12, p. 58] to the general case. This was already clear, however, from Morita's result that $Q \cong \operatorname{End}_R(Q_R)$ for left localizations Q [9, Theorem 7.1].

We end this section with three corollaries about right perfect rings. These results depend on the fact that for right perfect rings,

every hereditary torsion class of left R-modules is a TTF-class. See [1, Corollary 1.6] and [6, Theorem A].

COROLLARY 1.4. *If R is a right perfect ring, every left localization of R is a finite left localization.*

Proof. Every perfect torsion class is a perfect TTF-class.

COROLLARY 1.5. *If R is a right perfect ring, the faithful left localizations of R are precisely those rings of left quotients of R for which the minimal element in the corresponding filter is an idempotent, faithful, two-sided ideal RfR which is finitely generated and projective as a left R-module. In this case Q is Morita equivalent to fRf.*

Proof. Since R is right perfect, every idempotent ideal of R is of the form RfR for f an idempotent in R. The first statement then follows directly from Corollaries 1.3 and 1.4. Now RfR is the trace ideal of fR in R (see §2) and is finitely generated and projective as a left ideal of R. Thus it follows from [11, Corollary 4] and [2, Theorem 3.2] that fR is a left progenerator over $fRf \cong \mathrm{End}_R(fR)$. Since Q is isomorphic to the fRf-endomorphism ring of fR [5, Theorem 2.1], Q is Morita equivalent to fRf.

COROLLARY 1.6. *The flat epimorphic hull of a right perfect ring is isomorphic to the ring of left R-endomorphisms of the minimal member of the set of ideals of R described in the previous corollary.*

Proof. The flat epimorphic hull is the maximal faithful left localization of R. The result then follows from Corollaries 1.5 and 1.3.

REMARK. This shows that the flat epimorphic hull of a right perfect ring can be concretely constructed. This contrasts with the general construction using a direct limit argument [7].

2. **Finite projectors.** In this section we examine the torsion class and ring of left quotients determined by a finitely generated projective module P_R as in [5]. We ask when this torsion class is perfect. We will need some background and notation; details may be found in [5].

Let P_R be a finitely generated projective right R-module with $S = \mathrm{End}_R(P_R)$. The trace ideal T of P_R is $T = \Sigma f(P)$ as f ranges over $\mathrm{Hom}_R(P, R)$. The trace T is an idempotent two-sided ideal of R such that $PT = P$. The monule P_R determines a hereditary tor-

sion class $\mathscr{T}_T = \{M \in {}_R\mathscr{M} \mid P \otimes {}_R M = 0\}$; its [associated filter is $f_T = \{I$ left ideal of $R \mid I \supseteq T\}$. Thus $\mathscr{T}_T$ is always a TTF-class. We denote the corresponding torsion-free class by $\mathscr{F}_T$, localization by L_T, and ring of left quotients by Q_T.

The bimodule ${}_S P_R$ determines functors

$$F = {}_S P_R \otimes {}_R(\ ) \colon {}_R\mathscr{M} \longrightarrow {}_S\mathscr{M}$$

and

$$H = \operatorname{Hom}_S({}_S P_R, -) \colon {}_S\mathscr{M} \longrightarrow {}_R\mathscr{M}$$

such that F is the left adjoint of H. Then the localization functor L_T is naturally equivalent to the functor HF; thus $Q_T = L_T(R) \cong HF(R) = \hat{R}$, the double centralizer of P_R, as rings. We will consistently denote this ring by $\hat{R}$.

