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FUNCTIONALS ON CONTINUOUS FUNCTIONS

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Let $\mathscr{C}(M)$ be the space of continuous functions on a compact metric space M. In a previous paper a class of nonlinear functionals Φ on $\mathscr{C}([0, 1] \times [0, 1])$ was constructed, such that each Φ satisfied:

(i) $\lim_{||f||\to 0} \Phi(f) = 0$,

(ii) $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever $fg = 0$, and

(iii) $\Phi(f + \alpha) = \Phi(f) + \Phi(\alpha)$ for any constant α.

In this paper we show that the dimensionality of $[0, 1] \times [0, 1]$ is what makes the construction work. More precisely, we show that if Φ is a functional on $\mathscr{C}(M)$ satisfying (i), (ii), and (iii), and if the dimension of M is less than two, then Φ must be linear.

1. Introduction.

Let M be a compact metric space. Let $\mathscr{C}(M)$ be the space of continuous real-valued functions on M. In this paper, we will prove the following result:

THEOREM 1. *Let Φ: $\mathscr{C}(M) \to \boldsymbol{R}$ ($\boldsymbol{R}$ = the real numbers) be a functional such that*:

(i) $\lim_{||f||\to 0} \Phi(f) = 0$, $(||f|| = \sup_{x\in M} |f(x)|)$

(ii) $\Phi(f + g) = \Phi(f) + \Phi(g)$ *whenever* $fg = 0$

(iii) $\Phi(f + \alpha) = \Phi(f) + \Phi(\alpha)$ *for all* $f \in \mathscr{C}(M)$, $\alpha \in \boldsymbol{R}$.

Then if M has dimension no greater than one, Φ must be linear.

The additivity properties (ii) and (iii) may also be expressed by one condition:

(ii)′ $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever g is constant on $\{x \mid f(x) \neq 0\}$.

It is also easy to see that we must have $\Phi(\alpha) = \alpha\Phi(1)$ for all $\alpha \in \boldsymbol{R}$.

It has been shown in [2] that there exist nonlinear functionals Φ on $\mathscr{C}([0, 1] \times [0, 1])$ which are bounded, continuous, monotonic, and satisfy conditions (ii) and (iii). Thus Theorem 1 does not extend to spaces of dimension greater than one.

In [1], a proof of Theorem 1 is given for the special case $M = [0, 1]$. We will use this case of Theorem 1 to prove the general case. In § 2 it is shown that Theorem 1 is equivalent to the following result:

THEOREM 2. *For each $f \in \mathscr{C}(M)$, let $\mathscr{B}_f = \{f^{-1}(E) \mid E \subseteq \boldsymbol{R}, E$ Borel$\}$. Suppose a measure μ_f on $\mathscr{B}_f$ is given, for each $f \in \mathscr{C}(M)$, such that*:

(i) *the measures* μ_f *are uniformly bounded in total variation, and*

(ii) *the measures* μ_f *are consistent, in the sense that if* $\mathscr{B}_f \subseteqq \mathscr{B}_g$ *then* $\mu_f = \mu_g$ *on* $\mathscr{B}_f$.

Then if M has dimension no greater than one, a measure μ *on the Borel sets of M can be found, which is the common extension of all the* μ_f.

Theorem 2 is obvious if M is the unit interval, but not if M is the unit circle. Theorem 2 will be proved in § 3.

2. Construction of a set function. For each $f \in \mathscr{C}(M)$, let $\mathscr{L}_f$ be the space of continuous functions $g \in \mathscr{C}(M)$ which are measurable with respect to $\mathscr{B}_f$. It is easy to see that $g \in \mathscr{L}_f$ if and only if $g(x) = g(y)$ whenever $f(x) = f(y)$, and that this means g is of the form $h \circ f$, where h is a continuous function on $\boldsymbol{R}$.

LEMMA 1. *Φ satisfies conditions* (i), (ii), *and* (iii) *of Theorem* 1 *if and only if*:

(i) *Φ is bounded, that is, there exists k such that* $|\Phi(f)| \leqq k \|f\|$ *for all* $f \in \mathscr{C}(M)$,

(ii) *Φ is linear on each space* $\mathscr{L}_f$.

Proof. Assume Φ satisfies (i), (ii) and (iii) of Theorem 1. Fix $f \in \mathscr{C}(M)$. Let I be a compact interval containing $f(M)$.

Define Φ^* on $\mathscr{C}(I)$ by the equation $\Phi^*(h) = \Phi(h \circ f)$. Clearly Φ^* satisfies conditions (i), (ii), and (iii) of Theorem 1. By the special case of Theorem 1 that is proved in [1], Φ^* must be linear. It follows at once that Φ is linear on $\mathscr{L}_f$.

