CONNECTEDNESS IM KLEINEN AND LOCAL CONNECTEDNESS IN $2^X$ AND $C(X)$

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Let $X$ be a compact connected metric space and $2^X(C(X))$ denote the hyperspace of closed subsets (subcontinua) of $X$. In this paper the hyperspaces are investigated with respect to point-wise connectivity properties. Let $M \in C(X)$. Then $2^X$ is locally connected (connected im kleinen) at $M$ if and only if for each open set $U$ containing $M$ there is a connected open set $V$ such that $M \subseteq V \subseteq U$ (there is a component of $U$ which contains $M$ in its interior). This theorem is used to prove the following main result. Let $A \in 2^X$. Then $2^X$ is locally connected (connected im kleinen) at $A$ if and only if $2^X$ is locally connected (connected im kleinen) at each component of $A$. Several related results about $C(X)$ are also obtained.

A continuum $X$ will be a compact connected metric space. $2^X(C(X))$ denotes the hyperspace of closed subsets (subcontinua) of $X$, each with the finite (Vietoris) topology, and since $X$ is a continuum, each of $2^X$ and $C(X)$ is also a continuum (see [5]).

One of the earliest results about hyperspaces of continua, due to Wojdyslawski [7], was that each of $2^X$ and $C(X)$ is locally connected if and only if $X$ is locally connected. As a point-wise property, local connectedness is stronger than connectedness im kleinen, which in turn is stronger than aposyndesis. The author [1] has shown that if $X$ is any continuum, then each of $2^X$ and $C(X)$ is aposyndetic. It is the purpose of this paper to investigate the internal structure of $2^X$ and $C(X)$ with respect to these properties. In particular, we determine necessary and sufficient conditions (in terms of the neighborhood structure in $X$) that $2^X$ be locally connected at a point and that $2^X$ be connected im kleinen at a point. We also determine that $C(X)$ has, in general, stronger point-wise connectivity properties that either $2^X$ or $X$.

For notational purposes, small letters will denote elements of $X$, capital letters will denote subsets of $X$ and elements of $2^X$, and script letters will denote subsets of $2^X$. If $A \subseteq X$, then $A^\ast$ (int $A$) (bd $A$) will denote the closure (interior) (boundary) of $A$ in $X$.

Let $x \in X$. Then $X$ is locally connected (l.c.) at $x$ if for each open set $U$ containing $x$ there is a connected open set $V$ such that $x \in V \subseteq U$. $X$ is connected im kleinen (c.i.k.) at $x$ if for each open set $U$ containing $x$ there is a component of $U$ which contains $x$ in its interior. $X$ is aposyndetic at $x$ if for each $y \in X - x$ there is a
continuum $M$ such that $x \in \text{int} M$ and $y \in X - M$.

If $A_1, \ldots, A_n$ are subsets of $X$, then $N(A_1, \ldots, A_n) = \{B \subseteq 2^X \mid \text{for each } i = 1, \ldots, n, B \cap A_i \neq \emptyset, \text{ and } B \subseteq \bigcup_{i=1}^n A_i\}$. The collection of all sets of the form $N(U_1, \ldots, U_n)$, with $U_1, \ldots, U_n$ open in $X$, is a base for the finite topology. It is easy to establish that

$$N(U_1, \ldots, U_n)^* = N(U_1^*, \ldots, U_n^*)$$

and that $N(V_1, \ldots, V_m) \subseteq N(U_1, \ldots, U_n)$ if and only if $\bigcup_{i=1}^m V_i \subseteq \bigcup_{i=1}^n U_i$, and for each $U_i$ there exists a $V_j$ such that $V_j \subseteq U_i$ (see [5]). We remark also that the finite topology is equivalent to the Hausdorff metric topology on $2^X$ whenever $X$ is a compact metric space (theorem on page 47 of [4]).

If $\mathcal{A} \subseteq 2^X$, then $\bigcup \{A \mid A \in \mathcal{A}\}$ is open (closed) in $X$ whenever $\mathcal{A}$ is open (closed) in $2^X$ (see [5]). Furthermore, if $\mathcal{A} \cap C(X) \neq \emptyset$ and $\mathcal{A}$ is connected, then $\bigcup \{A \mid A \in \mathcal{A}\}$ is connected (Lemma 1.2 of [3]).

If $n$ is a positive integer, then $F_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ elements}\}$ and $F(X) = \bigcup_{n=1}^\infty F_n(X)$.

An order arc in $2^X(C(X))$ is an arc which is also a chain with respect to the partial order on $2^X(C(X))$ induced by set inclusion. If $A, B \in 2^X$, then there exists an order arc from $A$ to $B$ if and only if $A \subseteq B$ and each component of $B$ meets $A$ (Lemma 2.3 of [3]). It follows (Lemma 2.6 of [3]) that every order arc whose initial point is an element of $C(X)$ is entirely contained within $C(X)$.

It will be convenient to begin our study by considering points of $C(X)$.

**Theorem 1.** Let $M \in C(X)$. Then $2^X$ is c.i.k. at $M$ if and only if for each open set $U$ containing $M$ there is a component of $U$ which contains $M$ in its interior.