THEOREM 2.1. *Let R be a ring, P_R be a finitely generated projective right R-module, and $S = \operatorname{End}_R(P_R)$. The following statements are equivalent:*

(1) *$\mathscr{T}_T$ is a perfect torsion class,*
(2) *${}_S P$ is finitely generated and projective,*
(3) *the functor H is exact and commutes with direct sums.*

Proof. (1) $\Rightarrow$ (3). Let $\mathscr{T}_T$ be a perfect torsion class. Then the functor L_T, naturally equivalent to HF, is exact and commutes with direct sums. We claim this implies that H is exact. Let $0 \to N' \to N \xrightarrow{\delta} N'' \to 0$ be exact in ${}_S\mathscr{M}$. Then $0 \to H(N') \to H(N) \xrightarrow{H(\delta)} H(N'') \to$ coker $H(\delta) \to 0$ is exact in ${}_R\mathscr{M}$. Since HF is exact, $0 \to HFH(N') \to HFH(N) \to HFH(N'') \to HF(\text{coker } H(\delta)) \to 0$ is exact. Since FH is naturally equivalent to the identity functor on ${}_S\mathscr{M}$, $F(\text{coker } H(\delta)) = 0$ and $0 \to H(N') \to H(N) \xrightarrow{H(\delta)} H(N'') \to 0$ is exact. Finally, since the trace ideal T is the minimal element in the filter f_T, Theorem 1.1(4) shows that ${}_R T$ is finitely generated. Then [11, Corollary 4] shows that ${}_S P$ is finitely generated so that H commutes with direct sums.

(3) $\Rightarrow$ (1). This is clear since HF must be exact and commute with direct sums.

(3) $\Rightarrow$ (2). Since H is exact, ${}_S P$ must be projective. Moreover, since (3) is equivalent to (1), ${}_S P$ is finitely generated.

(2) $\Rightarrow$ (3). This is clear.

Condition (2) of Theorem 2.1 is obviously equivalent to the condition that the functor $F = {}_S P_R \otimes {}_R(\ )$ take finitely generated projective modules to finitely generated projective modules. Thus we follow Anderson [2] and call a finitely generated projective module satisfying the conditions of Theorem 2.1 a *finite projector*. We immediately get one more characterization of finite left localizations.

COROLLARY 2.2. *A ring of left quotients Q of R is a finite left localization of R if and only if Q is canonically isomorphic to the double centralizer $\hat{R}$ of a finite projector P_R.*

Proof. Let Q be a finite left localization of R. By [9, Theorem 7.1], $Q \cong \text{End}_R(Q_R)$. Hence Q_R is a finite projector and Q is clearly its double centralizer. Conversely, if P_R is a finite projector, $\mathscr{T}_T$ is a perfect TTF-class. Thus $\hat{R} \cong Q$ is a finite left localization of R by Theorem 1.1.

The many characterizations of perfect torsion classes give rise to many for finite projectors. The following corollary lists a few; proofs are straightforward and easily seen from [5], [13], [14], or Theorem 1.1, and are thus omitted.

COROLLARY 2.3. *For R, P, and S as in Theorem 2.1, the following statements are equivalent.*
  (1) *P_R is a finite projector,*
  (2) *$_R T$ is finitely generated and is projective relative to the class of epimorphisms $\{A \to B \to 0 \mid A, B \in \mathscr{F}_T\}$,*
  (3) *$_S P$ is a progenerator,*
  (4) *$\hat{R}$ and S are Morita equivalent via $_S P_{\hat{R}} \cong {}_S P_R \otimes {}_R \hat{R}$,*
  (5) *$\hat{R} \cdot T = \hat{R}$,*
  (6) *$L_T(M) \cong \hat{R} \otimes {}_R M$ for all $_R M$,*
  (7) *every left $\hat{R}$-module is in $\mathscr{F}_T$ viewed as an R-module.*

In view of Corollary 2.2, one might expect to add to this list the condition that $\hat{R}$ is a finite left localization of R. The following example shows that this condition is not sufficient.

EXAMPLE 2.4. Let S be a ring with a nonprojective left S-module $_S X$, let $P = S \otimes X$, and let $R = \text{End}_S({}_S P)$. Then P_R is finitely generated and projective since $_S P$ is a left S-generator. Further, $S \cong \text{End}_R(P_R)$ since generators have the double centralizer property. Hence $R = \hat{R}$ clearly, and $\hat{R}$ is a finite left localization of R. But $_S P$ is not projective; thus P_R cannot be a finite projector.

