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HOLLOW MODULES AND LOCAL ENDOMORPHISM RINGS

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This is a study of the conditions under which smallness of proper submodules will influence the structure of the endomorphism ring of a module. The case in which that endomorphism ring becomes local is of special interest. Further facts about hollow modules are also established.

1. Introduction. Throughout this paper, R will denote an associative ring with unit and R-Mod the category of left R-modules. In [3], we studied elements of R-Mod such that every strictly decreasing sequence of submodules eventually became small. Such modules were said to have finite spanning dimension and we showed that each such could be represented as the sum of a unique number of hollow submodules. A module was said to be hollow if every submodule were small. That is, X is hollow if X, Y_1, $Y_2 \in R$-Mod, $Y_1, Y_2 \subseteqq X$, and $Y_1 + Y_2 = X$ imply either $Y_1 = X$ or $Y_2 = X$. In this paper, we present some theorems about hollow modules and, in particular, we investigate the circumstances under which certain hollow modules have local endomorphism rings. By a local ring we simply mean a ring with a unique maximal left ideal and we assume neither chain conditions nor commutativity.

To a certain extent, hollow modules play the same role for modules with finite spanning dimension which simple modules play for modules with descending chain condition. For example, theorems about hollow modules having local endomorphism rings are extensions of Shur's lemma.

In § 2, we establish basic facts about hollow modules and in § 3 we introduce the idealizer and show its relation to endomorphism rings. In § 4, we present eight theorems which show that, under certain conditions, a hollow module has a local endomorphism ring. In § 5, we present some miscellaneous results on when a maximal left ideal is 2-sided and we show that the abelian group of p-adic integers is hollow.

2. Some preliminaries. We recall that if X is an element of R-Mod and Y_1 is a submodule of X, then Y_1 is called small if $Y_1 + Y_2 = X$ for some submodule Y_2 of X implies $Y_2 = X$. It is easy to show that the image of a small submodule is again small and we will have use for this fact later.

PROPOSITION 2.1. *If X is a hollow left R-module which is also finitely generated, then X is cyclic. Furthermore, if $X \cong R/I$, then I is contained in a unique maximal left ideal M. Conversely, if I is contained in a unique maximal left ideal, then R/I is hollow.*

Proof. Suppose $x_1, \cdots, x_n$ form a set of generators for X. Then $X = Rx_1 + \cdots + Rx_n$. Since every proper submodule of X is small, we may delete the summands one by one until we find $X = Rx_i$ for some i. Thus X is cyclic.

We now have $X \cong R/I$ where I is the annihilator of x_i. Suppose I is contained in two maximal ideals, say M_1 and M_2. Then $M_1 + M_2 = R$ so the sum of the images of M_1 and M_2 in R/I actually equal R/I. But it is clear that M_1/I and M_2/I are both proper submodules of R/I and so we have a contradiction. Thus I is contained in a unique maximal left ideal.

Finally, suppose I is contained uniquely in the maximal left ideal M. Then any submodule of R/I corresponds to a left ideal of R containing I. All such proper ideals must be contained in M so any finite sum of proper submodules of R/I is contained in M/I.

COROLLARY 2.2. *If X is hollow and finitely generated, then X contains a unique maximal submodule.*

Proof. We know from the theorem that $X \cong R/I$ where I is contained in a unique maximal left ideal, M, of R. So any proper submodule of R/I must be contained in M/I which is the unique maximal submodule of R/I.

Hidden in these proofs is the germ of a useful idea. We can dualize a notion of Goldie in the following way. Let $X \in R$-Mod and let Y_1, Y_2 be submodules of X. We say that Y_1 and Y_2 are correlated if $Y_1 + H = X$ if and only if $Y_2 + H = X$ for some submodule, H, of X. It is very easy to see that correlation is an equivalence relation. Furthermore, if R/I is hollow, then I must be correlated to the maximal left ideal M which contains it in R; for if M is a maximal left ideal which contains R then $I + H = R$ certainly implies $M + H = R$. On the other hand, if $M + H = R$, then, taking the canonical images of M and H in R/I, we have $M/I + (H + I)/I = R/I$. Since M/I is small in R/I we see that $(H + I)/I = R/I$ or $H + I = R$.