**Proof.** Suppose $2^X$ is c.i.k. at $M$. Let $U$ be an open set containing $M$. Then $M \in N(U)$, so there exists a component $C$ of $N(U)$ containing $M$ in its interior. It follows that $\bigcup \{A \mid A \in C\}$ is a connected set containing $M$ in its interior and lying in $U$.

Conversely, suppose that for each open set $U$ containing $M$ there is a component of $U$ which contains $M$ in its interior. Let $N(U_1, \ldots, U_n)$ be a basic open set containing $M$ and let $N(V_1, \ldots, V_m)$ be a basic open set such that $M \in N(V_1, \ldots, V_m) \subseteq N(U_1, \ldots, U_n)^* \subseteq N(U_1, \ldots, U_n)$. Let $V = \bigcup_{i=1}^m V_i$. Then there is a component $C$ of $V$ which contains $M$ in its interior. For each $i = 1, \ldots, m$, let $W_i = V_i \cap \text{int} C$. Then $M \in N(W_1, \ldots, W_m) \subseteq N(V_1, \ldots, V_m)$. If $A \in N(W_1, \ldots, W_m)$, then $A \subseteq C^*$, and $A, C^* \in N(V_1^*, \ldots, V_m^*) =$
$N(V_1, \cdots, V_m)^* \subset N(U_1, \cdots, U_n)$. Since $C^*$ is connected there exists an order arc in $N(U_1, \cdots, U_n)$ from $A$ to $C^*$. It follows that there is a component of $N(U_1, \cdots, U_n)$ which contains $M$ in its interior.

**Corollary 1.** Let $x \in X$. Then $2^X$ is c.i.k. at $\{x\}$ if and only if $X$ is c.i.k. at $x$.

**Lemma 1.** Let $V$ be a connected open set and $V_1, \cdots, V_n$ be open sets such that $\bigcup_{i=1}^n V_i = V$. Then $N(V_1, \cdots, V_n)$ is connected.

**Proof.** Let $p$ be the smallest positive integer such that $F^p(X) \cap N(V_1, \cdots, V_n) \neq \emptyset$. We will show that

$$\mathcal{F} = \bigcup_{\iota=1}^p (F^\iota(X) \cap N(V_1, \cdots, V_n))$$

is connected.

Let $\mathcal{A} = \{\{x_1, \cdots, x_n\} \mid$ for each $i = 1, \cdots, n$, $x_i \in V_i$, and $x_i = x_j$ if and only if $i = j\}$. We will first establish that $\mathcal{A}$ lies in a connected subset of $\mathcal{F}$. Let $\{x_1, \cdots, x_n\}, \{y_1, \cdots, y_n\} \in \mathcal{A}$. Define $\mathcal{A}_1 = \{\{x_1, \cdots, x_n, y\} \mid y \in V\}$ and $\mathcal{A}_i = \{\{y_1, x_2, \cdots, x_n, y\} \mid y \in V\}$. Then each of $\mathcal{A}_1$ and $\mathcal{A}_i$ is the continuous image of the connected set $V$, so $\mathcal{A}_1$ is a connected subset of $\mathcal{F}$ which contains $\{x_1, \cdots, x_n\}$ and $\{x_1, \cdots, x_n, y_i\}$. Similarly, $\mathcal{A}_i$ is a connected subset of $\mathcal{F}$ which contains $\{y_1, \cdots, x_n, y_i\}$ and $\{y_1, x_2, \cdots, x_n\}$. 

Define $\mathcal{A}_n = \{\{y_1, \cdots, y_{n-1}, x_n, y\} \mid y \in V\}$ and

$$\mathcal{B}_i = \{\{y_1, \cdots, x_{i+1}, \cdots, x_n, y\} \mid y \in V\}.$$ 

Then $\mathcal{A}_n$ is a connected subset of $\mathcal{F}$ which contains $\{y_1, \cdots, y_{n-1}, x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_{n-1}, x_{i+1}, \cdots, x_n\}$. Similarly, for each $i = 2, \cdots, n - 1$ define $\mathcal{A}_i = \{\{y_1, \cdots, y_{i-1}, x_i, \cdots, x_n, y\} \mid y \in V\}$ and

$$\mathcal{B}_i = \{\{y_1, \cdots, y_{i-1}, x_{i+1}, \cdots, x_n, y\} \mid y \in V\}.$$ 

Then $\mathcal{A}_n$ is a connected subset of $\mathcal{F}$ which contains $\{y_1, \cdots, y_{n-1}, x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_{n-1}, x_n\}$ and $\mathcal{B}_i$ is a connected subset of $\mathcal{F}$ which contains $\{y_1, \cdots, x_i, \cdots, x_n\}$ and $\{y_1, \cdots, x_{i+1}, \cdots, x_n\}$. Define $\mathcal{A}_n = \{\{y_1, \cdots, y_{n-1}, x_n, y\} \mid y \in V\}$ and

$$\mathcal{B}_i = \{\{y_1, \cdots, x_i, \cdots, x_n, y\} \mid y \in V\}.$$ 

Then $\mathcal{A}_n$ is a connected subset of $\mathcal{F}$ which contains $\{y_1, \cdots, y_{n-1}, x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_{n-1}, x_n\}$ and $\mathcal{B}_i$ is a connected subset of $\mathcal{F}$ which contains $\{y_1, \cdots, x_i, \cdots, x_n\}$ and $\{y_1, \cdots, x_{i+1}, \cdots, x_n\}$. It follows that $\bigcup_{i=1}^p (\mathcal{A}_i \cup \mathcal{B}_i)$ is a connected subset of $\mathcal{F}$ which contains $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$.