The following two propositions give us some sources of finite projectors.

PROPOSITION 2.5. *If R is left hereditary and left Noetherian, every finitely generated projective right R-module is a finite projector.*

Proof. If T is the trace ideal of P_R, then $_R T$ is finitely generated and projective.

Recall that Anderson [2] calls P_R a perfect projector (perfect

injector) if the functor $F = P \otimes_R (\ \ )$ preserves projective covers (injective hulls). See also [10].

PROPOSITION 2.6. *If R is semiperfect and P_R is a perfect projector, then P_R is a finite projector.*

Proof. By [10, Theorem 3.3] there is an idempotent $e \in R$ with $T = Re$. Since $_R T$ is finitely generated and projective, P_R is a finite projector.

The converse of this is false; there are triangular matrix ring examples which show that finite projectors need not be perfect projectors and that there is generally no relation between finite projectors and perfect injectors.

3. **Correspondence of torsion classes and quotient rings.** In this section we characterize finite projectors by special properties of certain correspondences from [5]. We begin with some background on these correspondences.

Let P_R be a finitely generated projective right R-module with $S = \mathrm{End}_R(P_R)$. Let F be the functor $F(M) = {}_S P \otimes {}_R M$: ${}_R \mathcal{M} \to {}_S \mathcal{M}$ as in the last section. For a hereditary torsion class $\mathcal{T}_R$ in ${}_R \mathcal{M}$, define

$$F(\mathcal{T}_R) = \{N \in {}_S \mathcal{M} \mid N \cong F(M) \text{ for some } M \in \mathcal{T}_R\}$$

and for a hereditary torsion class $\mathcal{T}_S$ in ${}_S \mathcal{M}$, define

$$F^{\leftarrow}(\mathcal{T}_S) = \{M \in {}_R \mathcal{M} \mid F(M) \in \mathcal{T}_S\} .$$

The mappings $\mathcal{T}_R \to F(\mathcal{T}_R)$ and $\mathcal{T}_S \to F^{\leftarrow}(\mathcal{T}_S)$ are inclusion-preserving, inverse one-to-one correspondences between the hereditary torsion classes in ${}_S \mathcal{M}$ and those hereditary torsion classes in ${}_R \mathcal{M}$ which contain $\mathcal{T}_T = \{M \in {}_R \mathcal{M} \mid F(M) = 0\}$. Further, if Q_R and Q_S are rings of left quotients of R and S with respect to hereditary torsion classes $\mathcal{T}_R$ and $\mathcal{T}_S$ which correspond, then $P \otimes_R Q_R$ is a finitely generated and projective right Q_R-module with $Q_S \cong \mathrm{End}_{Q_R}(P \otimes_R Q_R)$ and $Q_R \cong \mathrm{End}_{Q_S}(P \otimes_R Q_R)$.

THEOREM 3.1. *Let R be a ring, P_R be a finitely generated projective right R-module, and $S = \mathrm{End}_R(P_R)$. The following statements are equivalent:*

(1) *P_R is a finite projector,*

(2) *the correspondence of torsion classes above yields a one-to-one correspondence of the perfect torsion classes in ${}_S \mathcal{M}$ with the perfect torsion classes in ${}_R \mathcal{M}$ containing $\mathcal{T}_T$,*

(3) *if Q_R and Q_S are rings of left quotients with respect to torsion classes which correspond as above, then Q_R and Q_S are Morita equivalent via the module $P \otimes_R Q_R \cong Q_S \otimes_S P$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathscr{T}_R$ and $\mathscr{T}_S$ correspond, let $\mathscr{T}_R$ be perfect, and let N be a left Q_S-module. Since $Q_S \cong \mathrm{End}_{Q_R}(P \otimes_R Q_R)$, there is a left Q_R-module M with $(P \otimes_R Q_R) \otimes_{Q_R} M \cong P \otimes_R M \cong N$. Since $\mathscr{T}_R$ is perfect, $M \in \mathscr{F}_R$, the corresponding torsion-free class; hence by [5, Proposition 3.4], $N \in \mathscr{F}_S$. Since a hereditary torsion class is perfect just in case every module over the ring of quotients is torsion-free [14, Theorem 3.2(d)], $\mathscr{T}_S$ is perfect.