Since Φ is continuous at 0, there exists $r > 0$ such that

$$\|f\| \leqq r \text{ implies } |\Phi(f)| \leqq 1 .$$

Then for any $f \in \mathscr{C}(M)$, $f \neq 0$,

$$|\Phi(f)| = \left|\frac{\|f\|}{r}\Phi\left(\frac{rf}{\|f\|}\right)\right| \leqq \frac{1}{r}\|f\| .$$

Thus Φ is bounded.

Now assume Φ satisfies conditions (i) and (ii) of Lemma 1. Then condition (i) of Theorem 1 clearly holds.

To prove that condition (ii) of Theorem 1 holds, let us first assume that f and g are in $\mathscr{C}(M)$, with $f \geqq 0$, $g \leqq 0$, and $fg = 0$.

Then $f = (f + g) \vee 0$ and $g = (f + g) \wedge 0$, so that f and g are both in $\mathscr{L}_{f+g}$. Hence $\Phi(f + g) = \Phi(f) + \Phi(g)$.

Now assume that $f \geqq 0$, $g \geqq 0$, and $fg = 0$. Then by the preceding argument f and g are both in $\mathscr{L}_{f-g}$, so again $\Phi(f+g) = \Phi(f) + \Phi(g)$.

Finally, for arbitrary f and g in $\mathscr{C}(M)$ with $fg = 0$, let $f_1 = f \vee 0$, $f_2 = f \wedge 0$, $g_1 = g \vee 0$, $g_2 = g \wedge 0$. Then

$$\begin{aligned} \Phi(f+g) &= \Phi(f_1 + f_2 + g_1 + g_2) \\ &= \Phi(f_1 + g_1) + \Phi(f_2 + g_2) \quad \text{by the first case,} \\ &= \Phi(f_1) + \Phi(g_1) + \Phi(f_2) + \Phi(g_2) \quad \text{by the second case,} \\ &= \Phi(f_1 + f_2) + \Phi(g_1 + g_2) \quad \text{by the first case,} \\ &= \Phi(f) + \Phi(g) \,. \quad \text{Thus condition (ii) of Theorem 1 holds.} \end{aligned}$$

Condition (iii) of Theorem 1 clearly holds, so Lemma 1 is proved.

Using Lemma 1 and the Riesz representation theorem it is easy to see that for each functional Φ satisfying conditions (i), (ii), and (iii) of Theorem 1 we can find a system of measures μ_f satisfying conditions (i) and (ii) of Theorem 2, and such that $\Phi(f) = \int f d\mu_f$ for each $f \in \mathscr{C}(M)$. Conversely, if μ_f, $f \in \mathscr{C}(M)$, is a system of measures satisfying conditions (i) and (ii) of Theorem 2, then Lemma 1 implies that the functional Φ defined by $\Phi(f) = \int f d\mu_f$ must satisfy conditions (i), (ii), and (iii) of Theorem 1. It follows at once that Theorems 1 and 2 are equivalent.

In what follows we will use both Φ and the corresponding system of measures μ_f.

LEMMA 2. *Let f and g be in $\mathscr{C}(M)$. Let K be a closed set in $\mathscr{B}_f \cap \mathscr{B}_g$. Then $\mu_f(K) = \mu_g(K)$.*

Proof. $f(K)$ is a compact set in $\boldsymbol{R}$. It is easy to see that one can find a sequence of continuous functions h_n on $\boldsymbol{R}$ such that $0 \leqq h_n \leqq 1$, $h_n = 1$ on a neighborhood of $f(K)$, $h_n = 1$ on the support of h_{n+1}, and the intersection of the supports of the h_n is $f(K)$.

Let $f_n = h_n \circ f$. Then clearly $0 \leqq f_n \leqq 1$, $f_n = 1$ on a neighborhood of K, $f_n = 1$ on the support of f_{n+1}, and the intersection of the supports of the f_n is K.

Let $g_n = p_n \circ g$ be a sequence having the same properties as the f_n. Fix f_n. Then $f_n = 1$ on a neighborhood, A, of K. Since the intersection of the supports of the g_n is K, it follows that for sufficiently large m the support of g_m will be contained in A. Hence, by choosing subsequences and relabelling, we may assume that, in addition to the properties mentioned above, f_n and g_n are also such that $f_n = 1$ on a neighborhood of the support of g_n, and $g_n = 1$ on a neighborhood of the support of f_{n+1}.

