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CATEGORY THEORY APPLIED TO PONTRYAGIN DUALITY

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A proof of the Pontryagin duality theorem for locally compact abelian (LCA) groups is given, using category-theoretical ideas and homological methods. The proof is guided by the structure within the category of LCA groups and does not use any deep results except for the Peter-Weyl theorem. The duality is first established for the subcategory of elementary LCA groups (those isomorphic with $T^i \oplus Z^j \oplus R^k \oplus F$, where T is the circle group, Z the integers, R the real numbers, and F a finite abelian group), and through the study of exact sequences, direct limits and projective limits the duality is expanded to larger subcategories until the full duality theorem is reached.

Introduction. In this note we present a fairly economical proof of the Pontryagin duality theorem for locally compact abelian (LCA) groups, using category-theoretic ideas and homological methods. This theorem was first proved in a series of papers by Pontryagin and van Kampen, culminating in van Kampen's paper [5], with methods due primarily to Pontryagin. In [10, pp. 102-109], Weil introduced the simplifying notion of compactly generated group and explored the functorial nature of the situation by examining adjoint homomorphisms and projective limits. Proofs along the lines of Pontryagin-van Kampen-Weil appear in the books by Pontryagin [7, pp. 235-279] and Hewitt and Ross [2, pp. 376-380]. A different proof based on abstract Fourier analysis was given by Cartan and Godemont [1]; similar methods are also used by Rudin [9, pp. 27-29] and Heyer [3, pp. 148-161]. Negrepontis [6, pp. 239-252] presented the theorem in light of category theory, but for the most part used different methods and more structure theory than is used here.

The proof we present is based on ideas used previously by Hofmann [4, pp. 109-117] and the author [8]. It rests neither on the structure theorem for compactly generated LCA groups (see [2], [6], [7]) nor on the structure of the L^1-algebra of G (see [6] and [9]). The only deep result we require is the classical Peter-Weyl theorem, applied in our case to compact abelian groups, of course.

Definitions and preliminaries. L is the category of LCA groups, with continuous homomorphisms as morphisms. Groups will be written additively. Pointwise addition of functions makes each Hom (G, H) an abelian group and L an additive category. R denotes the additive real numbers with the usual topology, Z is the subgroup of R consisting of the integers with the induced (discrete) topology, T is the compact quotient group R/Z, and Z_n is Z/nZ. If $G \in L$, then the character group of G is $\chi(G) = G^* = \mathrm{Hom}(G, T)$, taken with the compact-open topology, which makes G^* an LCA group. For $f \in \mathrm{Hom}(G, H)$, the adjoint homomorphism $\chi(f) = f^* \in \mathrm{Hom}(H^*, G^*)$ is defined by $f^*(\alpha) = \alpha \circ f$ for $\alpha \in H^*$. Thus χ is a contravariant functor from L to L. There is a natural transformation ρ from the identity functor η in L to the covariant functor $\chi^2 = \chi \circ \chi$; we specify $\rho_G \colon G \to G^{**}$ by $[\rho_G(x)](\alpha) = \alpha(x)$ for $x \in G$, $\alpha \in G^*$. The Pontryagin duality theorem is now the statement that ρ is a natural isomorphism; i.e., that each ρ_G is an isomorphism between G and its second character group G^{**}. The method of proof will be to show that ρ restricted to a suitable subcategory is an isomorphism, and then enlarge the subcategory in successive steps until we have the theorem proved. We indicate this schematically in the following diagram, where each name stands for the full subcategory of L with the named property, and the labels on connecting lines indicate the methods used to extend the duality to the larger category.

$$
\begin{tikzcd}
 & & L \arrow[ddll, dash, "\text{exactness}"'] \arrow[dr, dash, "\text{exactness}"] & & \\
 & & & \text{COMPACTLY GENERATED} \arrow[ddl, dash, "\text{exactness}"'] \arrow[dr, dash, "\text{exactness}"] & \\
\text{DISCRETE} \arrow[dd, dash, "\overset{\text{lim}}{\longrightarrow}"'] & & & & \text{COMPACT} \arrow[dd, dash, "\overset{\text{lim}}{\longleftarrow}"] \\
 & & \text{ELEMENTARY} \arrow[dll, dash] \arrow[drr, dash] \arrow[dd, dash, "\oplus"] & & \\
\text{DISCRETE ELEMENTARY} & & & & \text{COMPACT ELEMENTRY} \\
 & & T,\ Z,\ R,\ Z_n & &
\end{tikzcd}
$$

First, $T \simeq Z^*$ with $x \in T$ corresponding to the character $n \mapsto nx$ of Z. Conversely, $Z \simeq T^*$ with $n \in Z$ corresponding to the character $x \mapsto nx$ of T. Also $R \simeq R^*$ with $x \in R$ corresponding to the character $y \mapsto xy + Z$ of R. And $Z_n \simeq Z_n^*$ by corresponding $x \in Z_n$ to the character $y \mapsto xy$ of Z_n. Thus the relation $G \simeq G^{**}$ holds in a natural way for $G = T, Z, R$, or Z_n, respectively, and in each case the isomorphism obtained is precisely the map ρ_G defined earlier.