Now let $\{x_1, \cdots, x_n\} \in \mathcal{F} - \mathcal{A}$. If $p \leq m < n$, choose $n - m$ distinct elements $x_{m+1}, \cdots, x_{m+1} \in \mathcal{F} - \mathcal{A}$. For each $i = 1, \cdots, n - m$ let $\mathcal{C}_i = \{\{x_1, \cdots, x_{m+i-1}, y\} \mid y \in V\}$. Then $\mathcal{C}_i$ is a connected subset of $\mathcal{F}$ containing $\{x_1, \cdots, x_{m+i-1}\}$ and $\{x_1, \cdots, x_{m+i}\}$. Hence $\bigcup_{i=1}^{n-m} \mathcal{C}_i$, is a connected subset of $\mathcal{F}$ containing $\{x_1, \cdots, x_{m+i}\}$ and $\{x_1, \cdots, x_n\}$.

If $m \geq n$, let $\{y_1, \cdots, y_n\} \in \mathcal{A}$. Let $\mathcal{D}_i = \{\{x_1, \cdots, x_{m+i}, y\} \mid y \in V\}$. 


Then $D_i$ is a connected subset of $\mathcal{F}$ containing $\{x_i, \ldots, x_m\}$ and $\{x_i, \ldots, x_m, y_i\}$. For each $i = 2, \ldots, n$, let $D_i = \{(x_i, \ldots, x_m, y_i, \ldots, y_{i-1}, y) \mid y \in V\}$. Then $D_i$ is a connected subset of $\mathcal{F}$ containing $\{x_i, \ldots, x_m, y_i, \ldots, y_{i-1}\}$ and $\{x_i, \ldots, x_m, y_i, \ldots, y_i\}$. Hence $\bigcup_{i=1}^{n} D_i$ is a connected subset of $\mathcal{F}$ containing $\{x_i, \ldots, x_m\}$ and $\{x_i, \ldots, x_m, y_i, \ldots, y_m\}$. With an analogous construction we can show that there is a connected subset of $\mathcal{F}$ which contains $\{y_i, \ldots, y_n\}$ and $\{x_i, \ldots, x_m, \ldots, y_n\}$. It follows that there is a connected subset of $\mathcal{F}$ which contains $\{x_i, \ldots, x_m\}$ and $\{y_i, \ldots, y_n\}$.

We have now established that $\mathcal{A}$ lies in a connected subset of $\mathcal{F}$ and that each member of $\mathcal{F} - \mathcal{A}$ lies in a connected subset of $\mathcal{F}$ which meets $\mathcal{A}$. Hence $\mathcal{F}$ is connected. Since $\mathcal{F}$ is dense in $N(V_i, \ldots, V_n)$, it follows that $N(V_i, \ldots, V_n)$ is connected.

**Theorem 2.** Let $M \in C(X)$. Then $2^X$ is l.c. at $M$ if and only if for each open set $U$ containing $M$ there exists a connected open set $V$ such that $M \subset V \subset U$.

**Proof.** Suppose $2^X$ is l.c. at $M$. Let $U$ be an open set containing $M$. Then $M \in N(U)$, so there exists a connected open set $V$ such that $M \in V \subset N(U)$. It follows that $M \subset \bigcup \{A \mid A \in V\} = V \subset U$, and $V$ is open and connected.

Conversely, suppose that for each open set $U$ containing $M$ there exists a connected open set $V$ such that $M \subset V \subset U$. Let $N(U_i, \ldots, U_n)$ be a basic open set such that $M \in N(U_i, \ldots, U_n)$ and let $U = \bigcup_{i=1}^{n} U_i$. Then there exists a connected open set $V$ such that $M \subset V \subset U$. Let $V_i = V \cap U_i$. Then $M \in N(V_i, \ldots, V_n) \subset N(U_i, \ldots, U_n)$, and by Lemma 1, $N(V_i, \ldots, V_n)$ is connected.

**Corollary 2.** Let $x \in X$. Then $2^X$ is l.c. at $\{x\}$ if and only if $X$ is l.c. at $x$.

We remark that if $M \in C(X)$ and $2^X$ is l.c. at $M$, then Lemma 1 and Theorem 2 imply the existence of a local base of connected sets at $M$, each of which is of the form $N(U_i, \ldots, U_n)$.

The next several results concern the relationships between $2^X$ and $C(X)$ with respect to local connectedness and connectedness im kleinen at points of $C(X)$.

