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THE NORM OF A CERTAIN DERIVATION

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J. C. Stampfli has asked whether the norm of the derivation $\mathfrak{D}_T: A \to TA - AT$ as a mapping of the subalgebra $\mathfrak{A}$ of $\mathfrak{B}(H)$ into $\mathfrak{B}(H)$ is given by $\inf\{2\|T - A'\|: A' \in \mathfrak{A}'\}$. That this need not be the case is shown through an example in 4×4 matrices.

H is a Hilbert space. $\mathfrak{B}(H)$ is the algebra of all bounded linear operators on H. $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}(H)$ and $\mathfrak{A}'$ is the commutant of $\mathfrak{A}$.

In [6], J. C. Stampfli proved that the norm of $\mathfrak{D}_T$ as a mapping of $\mathfrak{B}(H)$ into itself is precisely $2\ \inf_\lambda \|T - \lambda\|$. Thus the question about $\|\mathfrak{D}_T\|$ as a mapping from $\mathfrak{A}$ to $\mathfrak{B}(H)$ naturally arises. In addition, Kadison, Lance, and Ringrose [2, Theorem 3.1] show that if $T = T^*$ and $\mathfrak{D}_T$ maps $\mathfrak{A}$ into itself, then $\|\mathfrak{D}_T\| = \inf\{2\|T - A'\|: A' \in \mathfrak{A}'\}$. Our example will have T self-adjoint, which shows that their hypothesis $\mathfrak{D}_T(\mathfrak{A}) \subset \mathfrak{A}$ is not inessential.

For our example, we take H to be complex four-dimensional Hilbert spaces; elements of H are to be thought of as column 4-vectors, and elements of $\mathfrak{B}(H)$ as 4×4 matrices. We take $\mathfrak{A}$ to be the subalgebra of diagonal matrices, so $\mathfrak{A}' = \mathfrak{A}$.

For T we take the Hermitian matrix

$$T = \frac{1}{12}\begin{pmatrix} 1 & -4 & \frac{1}{\sqrt{14}}(-5+6i\sqrt{3}) & \frac{1}{\sqrt{14}}(-5-6i\sqrt{3}) \\ -4 & 4 & -2\sqrt{14} & -2\sqrt{14} \\ \frac{1}{\sqrt{14}}(-5-6i\sqrt{3}) & -2\sqrt{14} & \frac{7}{2} & \frac{1}{14}(-95+12i\sqrt{3}) \\ \frac{1}{\sqrt{14}}(-5+6i\sqrt{3}) & -2\sqrt{14} & \frac{1}{14}(-95-12i\sqrt{3}) & \frac{7}{2} \end{pmatrix}$$

T is of the form $P - Q$ where P and Q are self-adjoint projections. The range of P is two-dimensional and is spanned by the orthogonal unit vectors

$$p^{(1)} = \frac{1}{2\sqrt{3}}\left(1,\ -1 + i\sqrt{3},\ \frac{1}{\sqrt{14}}(4 - 5i\sqrt{3}),\ \frac{1}{\sqrt{14}}(-2 + i\sqrt{3})\right),$$

$$p^{(2)} = \frac{1}{2\sqrt{3}}\left(1,\ -1 - i\sqrt{3},\ \frac{1}{\sqrt{14}}(-2 - i\sqrt{3}),\ \frac{1}{\sqrt{14}}(4 + 5i\sqrt{3})\right);$$

the range of Q is one-dimensional and is spanned by the unit vector

$$q = \frac{1}{2\sqrt{3}}\left(1, 2, \sqrt{\frac{7}{2}}, \sqrt{\frac{7}{2}}\right).$$

First we show that $\|\mathfrak{D}_T\| = \sup\{\|TX - XT\|: X \in \mathfrak{A}, \|X\| = 1\} < 2$. As the unit sphere of $\mathfrak{A}$ is the convex hull of the unitary matrices in $\mathfrak{A}$, it suffices to consider $\|TX - XT\|$ only for diagonal unitary matrices X. As T has norm 1 and any X has norm 1, $\|TX - XT\| \leqq 2$. Suppose then that there were an X for which $\|TX - XT\| = 2$ (since the set of X under consideration is compact, the supremum defining $\mathfrak{D}_T$ is attained). Then there must be a unit vector $u \in H$ for which $\|(TX - XT)u\| = 2$, and since TX and XT are both of norm 1, we must have $\|TXu\| = 1 = \|XTu\|$; and since the norm of H is strictly convex, we must have $TXu = -XTu$. Further, since $\|Tu\| = 1$, we must have $u = Pu + Qu$. The next two relations are consequences of $TXu = -XTu$; start in the middle and work towards either end.

$$\begin{aligned} PXPu + PXQu &= PX(P + Q)u = PXu \\ &= P(P - Q)Xu = PTXu = -PXTu \\ &= -PXPu + PXQu\,, \end{aligned}$$

so $PXPu = 0$;

$$\begin{aligned} -QXPu - QXQu &= -QXu \\ &= QTXu = -QXTu \\ &= -QXPu + QXQu\,; \end{aligned}$$

so $QXQu = 0$.