It is easy to show that χ is additive (that is, for $f, g \in \mathrm{Hom}(G, H)$ we have $\chi(f + g) = \chi(f) + \chi(g)$); as a consequence χ preserves direct sums. (See Lemma 2, p. 162, in [8].) From this we see that Pontryagin duality holds at least for the elementary groups in L;

that is, groups isomorphic to $T^i \oplus Z^j \oplus R^k \oplus F$, where F is a finite abelian group. The full subcategory of elementary groups in L will be denoted by E. Then we have the following:

THEOREM 1. *$\rho \mid E: \eta \mid E \rightarrow \chi^2 \mid E$ is an isomorphism.*

Duality for compact groups in L. We define a morphism $f \in \operatorname{Hom}(G, H)$ to be *proper* if f is open considered as a function from G to its image in H. (Thus a "proper morphism" as the term is used here may not be a "proper map" as the term is used in general topology.)

PROPOSITION 1. *If $f: G \rightarrow H$ is proper and $f(G)$ is open in H, then f^* is proper.*

Proof. Let M be a compact neighborhood of 0 in H and W a closed neighborhood of 0 in T which contains no proper subgroups of T. (M, W) denotes the set of all characters α in H^* for which $\alpha(M) \subset W$. The set of (M, W) obtained from all possible such choices of M and W forms a base of compact neighborhoods of 0 in H^*. Suppose now that M is chosen to be a compact neighborhood of 0 in $f(G)$. Since f is proper and G is locally compact, we may find a compact neighborhood N of 0 in G such that $f(N) = M$. Then (N, W) is a neighborhood of 0 in G^* and $f^*(M, W) = (N, W) \cap f^*(H^*)$ is a compact neighborhood of 0 in $f^*(H^*)$. The properness of f^* follows from the open mapping theorem [2, Theorem 5.29, p. 42] applied to the open subgroup of H^* generated by (M, W).

PROPOSITION 2. *χ takes a short proper exact sequence $0 \rightarrow K \xrightarrow{i} G \xrightarrow{j} H \rightarrow 0$ to a sequence $0 \leftarrow K^* \xleftarrow{i^*} G^* \xleftarrow{j^*} H^* \leftarrow 0$ in which j^* is proper and exactness holds at G^* and H^*. If in addition K is an open subgroup of G, then the sequence induced by χ is also proper exact.*

Proof. It is now easy to see that H^* is isomorphic to the closed subgroup of G^* consisting of those characters of G which are trivial on K, via the proper injection j^*. The kernel of i^* is also the set of characters on G which are trivial on K. This proves exactness at G^* and H^*. Now, because T is a divisible group, any character on H extends to a homomorphism (not necessarily continuous) from G to T. If H is an open subgroup, then any such extension will be continuous on G, so the proper map i^* will be a surjection.

Using the (M, W) notation of the proof of Proposition 1, we see that if G is discrete and $M = \{0\}$ in G, then $(M, W) = G^*$, so G^* is compact. On the other hand, if G is compact and $M = G$, then $(M, W) = \{0\}$, so G^* is discrete. We now let A be the subcategory of discrete groups in L and A_0 the subcategory of discrete elementary groups (isomorphic to $Z^j \oplus F$). C is the subcategory of compact groups and C_0 the subcategory of compact elementary groups (isomorphic to $T^i \oplus F$) in L.

We can make a directed set I into a category by declaring, for $i, j \in I$, that $\mathrm{Hom}(i, j)$ consist of exactly one element if $i \leqq j$ and be empty otherwise. Then, for our purposes, a direct system in A_0 is a covariant functor U from a directed set to A_0. We shall often write U_i for $U(i)$ and u_{ij} for $U(\mathrm{Hom}(i, j))$. We let DA_0 be the collection of direct systems in A_0 whose morphisms are all injective. DA_0 becomes a category when we define a morphism from $U: I \to A_0$ to $V: J \to A_0$ to be a pair (m, φ), where $m: I \to J$ is a functor (order-preserving map) and φ is a natural transformation from U to $V \circ m$.