**Theorem 3.** Let $M \in C(X)$. If $2^X$ is c.i.k. at $M$, then $C(X)$ is c.i.k. at $M$.

**Proof.** Let $N(U_i, \ldots, U_n) \cap C(X)$ be an open set containing $M$. Let $N(V_i, \ldots, V_n)$ be an open set such that $M \in N(V_i, \ldots, V_n)$.
$N(V, \ldots, V_m) \subset N(U, \ldots, U_\alpha)$. Since $2^X$ is c.i.k. at $M$, there exists an open set $N(W, \ldots, W_\beta)$ such that

$$M \in N(W, \ldots, W_\beta) \subset N(V, \ldots, V_m)$$

and with the property that $B \in N(W, \ldots, W_\beta)$ implies $N(V, \ldots, V_m)$ contains a connected set containing $B$ and $M$. Then $N(U, \ldots, U_\alpha)$ contains a continuum containing $B$ and $M$.

Let $K \in N(W, \ldots, W_\beta) \cap C(X)$. Then there exists a continuum $L$ in $N(U, \ldots, U_\alpha)$ containing $K$ and $M$. Now $\bigcup \{A \mid A \in L\} = L \in C(X)$, and $L \in N(U, \ldots, U_\alpha)$, since $L \subset N(U, \ldots, U_\alpha)$. It follows that there exist order arcs $L_k$ and $L_m$ in $N(U, \ldots, U_\alpha) \cap C(X)$ from $K$ to $L$ and from $M$ to $L$. So $L_k \cup L_m$ is a continuum in $N(U, \ldots, U_\alpha) \cap C(X)$ containing $K$ and $M$. Hence $C(X)$ is c.i.k. at $M$.

**Corollary 3.** Let $M \in C(X)$. If for each open set $U$ containing $M$ there is a component of $U$ which contains $M$ in its interior, then $C(X)$ is c.i.k. at $M$.

Corollary 3 is a generalization of Theorem 6 of [6]. The example following Theorem 6 of [6] shows that the converse of Corollary 3 is false. It also shows that the converse of Question 1 below is false.

**Question 1.** Let $M \in C(X)$. If $2^X$ is l.c. at $M$, is $C(X)$ l.c. at $M$?

**Corollary 4.** Let $x \in X$. Then $X$ is c.i.k. at $x$ if and only if $C(X)$ is c.i.k. at $\{x\}$.

**Proof.** If $X$ is c.i.k. at $x$, then by Corollary 1, $2^X$ is c.i.k. at $\{x\}$, and by Theorem 3, $C(X)$ is c.i.k. at $\{x\}$.

Suppose $C(X)$ is c.i.k. at $\{x\}$. Let $U$ be an open set containing $x$. Then $\{x\} \in N(U) \cap C(X)$, so there exists an open set $N(V) \cap C(X)$, $\{x\} \in N(V) \cap C(X) \subset N(U) \cap C(X)$, with the property that $M \in N(V) \cap C(X)$ implies $N(U) \cap C(X)$ contains a connected set containing $M$ and $\{x\}$.

Now $x \in V \subset U$. Let $y \in V$. Then $\{y\} \in N(V) \cap C(X)$, so $N(U) \cap C(X)$ contains a connected set $L$ containing $\{y\}$ and $\{x\}$. It follows that $\bigcup \{L \mid L \in L\}$ is a connected subset of $U$ containing $x$ and $y$. Hence $X$ is c.i.k. at $x$.

**Corollary 5.** Let $x \in X$. If $X$ is l.c. at $x$, then $C(X)$ is l.c. at $\{x\}$.
This follows from the observation that if $V$ is connected, then $N(V) \cap C(X)$ is connected, since each point of $(N(V) \cap C(X)) - F_i(V)$ can be joined by an order arc in $N(V) \cap C(X)$ to a point of $F_i(V)$, and $F_i(V)$ is connected.

The next example shows that the converse of Corollary 5 is false.

**Example 1.** This example is from page 113 of [2]. For each positive integer $n$ and each positive integer $m$ let $L_{n,m}$ denote the line segment in the plane from $(1/(n+1), (-1)^{n+1}/m(n+1))$ to $(1/n, 0)$. Let $A_n = (\bigcup_{k=1}^{n} L_{n,k})^*$ and let $X = (\bigcup_{n=1}^{\infty} A_n)^*$. Then $X$ is c.i.k. at $(0, 0)$ but is not i.e. at $(0, 0)$.

We now give a brief argument that $C(X)$ is i.e. at $(0, 0)$. For each $n \geq 2$ choose $q_n, r_n, s_n$ so that $1/(n+1) < q_n < r_n < 1/n < s_n < 1/(n-1)$. Let $U_n = \{(x, y) \mid x < r_n\}$ and $V_n = \{(x, y) \mid q_n < x < s_n\}$. Then $N(U_n) \cup N(U_n, V_n)$ is a continuum-wise connected open set in $C(X)$ containing $(0, 0)$, for if $M, N \in N(U_n) \cup N(U_n, V_n)$, then $M, N \subset \{(x, y) \mid x < 1/n\} \cup \{(x, 0) \mid 1/n \leq x \leq s_n\}$ and a continuum can be constructed in $C(X)$ containing $M$ and $N$ and lying in $N(U_n) \cup N(U_n, V_n)$. Clearly $\{N(U_n) \cup N(U_n, V_n) \mid n = 2, 3, \ldots\}$ is a neighborhood base at $(0, 0)$.

The following definition and Lemma 2 concern the finite topology and will be used in proving our main results, in which we obtain necessary and sufficient conditions that $2^X$ be i.e. (c.i.k.) at an arbitrary point.