Since the f_n are uniformly bounded, and $f_n \to \chi_K$ pointwise as

$n \to \infty$, we have $\Phi(f_n) = \int f_n d\mu_f \to \mu_f(K)$ as $n \to \infty$. Similarly $\Phi(g_n) \to \mu_g(K)$ as $n \to \infty$. Suppose $\mu_f(K) > \mu_g(K)$. Choose $\delta > 0$, $\delta < \mu_f(K) - \mu_g(K)$. For sufficiently large n we must have $\Phi(f_n) > \Phi(g_n) + \delta$. By relabelling we may assume that $\Phi(f_n) > \Phi(g_n) + \delta$ for all n.

Let u_n be a continuous function on M such that $0 \leqq u_n \leqq 1$, $u_n = 0$ on the support of g_n, and $u_n = 1$ on $\{x \mid f_n(x) < 1\}$. Let

$$v_n = f_n - u_n f_n - g_n .$$

It is easy to check that $0 \leqq v_n \leqq 1$, and the support of v_n is contained in

$$\{x \mid f_n(x) = 1\} - \{x \mid g_n(x) = 1\} .$$

Hence $\Phi(-v_n + f_n) = \Phi(-v_n) + \Phi(f_n)$, by the additivity property (ii)′ of Φ. That is, $\Phi(u_n f_n + g_n) = \Phi(-v_n) + \Phi(f_n)$. Since $u_n f_n = 0$ on the support of g_n, we have $\Phi(u_n f_n + g_n) = \Phi(u_n f_n) + \Phi(g)$ by the additivity of Φ again. Thus $\Phi(u_n f_n) + \Phi(g_n) = \Phi(-v_n) + \Phi(f_n)$. Hence $\Phi(u_n f_n) > \Phi(-v_n) + \delta$, and so $\sum_{n=1}^{m} \Phi(u_n f_n) > \sum_{n=1}^{m} \Phi(-v_n) + m\delta$, for all m.

It is easy to check that the supports of the $u_n f_n$ are pairwise disjoint, as are the supports of the v_n. Hence

$$\Phi\Big(\sum_{n=1}^{m} u_n f_n\Big) > \Phi\Big(\sum_{n=1}^{m} (-v_n)\Big) + m\delta ,$$

by additivity, for all m.

The functions $\sum_{n=1}^{m} u_n f_n$ and $\sum_{n=1}^{m} (-v_n)$ are uniformly bounded in m. Hence the last inequality contradicts the boundedness of Φ. Hence our original supposition, $\mu_f(K) > \mu_g(K)$, was false. This proves Lemma 2.

Since M is a metric space, it is easy to see that every closed set E and every open set E occurs in some $\mathscr{B}_f$.

DEFINITION 1. Let us write $\mu_f(E) = \mu(E)$ for E closed or E open, since the number has been shown to be independent of f.

LEMMA 3. *The set function μ is bounded and additive wherever defined.*

Proof. μ is bounded because the total variation of the μ_f's is uniformly bounded.

Let E_1 and E_2 be sets, with $E_1 \cap E_2 = \phi$, such that $\mu(E_1)$, $\mu(E_2)$, and $\mu(E_1 \cup E_2)$ are defined. We may have E_1, E_2 open, E_1, E_2 closed, E_1 open, E_2 closed, and $E_1 \cup E_2$ open, or E_1 open, E_2 closed, and $E_1 \cup E_2$ closed. In each of the four possible cases it is easy to find a function $f \in \mathscr{C}(M)$ such that E_1 and E_2 are in $\mathscr{B}_f$. This proves Lemma 3.

LEMMA 4. *Let G_n be a monotone increasing sequence of open sets, with union G. Let F_n be a sequence of closed sets such that $G_n \subseteqq F_n \subseteqq G$ for all n. Then $\mu(G_n) \to \mu(G)$* and *$\mu(F_n) \to \mu(G)$ as $n \to \infty$.*

Proof. Suppose $\mu(G_n) \not\to \mu(G)$ or $\mu(F_n) \not\to \mu(G)$. Then there exists a $\delta > 0$ and a subsequence n_j such that

$$| \mu(G_{n_j}) - \mu(G) | + | \mu(F_{n_j}) - \mu(G) | > \delta$$

for all j. Since the F_n are compact we can choose n_j so that $F_{n_j} \subseteqq G_{n_{j+1}}$. It is then a straightforward matter to construct $f \in \mathscr{C}(M)$ such that $G_{n_j}, E_{n_j} \in \mathscr{B}_f$ for all j. This contradiction proves the lemma.

3. Proof of the theorems.

In this section we will prove:

THEOREM 3. *Let μ be a real-valued set function defined for closed subsets and for open subsets of M, such that:*

(i) *μ is bounded and additive wherever defined, and*

(ii) *μ has the continuity property described in Lemma 4.*

Then if M has dimension no greater than one, μ can be extended to a measure on the Borel sets of M.