Well-known properties of abelian groups include the fact that each element U of DA_0 has a direct limit $\varinjlim U$(or $\varinjlim U_i$) which will be an object in D. In fact, a necessary and sufficient condition for a discrete group G to be isomorphic to $\varinjlim U$ is the existence of injective morphisms $g_i: U_i \to G$, one for each i, such that $g_j \circ u_{ij} = g_i$ whenever $i \leqq j$, and the union of the images $g_i(U_i)$ is all of G. The result is that $\varinjlim: DA_0 \to A$ can be regarded as a covariant functor, since if (m, φ) is a functor from U to V in DA_0, the universal property of $\varinjlim U$ guarantees existence of a unique morphism $\varinjlim (m, \varphi): \varinjlim U \to \varinjlim V$ making the following diagram commutative for every $i \in I$:

$$\begin{array}{ccc} U_i & \longrightarrow & \varinjlim U \\ \Big\downarrow{\scriptstyle \varphi_i} & & \Big\downarrow{\scriptstyle \varinjlim (m,\varphi)} \\ V_{m(i)} & \longrightarrow & \varinjlim V . \end{array}$$

Similarly, an inverse system in C_0 is a contravariant functor U from a directed set to C_0. The category of all inverse systems in C_0 all of whose morphisms are surjective is denoted by IC_0. A morphism from $U: I \to C_0$ to $V: J \to C_0$ in IC_0 is a pair (m, φ) where $m: J \to I$ is a functor and $\varphi: U \circ m \to V$ is a natural transformation. Also, any inverse system in IC_0 has a projective limit $\varprojlim U$ which will be an object of C. Further, any object G of C is isomorphic to $\varprojlim U$ if and only if there exists a surjective morphism $g_i: G \to U_i$ for each i such that $u_{ij} \circ g_j = g_i$ whenever $i \leqq j$ and the intersection of the kernels of the g_i is $\{0\}$ in G. (The usual condition [10, p. 25] is that every neighborhood of 0 in G contain $\mathrm{Ker}\, g_i$ for some i, but if the intersection of the $\operatorname{Ker} g_i$ is $\{0\}$ and $N^{\cdot}$ is any open neighborhood of 0 in G, then by compactness we can find a finite number of g_i the intersection of whose kernels is contained in N. Picking an index j greater than these i gives us $\operatorname{Ker} g_j \subset N$.) Again in this situation $\varprojlim(m, \varphi)$ for a morphism (m, φ) in IC_0 is defined by the universal property, and $\varprojlim$ becomes a covariant functor. We call DA_0 and IC_0 convergence structures on A and C, respectively.

By Proposition 2, if $U \in DA_0$ then $\chi \circ U \in IC_0$. This correspondene $U \mapsto \chi \circ U$ gives us a functor which we denote by $D\chi\colon DA_0 \to IC_0$. Similarly, χ induces the functor $I\chi\colon IC_0 \to DA_0$.

PROPOSITION 3. *χ is continuous on A with respect to DA_0; that is, the two functors $\chi \circ \varinjlim$ and $\varprojlim \circ D\chi\colon DA_0 \to C$ are naturally isomorphic.*

Proof. Let $U \in DA_0$ and $G = \varinjlim U_i$ with $g_i\colon U_i \to G$ the associated injections. We must show that $G^* \simeq \varprojlim(U_i^*)$. It is clear that each $g_i^*\colon G^* \to U_i^*$ is surjective. Let $0 \neq \alpha \in G^*$. We shall show the existence of an index i with $g_i^*(\alpha) \neq 0$. We know $\alpha(x) \neq 0$ for some $x \in G$, and then $x = g_i(y)$ for some i and $y \in G_i$. Then for this i, $g_i^*(\alpha)(y) = \alpha(g_i(y)) \neq 0$. So $(\varinjlim U_i)^* \simeq \varprojlim(U_i^*)$. The fact that we have a natural isomorphism follows from the universal property.