Let $A \in 2^X$. A basic open set $N(U_1, \ldots, U_n)$ is essential with respect to $A$ if $A \in N(U_1, \ldots, U_n)$ and for each $i = 1, \ldots, n$, $A - \bigcup_{j \neq i} U_j \neq \emptyset$.

**Lemma 2.** Let $A \in 2^X$ and $N(U_1, \ldots, U_n)$ be an open set containing $A$. Then there exists an open set $N(V_1, \ldots, V_n)$ such that $A \in N(V_1, \ldots, V_n) \subset N(U_1, \ldots, U_n)$ and $N(V_1, \ldots, V_n)$ is essential with respect to $A$.

**Proof.** Choose $x_1, \ldots, x_n \in A$ such that $x_i \in U_i$. Let $V_i, \ldots, V_n$ be open sets such that for each $i = 1, \ldots, n$, $x_i \in V_i \subset \bigcap \{U_j \mid x_i \in U_j\}$ and with the additional property that $V_i = V_j$ if $x_i = x_j$ and $V_i \cap V_j = \emptyset$ if $x_i \neq x_j$. Let $\{V_1, \ldots, V_n\}$ (relabeling if necessary) be the set of $V_i$'s which are distinct. For each $y \in A - \bigcup_{i=1}^{n} V_i$ let $O_y$ be an open set such that $y \in O_y \subset \bigcap \{U_j \mid y \in U_j\}$ and such that $O_y \cap \{x_1, \ldots, x_n\} = \emptyset$. Since $A - \bigcup_{i=1}^{n} V_i$ is compact, there exist $y_1, \ldots, y_s$ such that $A - \bigcup_{i=1}^{n} V_i \subset \bigcup_{i=1}^{s} O_{y_i}$. We may assume that all the $O_{y_i}$'s are distinct. Let $\{O_{y_1}, \ldots, O_{y_s}\}$ (relabeling if necessary) be the subset of
\{O_{y_1}, \ldots, O_{y_q}\} consisting of all the \(O_{y_i}\)'s with the property that 
\((A - \bigcup_{i=1}^{k} V_i) - \bigcup_{j \neq i} O_{y_j} \neq \emptyset\).

For notational purposes, for each \(j = 1, \ldots, q\) let \(O_{y_j} = V_{k+j}\) and 
let \(k + q = m\). Then \(A \in \mathcal{N}(V_1, \ldots, V_k, V_{k+1}, \ldots, V_m)\). Clearly

\[ \mathcal{N}(V_1, \ldots, V_k, V_{k+1}, \ldots, V_m) \subset \mathcal{N}(U_1, \ldots, U_n) \, . \]

For each \(j = 1, \ldots, k\) there exists \(x_i \in A\) such that \(x_i \in V_j\) and \(x_i \in (\bigcup_{p=1}^{k} V_p) - V_j\). For each \(j = k+1, \ldots, m\), \n\[
(A - \bigcup_{i=1}^{k} V_i) - \bigcup_{i=1}^{m} V_i \neq \emptyset ,
\]

so there exists \(a_j \in V_j \cap (A - \bigcup_{i=1}^{k} V_i)\) such that \(a_j \in \bigcup_{i=1}^{k} V_i\). It
follows that \(\mathcal{N}(V_1, \ldots, V_m)\) is essential with respect to \(A\).

**Theorem 4.** Let \(A \in 2^X\). Then \(2^X\) is c.i.k. at \(A\) if and only if \(2^X\) is c.i.k. at each component of \(A\).

**Proof.** Suppose that \(2^X\) is c.i.k. at \(A\). Let \(A\), be a component of \(A\) and let \(W\) be an open set containing \(A\). Let \(U\) be an open set such that \(A_1 \subset U \subset U^* \subset W\) and such that \((\text{bd } U) \cap A = \emptyset\). Let \(\{U_1, \ldots, U_n\}\) be a finite cover of \(A - U\) by open sets such that for each \(i = 1, \ldots, n\), \(U \cap U_i = \emptyset\) and \(A \cap U_i \neq \emptyset\). Then \(A \in \mathcal{N}(U, U_1, \ldots, U_n)\).

Let \(\mathcal{C}\) be a component of \(\mathcal{N}(U, U_1, \ldots, U_n)\) which contains \(A\) in its interior. Define \(f: \mathcal{C} \to \mathcal{N}(U)\) by \(f(B) = B \cap U\). If \(\mathcal{N}(V_1, \ldots, V_m) \subset \mathcal{N}(U)\), then \(f^{-1}(\mathcal{N}(V_1, \ldots, V_m)) = \mathcal{N}(V_1, \ldots, V_m, U_1, \ldots, U_n) \cap \mathcal{C}\), so \(f\) is continuous. Hence \(f(\mathcal{C})\) is connected.