Conversely, let $\mathscr{T}_S$ be perfect and let M be a left Q_R-module. Since $\mathscr{T}_R \cong \mathscr{T}_T$, the natural map $R \rightarrow Q_R$ can be extended to an R-homomorphism $\hat{R} \rightarrow Q_R$. An elementary argument shows that this map is, in fact, a ring homomorphism. Thus we may view M as a left $\hat{R}$-module. Since $\mathscr{T}_T$ is perfect, $M \in \mathscr{F}_T$. Further, $P \otimes_R M \cong (P \otimes_R Q_R) \otimes_{Q_R} M$ is a left Q_S-module, and is thus in $\mathscr{F}_S$. Again by [5, Proposition 3.4], $M \in \mathscr{F}_R$. Thus $\mathscr{T}_R$ is perfect.

(2) $\Rightarrow$ (1). Since $\{0\} \subseteq {}_S \mathscr{M}$ is clearly perfect, and $\{0\}$ corresponds to $\mathscr{T}_T$, $\mathscr{T}_T$ is perfect. Hence P_R is a finite projector.

(1) $\Rightarrow$ (3). If P_R is a finite projector, ${}_S P$ is a progenerator. Hence by [5, §4], $P \otimes_R Q_R \cong Q_S \otimes_S P$ is a left Q_S-progenerator and $Q_R \cong \mathrm{End}_{Q_S}(Q_S \otimes_S P)$. Thus Q_R and Q_S are Morita equivalent.

(3) $\Rightarrow$ (1). This is clear by Corollary 2.3.

REFERENCES

1. J. S. Alin and E. P. Armendariz, *TTF-classes over perfect rings*, J. Aust. Math. Soc., **11** (1970), 499–503.
2. F. W. Anderson, *Endomorphism rings of projective modules*, Math. Z., **111** (1969), 322–332.
3. S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc., **97** (1960), 457–473.
4. R. R. Colby and E. A. Rutter, Jr., *Π-flat and Π-projective modules*, Arch. der Math., **22** (1971), 246–251.
5. R. S. Cunningham, E. A. Rutter, Jr., and D. R. Turnidge, *Rings of quotients of endomorphism rings of projective modules*, Pacific. J. Math., **41** (1972), 647–668.
6. V. Dlab, *A characterization of perfect rings*, Pacific. J. Math., **33** (1970), 79–88.
7. G. D. Findlay, *Flat epimorphic extensions of rings*, Math. Z., **118** (1970), 281–288.
8. J. P. Jans, *Some aspects of torsion*, Pacific. J. Math., **15** (1965), 1249-1259.
9. K. Morita, *Localizations in categories of modules I*, Math. Z., **114** (1970), 121–144.
10. E. A. Rutter, Jr., *Perfect Projectors and Perfect Injectors. Ring Theory*, Academic Press, New York-London, 1972.
11. F. L. Sandomierski, *Modules over the endomorphism ring of a finitely generated projective module*, Proc. Amer. Math. Soc., **31** (1972), 27–31.
12. L. Silver, *Noncommutative localizations and applications*, J. Algebra, **7** (1967), 44-76.

13. B. Stenström, *Rings and Modules of Quotients*, Springer, Lecture Notes in Mathematics, 237, 1971.

14. C. L. Walker and E. A. Walker, *Quotient categories and rings of quotients*, Rocky Mountain J. Math., **2** (1972), 513-555.

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