We can apply Theorem 3 to the set function μ constructed in the previous section. The Borel measure $\hat{\mu}$ which is an extension of μ agrees with each measure μ_f on all closed sets in $\mathscr{B}_f$. Since each μ_f is obviously regular, $\hat{\mu}$ must be an extension of μ_f. Thus Theorem 2 is proved, and hence Theorem 1 also.

From now on let μ be any set function satisfying conditions (i) and (ii) of Theorem 3.

LEMMA 5. *Let F_n be a monotone decreasing sequence of closed sets, having intersection F. Let G_n be a sequence of open sets such that $F_n \supseteqq G_n \supseteqq F$ for all n. Then $\mu(F_n) \to \mu(F)$ and $\mu(G_n) \to \mu(F)$* as *$n \to \infty$.*

Proof. Follows from condition (ii) by taking complements and using the additivity property.

DEFINITION 2. For any set $E \subseteqq M$, define

$$\nu(E) = \sup \{\mu(F) \mid F \subseteqq E, F \text{ closed}\} .$$

Since μ is bounded, so is ν. Clearly ν is monotone.

LEMMA 6. *Let E_1 and E_2 be disjoint subsets of M. Then $\nu(E_1 \cup E_2) \geqq \nu(E_1) + \nu(E_2)$. If E_1 and E_2 are either both open or both closed,* *then* $\nu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2)$.

Proof. Follows from the additivity of μ.

LEMMA 7. *Let G be open. Then*

$$\nu(G) = \sup\{\mu(H) \mid H \subseteqq G,\ H \text{ open}\}\ .$$

Proof. Follows from the continuity of μ.

We pause now for a general topological lemma.

LEMMA 8. *Let X be a locally compact separable metric space of dimension* 0. *Then X is a countable union of monotone increasing sets that are both compact and open.*

Proof. From the definition of dimension 0, each point x has arbitrarily small neighborhoods G_x which are both closed and open.

By choosing G_x small enough, it can therefore be made both compact and open.

Since $X = \bigcup_{x \in X} G_x$, and X has a countable base, we can find $x_1, x_2, \cdots$ such that $X = \bigcup_{n=1}^{\infty} G_{x_n}$. Let $K_n = \bigcup_{j=1}^{n} G_{x_j}$. Then each K_n is both compact and open, and $K_n \uparrow X$.

Now we return to M, μ, and ν.

LEMMA 9. *Let G be open. Let E be open,* $E \subseteqq G$, *such that* $\partial E \cap G$ *has dimension* 0. *Then* $\mu(G) \leqq \nu(E) + \nu(G - E)$.

Proof. Let $D = \partial E \cap G$. Let $H = G - \bar{E}$. Then the sets E, D, and H are mutually disjoint, and $G = E \cup D \cup H$.

Since D is a closed subset of the locally compact separable metric space G, D is a locally compact separable metric space also.

By Lemma 8, we can find sets K_n which are both compact and open in D, such that $K_n \uparrow D$.

Let $K_n = A_n \cap D$, where A_n is open. Since K_n is compact we may choose A_n such that $\bar{A}_n \subseteqq G$. By taking unions if necessary we may choose the A_n to be increasing.

Let E_n and H_n be open sets such that $\bar{E}_n \subseteqq E$, $\bar{H}_n \subseteqq H$ for all n, $E_n \uparrow E$ and $H_n \uparrow H$. Let $G_n = E_n \cup A_n \cup H_n$. Then G_n is open, $\bar{G}_n \subseteqq G$, and $G_n \uparrow G$. Then $\mu(G_n) \to \mu(G)$ as $n \to \infty$, by continuity.

But for all n, $G_n = (G_n \cap E) \cup (G_n \cap D) \cup (G_n \cap H)$

$$= (G_n \cap E) \cup K_n \cup (G_n \cap H)\ .$$

$$\begin{aligned}\text{Thus } \mu(G_n) &= \mu(G_n \cap E) + \mu(K_n) + \mu(G_n \cap H), \text{ by additivity,}\\ &\leqq \nu(G_n \cap E) + \nu(K_n) + \nu(G_n \cap H)\\ &\leqq \nu(E) + \nu(D) + \nu(H) \leqq \nu(E) + \nu(G - E)\ .\end{aligned}$$

This proves Lemma 9.