Let \(\mathcal{N}(V_1, \ldots, V_q)\) be an open set such that \(A \in \mathcal{N}(V_1, \ldots, V_q) \subset \mathcal{C}\). Let \(\{V_1, \ldots, V_q\}\) (relabeling if necessary) be the largest subset of \(\{V_1, \ldots, V_q\}\) with the property that for each \(j = 1, \ldots, m\), \(V_j \cap U \neq \emptyset\). Let \(\{V_1, \ldots, V_k\}\) (relabeling if necessary) be the largest subset of \(V_1, \ldots, V_m \cap (U_{k+1} U_i) = \emptyset\). For each \(j = 1, \ldots, k\), let \(V_j = V_j \cap U\) and \(V_j = V_j \cap (U_{k+1} U_i)\). Then

\[
A \in \mathcal{N}(V_1, \ldots, V_k, V_{k+1}, \ldots, V_m, V_{1}, \ldots, V_m) = \mathcal{C} \subset \mathcal{N}(V_1, \ldots, V_q) \subset \mathcal{C}\, .
\]

Now if \(B \in \mathcal{C}\); then

\[
f(B) = B \cap U
= \left[ \bigcup_{j=1}^{k} V_j \right] \cup \left( \bigcup_{j=k+1}^{m} V_j \right) \in \mathcal{N}(V_1, \ldots, V_k, V_{k+1}, \ldots, V_m) .
\]

Conversely, suppose \(C \in \mathcal{N}(V_1, \ldots, V_k, V_{k+1}, \ldots, V_m)\). For each \(j = 1, \ldots, k\), let \(x_j \in V_j\) and for each \(j = m + 1, \ldots, q\) let \(x_j \in V_j\). Then
Let $C = \bigcup \{ f(B) \mid B \in \mathcal{E} \}$. Then $C^* \subset U^* \subset W$. Let $C(A_i)$ be the component of $C^*$ which contains $A_i$. Let $N(V_1, \ldots, V_m, V_{m+1}, \ldots, V_p)$ be an open set such that $A \in N(V_1, \ldots, V_m, V_{m+1}, \ldots, V_p)$ and such that $\bigcup_{i=1}^{m+1} V_i \subset U$ and $\bigcup_{i=m+1}^{p} V_i \subset \bigcup_{i=1}^{p} U_i$. Let $\{ V_i, \ldots, V_k \}$ (relabeling if necessary) be the largest subset of $\{ V_i, \ldots, V_m \}$ with the property that for each $i = 1, \ldots, k$, $V_i \cap C(A_i) = \emptyset$. (A slight modification of the following argument is necessary in the case that $\{ V_i, \ldots, V_m \} = \emptyset$.) Let $0$ be an open set containing $C(A_i)$ such that $0 \cap (U_{k+1} \cup U_k) = 0$ and such that $(\text{bd } 0) \cap C^* = 0$.

Let $C(A_i)$ be a connected subset of $C^*$ which contains $A_i$ in its interior. Hence, by Theorem 1, $2^x$ is c.i.k. at $A_i$.

For the converse, suppose that $2^x$ is c.i.k. at each component of $A$. Let $\mathcal{U}$ be an open set containing $A$ and $N(U_1, \ldots, U_n)$ be a basic open set such that $A \in N(U_1, \ldots, U_n) \subset N(U_1, \ldots, U_n) \subset \mathcal{U}$. By Lemma 2 we may assume that $N(U_1, \ldots, U_n)$ is essential with respect to $A$. For each component $A_i$ of $A$ let $\{ U_{i_1}, \ldots, U_{i_n} \}$ be the largest subset of $\{ U_1, \ldots, U_n \}$ with the property that for $j = 1, \ldots, n$, $A_i \cap U_{i_j} = \emptyset$. Then $A_i = N(U_{i_1}, \ldots, U_{i_n}) = U_{i_1} \cup \cdots \cup U_{i_n}$.

Now $A \subset \bigcup_{i \in I_i} (\bigcup_{j=1}^{n_i} V_{i,j})$ and since $A$ is compact there exists a finite subcover of $A$ of the form $\bigcup_{i \in I} (\bigcup_{j=1}^{n_i} V_{i,j})$. Then
The last inclusion follows from the construction and the fact that
$N(U_1, \ldots, U_s)$ is essential with respect to $A$. Let $M = \bigcup_{i=1}^s M_i$.
Then $M \in N(U_1, \ldots, U_s)$. Let $B \in N(V_{a_{s_1}}, \ldots, V_{a_{s_t}}, \ldots, V_{a_{s_m}})$. Note that $B = \bigcup_{i=1}^{s_t} (B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t})$. Now $B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t} \subset M_i^*$. Define $f_i : \mathcal{B}_i \to \mathcal{U}$ by $f_i(C) = C \cup \bigcup_{j=1}^{a_{s_t}} (B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t})$. Since union is continuous, $f_i(\mathcal{B}_i)$ is connected, and $B, M_i^* \cup (\bigcup_{s=1}^{s_t-1} (B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t})) \in f_i(\mathcal{B}_i)$. For each $i = 2, \ldots, m$, there exists an order arc $A_i$ from $B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t}$ to $M_i^*$. For each $i = 2, \ldots, m - 1$, define $f_i(A_i) = \mathcal{U}$ by

$$f_i(C) = \left( \bigcup_{k=1}^{i-1} M_{a_k} \right) \cup C \cup \left( \bigcup_{k=i+1}^{m} \left( B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t} \right) \right).$$