PROPOSITION 4. *χ is continuous on C with respect to IC_0; that is, the functors $\chi \circ \varprojlim$ and $\varinjlim \circ I\chi\colon IC_0 \to A$ are naturally isomorphic.*

Proof. Let $U \in IC_0$ and $G = \varprojlim U_i$ with $g_i\colon G \to U_i$ the associated surjections. Clearly each g_i^* is injective. To show that G^* is isomorphic with $\varinjlim(U_i^*)$, we must show that every $\alpha \in G^*$ is equal to $g_i^*(\beta)$ for some i and some $\beta \in U_i^*$. Let W be a neighborhood of 0 in T containing no proper subgroups of T. Let M be a neighborhood of 0 in G with $\alpha(M) \subset W$. Then we may find U_i with $\operatorname{Ker} g_i \subset M$. Then $\alpha(\operatorname{Ker} g_i) = 0$, so α factors; $\alpha = \beta \circ g_i$ for some $\beta \in U_i^*$. But $\beta \circ g_i = g_i^*(\beta)$, so we are done.

We remark that Proposition 3 above could be proved as in [6] by showing that χ has a left adjoint functor, which implies that χ takes direct limits to limits. However, Proposition 4 does not admit an analogous proof.

PROPOSITION 5. *The category C_0 is dense in C. That is, there is a functor $S\colon C \to IC_0$ such that the functor $\varprojlim \circ S$ and the identity functor on C are naturally isomorphic.*

Proof. This follows from the Peter-Weyl theorem [7, Theorem 33, p. 229], which says for our abelian case that the characters of any G in C separate the points of G. Let $G \in C$ and define $S(G) \in IC_0$ to be the collection of quotient groups G/K of G which are in C_0. We order them by $G/K \leqq G/N$ if $N \subset K$. Note that $G/(K \cap N)$ is isomorphic to a subgroup of $(G/K) \oplus (G/N)$, so we have a directed set. We can define $g_{KN}: G/N \to G/K$ to be the natural projection when $N \subset K$. For a morphism $f: G \to H$ in C, $S(f)$ is defined as follows: H/K in $S(H)$ corresponds to $G/f^{-1}(K)$ in $S(G)$, and the map $G/f^{-1}(K) \to H/K$ is the natural one induced by f. Then if $x \in G$, let $\alpha \in G^*$ with $\alpha(x) \neq 0$. Then G/K is in $S(G)$, where $K = \operatorname{Ker} \alpha$, and $g_K(x) \neq 0$, where $g_K: G \to G/K$ is the natural map. The collection of g_K exhibits G as $\varprojlim S(G)$.

We are now ready to consider our two covariant functors η and χ^2 again.

THEOREM 2. *Pontryagin duality holds in C; that is, $\rho \mid C: \eta \mid C \to \chi^2 \mid C$ is an isomorphism.*

Proof. $\rho \mid C_0$ is already an isomorphism. By Proposition 1.18 in [4, p. 115], $\rho \mid C_0$ extends uniquely to a natural transformation between $\eta \mid C$ and $\chi^2 \mid C$. This extension must also be an isomorphism. But $\rho \mid C$ already extends $\rho \mid C_0$ so $\rho \mid C$ must be an isomorphism.

Duality of compactly generated groups in L. Let CG be the full subcategory of the compactly generated groups in L.

LEMMA. *Suppose $G \in CG$ is generated by the compact neighborhood M of 0 in G. Then there is a subgroup K of G, $K \simeq Z^n$ for some n, such that $K \cap M = \{0\}$ and G/K is compact.*

Proof. This is Lemma 2.42 in [9, p. 41].

PROPOSITION 6. *If $G \in CG$, then ρ_G is injective.*

Proof. This is an easy consequence of the above lemma. Let $x \in G$, $x \neq 0$. Apply the lemma to $M \cup \{x\}$, which is also a compact neighborhood of 0 which generates G. The coset $x + K$ is not the identity element in the compact group G/K. Therefore, there is a character α of G/K such that $\alpha(x + K) \neq 0$. Composing α with the natural projection $G \to G/K$ gives us a character on G which is not trivial on x. Therefore $\rho_G(x) \neq 0$.

THEOREM 3. *Pontryagin duality holds in CG.*

Proof. Let $G \in CG$. Let M be a compact neighborhood of 0 in G and let $S \simeq Z^n$ be a subgroup of G such that $S \cap M = \{0\}$ and G/S is compact, as guaranteed by the lemma above. Let $Q = G/S$ and $p: G \to Q$ the natural map. Let N be a compact symmetric neighborhood of 0 in G such that $N + N + N \subset M$. Then p maps N homeomorphically onto $p(N)$. Since Q is compact and p is proper, there is a compact subgroup Q_1 of$Q$ such that $Q_1 \subset p(N)$ and Q/Q_1 is compact elementary, by Proposition 5. Then $p^{-1}(Q_1)$ is a closed subgroup of G contained in $S + N$. Letting $K = p^{-1}(Q_1) \cap N$, we have by the choice of N that K is a compact subgroup of G satisfying $p(K) = Q_1$.