Next we observe that XPu is in the range of Q and XQu is in the range of P; for if one of these were not the case, we should have the strict inequality below:

$$\begin{aligned} 1 &= \|u\|^2 = \|Qu\|^2 + \|Pu\|^2 = \|XQu\|^2 + \|XPu\|^2 \\ &> \|PXQu\|^2 + \|QXPu\|^2 \\ &= \|PX(P + Q)u\|^2 + \|QX(P + Q)u\|^2 \\ &= \|PXu\|^2 + \|QXu\|^2 = \|TXu\|^2\,. \end{aligned}$$

But if $\|\mathfrak{D}_T\|$ is to be 2, we cannot allow $\|TXu\| < 1$.

Since $\|XPu\|^2 + \|XQu\|^2 = 1$, not both of XPu and XQu may be zero. Observe that operation on a vector by the diagonal unitary X does not change the absolute value of any component. If $XPu \neq 0$, then XPu is in the range of Q and the conclusion we draw is that there must be a nonzero vector in the range of p with moduli of components the same as that of q. If $XPu = 0$, then $XQu \neq 0$ and XQu is in the range of P; we draw the same conclusion.

To finally reach the desired contradiction to the assumption $||\mathfrak{Q}_T|| = 2$, we need only show that no vector in the range of p has components of the same modulus as q. Indeed, if there were, such a vector must be of the form $p = e^{i\beta}(\cos\Theta p^{(1)} + e^{i\Phi}\sin\Theta p^{(2)})$ for some real β, Θ, Φ. Equating the squares of the moduli of the first two components yields

$$\begin{aligned} 1 &= |\cos\Theta + e^{i\Phi}\sin\Theta|^2 = 1 + 2\cos\Theta\sin\Theta\cos\Phi , \\ 4 &= |\cos\Theta(-1 + i\sqrt{3}) + e^{i\Phi}\sin\Theta(-1 - i\sqrt{3})|^2 \\ &= 4 + 8\cos\Theta\sin\Theta\cos\left(\Phi + \frac{2\pi}{3}\right) . \end{aligned}$$

Thus $\cos\Theta\sin\Theta = 0$ and p must be a multiple of $p^{(1)}$ or $p^{(2)}$; but neither of these has the moduli of their last two components the same as q.

Having demonstrated that $||\mathfrak{Q}_T|| < 2$, we show now that $||T - A'|| \geqq 1$ for every $A' \in \mathfrak{A}'$. As $||T|| = 1$, this is equivalent to showing that $||T - D|| \geqq 1$ for every diagonal matrix D. Suppose, then, that there were a diagonal matrix D-with diagonal entries d_1, d_2, d_3, d_4-for which $||T - D|| < 1$. We may assume D real, since $||T - \operatorname{Re} D|| = ||\operatorname{Re}(T - D)|| \leqq ||T - D|| < 1$, where by $\operatorname{Re} A$ we mean $1/2(A + A^*)$.

Let p be any unit vector in the range of p, and q as before. Consider the inner product

$$((T - D)(\cos\Theta p + e^{i\Phi}\sin\Theta q), (\cos\Theta p - e^{i\Phi}\sin\Theta q)) .$$

This is equal to

$$1 - (D(\cos\Theta p + e^{i\Phi}\sin\Theta q), (\cos\Theta p - e^{i\Phi}\sin\Theta q)) ,$$

but has absolute value less than 1. Hence

$$\operatorname{Re}(D(\cos\Theta p + e^{i\Phi}\sin\Theta q), (\cos\Theta p - e^{i\Phi}\sin\Theta q)) > 0 ,$$

for any choice of p, Θ, and Φ. The choices $\Theta = 0$, $p = p^{(1)}$, and $p = p^{(2)}$ give

$$\left(d_1 + 4d_2 + \frac{13}{2}d_3 + \frac{1}{2}d_4\right) > 0 ,$$

$$\left(d_1 + 4d_2 + \frac{1}{2}d_3 + \frac{13}{2}d_4\right) > 0 ,$$

and hence

$$\left(d_1 + 4d_2 + \frac{7}{2}d_3 + \frac{7}{2}d_4\right) > 0 .$$

But the choice $\Theta = \pi/2$ gives

$$-\left(d_1 + 4d_2 + \frac{7}{2}d_3 + \frac{7}{2}d_4\right) > 0 \ .$$

This incompatibility is a contradiction to $\|T - D\| < 1$.

The reader will observe the similarity with Example 5.5 of [3]. In spirit, we have the logarithmic analogue of the problem of conditioning matrices. One can ascertain conditions that $\|T - A'\| \geqq \|T\|$ for all $A' \in \mathfrak{A}'$ by consideration of the norms $\|T\|_p = [\text{trace}(T^* T)^{p/2}]^{1/p}$ as $p \longrightarrow \infty$, as in [4, Lemma 4.7, Theorem 4.8] or, more generally, [5, § 6]. For T self-adjoint, the relevant condition to have $\|T - A'\| \geqq \|T\|$ for all diagonal A' is that both numbers $-\|T\|$ and $\|T\|$ are eigenvalues of T and that the spectral projections associated with these eigenvalues have proportional diagonals. Conditions involving suprema of norms over the group of diagonal unitaries are related to the moduli of components of certain vectors; see [4, Theorem 5.4], as well as [1, 2]. Finally, we note that for the analogous problem of conditioning matrices, examples such as we have constructed are not available in 3×3 matrices, nor with 4×4 real matrices.

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