LEMMA 10. *Let G be an open set. Let E be open, $E \subseteqq G$, such that $\partial E \cap G$ has dimension 0. Then $\nu(G) = \nu(E) + \nu(G - E)$.*

Proof. Let $\varepsilon > 0$ be given. Choose H open, $H \subseteqq G$, such that $\mu(H) \geqq \nu(G) - \varepsilon$. This is possible by Lemma 7.

Then $\partial(E \cap H) \cap H = \partial E \cap H \subseteqq \partial E \cap G$. Hence $\partial(E \cap H) \cap H$ has dimension 0. By Lemma 7, $\mu(H) \leqq \nu(E \cap H) + \nu(H - E \cap H) \leqq \nu(E) + \nu(G - E)$. Hence $\nu(G) \leqq \nu(E) + \nu(G - E)$.

The reverse inequality holds by Lemma 6, so Lemma 10 is proved.

From now on in this section, let M have dimension at most one.

LEMMA 11. *Let G_1 and G_2 be open, with union G. Then $\nu(G) \leqq \nu(G_1) + \nu(G_2)$.*

Proof. $G_1 - G_2$ and $G_2 - G_1$ are disjoint and relatively closed in G. G is a separable metric space of dimension no larger than 1. Hence by Theorem 1 in [3], section 27II, page 290, we can find an open set $E \subseteqq G$ such that $E \supseteqq G_1 - G_2$, $\bar{E} \cap (G_2 - G_1) = \varnothing$, and $\partial E \cap G$ has dimension 0.

By Lemma 10,

$$\nu(G) = \nu(E) + \nu(G - E) \leqq \nu(G_1) + \nu(G_2)\ .$$

LEMMA 12. *Let G_n be a sequence of open sets. Let $G = \bigcup_{n=1}^{\infty} G_n$. Then $\nu(G) \leqq \sum_{n=1}^{\infty} \nu(G_n)$.*

Proof. Let $\varepsilon > 0$ be given. Choose F closed, $F \subseteqq G$ such that $\mu(F) \geqq \nu(G) - \varepsilon$.

Then there exists n such that $F \subseteqq \bigcup_{j=1}^{n} G_j$. Hence $\sum_{j=1}^{\infty} \nu(G_j) \geqq \sum_{j=1}^{n} \nu(G_j) \geqq \nu(\bigcup_{j=1}^{n} G_j)$, by Lemma 11, $\geqq \mu(F)$ by definition.

This proves Lemma 12.

DEFINITION 3. For any set $E \subseteqq M$, define $\nu^*(E) = \inf\{\nu(G) \mid E \subseteqq G, G \text{ open}\}$. Clearly $\nu^*(E) = \nu(E)$ when E is open.

LEMMA 13. *ν^* is an outer measure.*

Proof. Follows from Lemma 12.

LEMMA 14. *Every open set is measurable with respect to* ν^*, *in the sense of Caratheodory.*

Proof. Let G be open. Let E be any set. We know

$$\nu^*(E) \leqq \nu^*(E \cap G) + \nu^*(E - G) ,$$

since ν^* is an outer measure. We must show that

$$\nu^*(E) \geqq \nu^*(E \cap G) + \nu^*(E - G) .$$

Choose any open set H such that $E \subseteqq H$. Let $\varepsilon > 0$ be given. Choose F closed, $F \subseteqq G \cap H$, such that $\nu(F) \geqq \nu(G \cap H) - \varepsilon$. Then $\nu(H) \geqq \nu(F) + \nu(H - F)$, by Lemma 6, $\geqq \nu(G \cap H) - \varepsilon + \nu(H - F) \geqq \nu^*(E \cap G) - \varepsilon + \nu^*(E - G)$ by definition.

Hence $\nu(H) \geqq \nu^*(E \cap G) + \nu^*(E - G)$. By definition, then, $\nu^*(E) \geqq \nu^*(E \cap G) + \nu^*(E - G)$, and Lemma 14 is proved.

Because of Lemma 14 we know that ν^* defines a measure on a σ-algebra of sets that includes the Borel sets of M.

Proof of Theorem 3. First suppose that μ is nonnegative. Let G be open. By Lemma 7, $\mu(G) \leqq \nu(G)$. On the other hand, for any closed subset F of G, $\mu(F) \leqq \mu(F) + \mu(G - F) = \mu(G)$. Thus $\mu(G) = \nu(G)$. ν^* is a measure on the Borel sets of M which agrees with μ on open sets and hence on all sets in the domain of μ.

Now let μ be arbitrary. Consider the set function $\omega = \nu^* - \mu$, defined for closed subsets of M and for open subsets of M. ω is nonnegative by Lemma 7. By what has already been proved, ω can be extended to a Borel measure. But then $\mu = \nu^* - \omega$ can be extended also, so the theorem is proved.

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