Now let $H = G/K$; we shall show that H is an elementary group. First, p gives rise to proper surjection $H \to Q/Q_1$ with kernel $S + K$ which is discrete in H by the construction of K. Therefore H is locally isomorphic with Q/Q_1, wich in turn is locally isomorphic with R^n for some n. This means we have an isomorphism $f: B \to V$, where B is an open ball about 0 in R^n and V is a neighborhood of 0 in H. Then we can extend f to a proper surjective homorphism $g: R^n \to H_1$, where H_1 is the open subgroup of H generated by V, by defining $g(x) = n \cdot f(x/n)$, for $x \in R^n$ and n large enough so that $x/n \in B$. Thus $H_1 \simeq R^a \oplus T^b$ for some integers a and b (a quotient group of R^n). Since H_1 is a divisible open subgroup of H, we can obtain a morphism $H \to H_1$ which is the identity on H_1, so $H \simeq H_1 \oplus H/H_1$. But H/H_1 is an elementary group since it is discrete and compactly generated. Therefore H is also elementary. Now we examine the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & G & \xrightarrow{j} & H & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \rho_K} & & \downarrow{\scriptstyle \rho_G} & & \downarrow{\scriptstyle \rho_H} & & \downarrow \\ 0 & \longrightarrow & K^{**} & \xrightarrow{i^{**}} & G^{**} & \xrightarrow{j^{**}} & H^{**} & \longrightarrow & 0 \end{array} \tag{1}$$

Now $K \in C$ and $H \in E$, so ρ_K and ρ_H are isomorphisms, while ρ_G is injective by Proposition 6. Since i is injective, we conclude that i^{**} is injective. Now briefly consider $i^*: G^* \to K^*$. If i^* were not surjective, then there would be a nontrivial character on $K^*/i^*(G^*)$. Composing with the natural projection $K^* \to K^*/i^*(G^*)$, we get a character on K^* which is trivial on $i^*(G^*)$. This character would then be in the kernel of i^{**}, contradicting the injectivity of i^{**}. Therefore, i^* is surjective and Proposition 2 now tells us that the induced sequence

$$0 \longleftarrow K^* \xleftarrow{i^*} G^* \xleftarrow{j^*} H^* \longleftarrow 0$$

is a proper exact sequence, since K^* is discrete. Thus H^* can be regarded as an open subgroup of G^*, and so the lower sequence in diagram (1) is also proper exact. The 5-lemma and the open mapping theorem show that ρ_G is an isomorphism, and we are done.

Duality for arbitrary locally compact groups. We begin with discrete groups.

PROPOSITION 7. *The category A_0 is dense in A. That is, there is a functor $T\colon A \to DA_0$ such that $\varinjlim \circ T$ and the identity functor on A are naturally isomorphic.*

Proof. Every abelian group is the direct limit of its finitely generated subgroups. The functor T assigns to each G in A the direct system (U_i) of finitely generated subgroups of G ordered by $i \leqq j$ if $U_i \subset U_j$. A morphism $f\colon G \to H$ in A is carried by T to $T(f)$, in which each finitely generated subgroup of G is mapped via the restriction of f to its image in H.

THEOREM 4. *Pontryagin duality holds in A.*

Proof. This is completely analogous to the proof of Theorem 2.

THEOREM 5. *The Pontryagin duality theorem. ρ is a natural epuivalence.*

Proof. Let $G \in L$. Let M be a compact neighborhood of 0 in G and let K be the subgroup of G generated by M. Let $H = G/K$. Since K is open, the induced sequence

$$0 \longleftarrow K^* \longleftarrow G^* \longleftarrow H^* \longleftarrow 0$$

is proper exact by Proposition 2. Consider diagram (1) for this K, G, and H. We have exactness at K^{**} and G^{**} in the bottom row. H is discrete since K is open, so ρ_H is an isomorphism, and so j^{**} is surjective. Both rows of the diagram are proper exact and ρ_K is also an isomorphism. Again the 5-lemma applies and ρ_G is algebraically an isomorphism. But ρ_G restricted to the open subgroup K is the isomorphism ρ_K, so ρ_G is also an isomorphism.

Had we extended the concepts of convergence to include direct and projective limits of nondiscrete and noncompact groups, respectively, and had we modified Propositions 3 and 4 and their proofs accordingly, we could have proved Theorem 5 differently by showing that CG is dense in L and χ is continuous on L with respect to the modifind convergence. Theorem 3 may be treated similarly; in fact, the proof of Theorem 3 already shows that E is dense in CG.

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