Then $f_i(\mathcal{B}_i)$ is a connected subset of $\mathcal{U}$ containing $(\bigcup_{k=1}^{i-1} M_{a_k}) \cup (\bigcup_{k=i+1}^{m} \left( B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t} \right))$. Define $f_m(\mathcal{B}_m) \to \mathcal{U}$ by $f_m(C) = (\bigcup_{k=1}^{m-1} M_{a_k}) \cup C$. Then $f_m(\mathcal{B}_m)$ is a connected subset of $\mathcal{U}$ containing $(\bigcup_{k=1}^{m-1} M_{a_k}) \cup (B \cap \bigcup_{j=1}^{a_{s_t}} V_j^{s_t})$ and $M$. Hence $\bigcup_{i=1}^{s_t} f_i(\mathcal{B}_i)$ is a connected subset of $\mathcal{U}$ containing $B$ and $M$. It follows that $2^X$ is c.i.k. at $A$.

**Theorem 5.** Let $A \in 2^X$. Then $2^X$ is l.c. at $A$ if and only if $2^X$ is l.c. at each component of $A$.

**Proof.** Suppose that $2^X$ is l.c. at $A$. Let $A$, be a component of $A$ and let $W$ be an open set containing $A$. Let $U$ be an open set such that $A \subset U \subset W$ and such that $(\text{bd } U) \cap A = \emptyset$. Let $\{U_1, \ldots, U_s\}$ be a finite cover of $A - U$ by open sets such that for each $i = 1, \ldots, n$, $U_i \cup U_i = \emptyset$ and $A \cap U_i = \emptyset$. Then $A \in N(U_i, U_2, \ldots, U_s)$. Let $\mathcal{V} = \{f(B) \mid B \in \mathcal{V}\}$. Then $V \subset U$. Let $Q(A_i)$ be the quasicomponent of $V$ which contains $A_i$ and let $x \in Q(A_i)$. Let $B \in \mathcal{V}$ such that $x \in f(B)$. Then there exists an open set $N(V_1, \ldots, V_m, V_{m+1}, \ldots, V_p)$ such that $B \in N(V, \ldots, V_m, V_{m+1}, \ldots, V_p) \subset N(V_1^*, \ldots, V_m^*, V_{m+1}^*, \ldots, V_p^*)$. Let $\bigcup_{i=1}^{m} V_i^* \subset U$ and $\bigcup_{i=m+1}^{p} V_i \subset U$. Let $\{V_1, \ldots, V_k\}$ be the largest subset of $\{V_1, \ldots, V_m\}$ with the property that for each $i = 1, \ldots, k$, $V_i^* \cap Q(A_i) = \emptyset$. (A slight modification of the following argument is necessary in the case that
\{V_1, \cdots, V_k\} = \emptyset \). Since \( U_{i=1}^k V_i^* \) is compact, there exist disjoint open-closed sets \( S \) and \( T \) such that \( U_{i=1}^k V_i^* \subset S, Q(A_i) \subset T \) and \( S \cup T = V \).

Suppose \( x \in \text{int} Q(A_i) \). Let \( O \) be an open set containing \( x \) such that \( O \subset T \cap (\bigcap \{V_i, x \in V_i\}) \). Let \( y \in O \) such that \( y \notin Q(A_i) \). Then there exist disjoint open-closed sets \( T' \) and \( T'' \) such that \( Q(A_i) \subset T', y \in T'' \), and \( T' \cup T'' = T \).

Now \( T', T'' \), and \( S \) are disjoint open sets whose union is \( V \). Consequently the sets \( N(T'), N(T''), N(S), N(T', T''), N(T', S), N(T'', S) \), and \( N(T', T'', S) \) are pairwise disjoint and \( f(\mathcal{V}) \) is contained in the union of these sets.

For each \( i = 1, \cdots, k \), let \( x_i \in V_i \). For each \( i = k + 1, \cdots, m \), \( Q(A_i) \cap V_i^* \neq \emptyset \), and since \( T' \) is an open set containing \( Q(A_i) \), there exists \( x_i \in T' \) such that \( x_i \in V_i \). Then

\[ \{x_1, \cdots, x_m\}, \{x_1, \cdots, x_m, y\} \in N(V_1, \cdots, V_m) \subset f(\mathcal{V}) . \]

Furthermore, \( \{x_1, \cdots, x_m\} \in N(T', S) \) and \( \{x_1, \cdots, x_m, y\} \in N(T', T'', S) \). Hence \( f(\mathcal{V}) \) is not connected, a contradiction, so the assumption that \( x \in \text{int} Q(A_i) \) was false.

We have now established that \( Q(A_i) \) is open. So \( Q(A_i) \) and \( V - Q(A_i) \) are disjoint open-closed subsets of \( V \). If \( Q(A_i) \) were not connected, there would exist a proper open-closed subset of \( Q(A_i) \) (and hence of \( V \)) containing \( A_i \), which is impossible. It follows that \( Q(A_i) \) is an open connected subset of \( V \) containing \( A_i \). Hence, by Theorem 2, \( 2^x \) is l.c. at \( A \).

For the converse, suppose that \( 2^x \) is l.c. at each component of \( A \). Let \( N(U_{i_1}, \cdots, U_{i_n}) \) be a basic open set containing \( A \). By Lemma 2 we may assume that \( N(U_{i_1}, \cdots, U_{i_n}) \) is essential with respect to \( A \). For each component \( A_a \) of \( A \) let \( \{U_{i_{a1}}, \cdots, U_{i_{an}}\} \) be the largest subset of \( \{U_{i_1}, \cdots, U_{i_n}\} \) with the property that for each \( j = 1, \cdots, n_a \), \( U_{i_j} \cap A_a \neq \emptyset \). Then \( A_a \in N(U_{i_{a1}}, \cdots, U_{i_{an}}) \). Let \( U_a = \bigcup_{j=1}^{n_a} U_{i_j} \). By Theorem 2 there is a connected open set \( V_a \) such that \( A_a \subset V_a \subset U_a \). For each \( j = 1, \cdots, n_a \) let \( V_j^a = V_a \cap U_{i_j} \). Then

\[ A_a \in N(V_1^a, \cdots, V_{n_a}^a) \subset N(U_{i_1}, \cdots, U_{i_{n_a}}) \]

and by Lemma 1, \( N(V_1^a, \cdots, V_{n_a}^a) \) is connected. Now \( A \subset \bigcup_a \bigcup_{j=1}^{n_a} V_j^a \), and since \( A \) is compact, there exist \( \alpha_1, \cdots, \alpha_m \) such that \( A \subset \bigcup_{i=1}^m (\bigcup_{j=1}^{n_i} V_j^i) \). Then

\[ A \in N(V_1^{\alpha_1}, \cdots, V_{n_{\alpha_1}}^{\alpha_1}, \cdots, V_1^{\alpha_m}, \cdots, V_{n_{\alpha_m}}^{\alpha_m}) = \mathcal{V} \subset N(U_1, \cdots, U_m) . \]

The last inclusion follows from the construction and the fact that \( N(U_1, \cdots, U_m) \) is essential with respect to \( A \).
Let $B, C \in \mathcal{Y} \cap F(X)$ and for and $i = 1, \ldots, m$ let $B_i = B \cap (\bigcup_{r=1}^{n_i} V_i^r)$ and $C_i = C \cap (\bigcup_{r=1}^{n_i} V_i^r)$. Then $B_i, C_i \in N(V_i^1, \ldots, V_i^{n_i}) \cap F(X)$. As in the proof of Theorem 2, for each $i = 1, \ldots, m$ there exists a connected set $\mathcal{C}_i$ in $N(V_i^1, \ldots, V_i^{n_i}) \cap F(X)$ which contains $B_i$ and $C_i$. Define $f_i: \mathcal{C}_i \to \mathcal{Y} \cap F(X)$ by $f_i(D) = D \cup (\bigcup_{r=1}^{n_i} B_i)$. Since $f_1$ is continuous, $f_1(\mathcal{C}_1)$ is a connected subset of $\mathcal{Y} \cap F(X)$ containing $B$ and $C, \cup (\bigcup_{r=1}^{n_i} B_i)$. For each $i = 2, \ldots, m - 1$ define $f_i: \mathcal{C}_i \to \mathcal{Y} \cap F(X)$ by $f_i(D) = (\bigcup_{r=1}^{n_i} C_i) \cup D \cup (\bigcup_{r=n_{i+1}}^{n_i} B_i)$. Then $f_i(\mathcal{C}_i)$ is a connected subset of $\mathcal{Y} \cap F(X)$ containing $(\bigcup_{r=1}^{n_i} C_i) \cup (\bigcup_{r=n_i+1}^{n_i} B_i)$ and $(\bigcup_{r=1}^{n_i} C_i) \cup (\bigcup_{r=n_i+1}^{n_i} B_i)$. Define $f_m: \mathcal{C}_m \to \mathcal{Y} \cap F(X)$ by $f_m(D) = (\bigcup_{r=1}^{n_m-1} C_i) \cup D$. Then $f_m(\mathcal{C}_m)$ is a connected subset of $\mathcal{Y} \cap F(X)$ containing $(\bigcup_{r=1}^{n_m-1} C_i) \cup B_m$ and $C$. Hence $\bigcup_{i=1}^{n_m} f_i(\mathcal{C}_i)$ is a connected subset of $\mathcal{Y} \cap F(X)$ containing $B$ and $C$. It follows that $\mathcal{Y} \cap F(X)$ is connected, and since $\mathcal{Y} \cap F(X)$ is dense in $\mathcal{Y}$, $\mathcal{Y}$ is connected. Hence $2^X$ is l.c. at $A$.

**COROLLARY 6.** Let $A \in 2^X$. If $X$ is c.i.k. (l.c.) at each point of $A$, then $2^X$ is c.i.k. (l.c.) at $A$.

The converses of Corollary 6 are false. It is easy to verify (see Lemma 2 of [1]) that for any continuum $X$, $2^X$ is l.c. at $X$.

**COROLLARY 7.** The following are equivalent:

1. For each $i = 1, \ldots, n$, $X$ is c.i.k. (l.c.) at $p_i$.
2. For each $i = 1, \ldots, n$, $2^X$ is c.i.k. (l.c.) at $\{p_i\}$.
3. $2^X$ is c.i.k. (l.c.) at $\{p_i, \ldots, p_n\}$.

**REFERENCES**


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