Conversely, if I is correlated to M, then I must be contained uniquely in M. This is because the containment of I in another maximal left ideal M' would imply $I + M' = R$ from the relationship $M + M' = R$. But since $I \subsetneqq M'$, $I + M' = M'$. We thus have Theorem 2.3 which is stated below in an equivalent but more useful form.

THEOREM 2.3. *Let I be a left ideal of R. Then I is contained in a unique maximal left ideal M if and only if $Rx + I = R$ for all $x \in R$ and $x \notin M$.*

The following propositions show that there are plenty of left ideals which are contained uniquely in maximal ideals.

PROPOSITION 2.4. *If I is a left ideal of R and is contained uniquely in the maximal left ideal M, then I^2 is contained uniquely in M.*

Proof. Suppose I^2 is contained in the maximal left ideal $M' \neq M$. Then there is $m_1 \in M'$, $m_1 \notin M$, so $Rm_1 + I = R$. Thus there is $r \in R$, $i \in I$ and $rm_1 + i = 1$. Let i' be any element of I. Then $i'rm_1 + i'i = i'$. Both the first and second terms in the sum are in M'. Thus $I \subseteqq M'$ which is a contradiction.

COROLLARY 2.5. *If I is a left ideal of R which is contained uniquely in a maximal left ideal, M, then I^n is contained uniquely in M for all $n \geqq 1$.*

COROLLARY 2.6. *If M is a maximal left ideal of R, then M^n is contained uniquely in M for all $n \geqq 1$.*

COROLLARY 2.7. *If I is a left ideal of R which contains some power of the maximal left ideal M, then I is contained uniquely in M.*

COROLLARY 2.8. *If I is a nilpotent left ideal of R which is contained uniquely in a maximal left ideal M, then R is local.*

3. **The idealizer.** From now on, we will assume that all modules are hollow and cyclic unless otherwise stated. We shall be interested in $\mathrm{Hom}_R(R/I, R/I) = \mathrm{End}_R(R/I)$ where R/I is hollow. Each R-homomorphism $f: R/I \rightarrow R/I$ is uniquely determined by its image on $1 + I$. Thus $f(1 + I) = a + 1$, so $f(r + I) = r(f(1 + I) = ra + I$ and f corresponds to $a + I$. Furthermore, $f(0 + I) = 0 + I$ so if we choose any other representative for zero, say $i + I$, then $f(i + I) = ia + I = 0 + I$. That is, $ia \in I$. We then set $\mathscr{I}(I) = \{x \in R: Ix \subseteqq I\}$. $\mathscr{I}(I)$ is called the idealizer of I and $\mathrm{End}_R(R/I)$ is obviously isomorphic to $\mathscr{I}(I)/I$. (Cohn, [1], Proposition 4.1, p. 17).

Since every R-endomorphism of R/I corresponds to an element of $\mathscr{I}(I)/I$, we may think of an R-endomorphism as right multiplication of R/I an element of R. This observation will be useful later.

PROPOSITION 3.1. *If the left ideal I of R is contained uniquely in the maximal left ideal M, then $\mathscr{I}(I) \subseteqq \mathscr{I}(M)$.*

Proof. Let $r \in \mathscr{I}(I)$. Thus $Ir \subseteqq I$. Now r induces an R-endomorphism of R/I by sending $a + I$ to $ar + I$. We know the image of a small submodule under an R-homomorphism is again small. Thus the image of M/I is small in R/I so it must be contained in M/I. Thus if $m \in M$, $mr + I \in M/I$ which, in turn, implies $mr \in M$.

COROLLARY 3.2. *If I is an ideal of R and I is contained uniquely in the maximal left ideal M, then M is an ideal.*

Proof. We know that $R \supseteqq \mathscr{I}(M) \supseteqq \mathscr{I}(I) = R$. Thus $\mathscr{I}(M) = R$ so M is an ideal.

PROPOSITION 3.3. *Let R/I be hollow and $x \in \mathscr{I}(I)$. Then the endomorphism induced by x on R/I is epic if and only if $x \notin M$ where M is the unique maximal left ideal containing I.*

Proof. The endomorphism induced by x is epic if and only if $R/I = (Rx + I)/I$. But that is true if and only if $Rx + I = R$ which is true if and only if $x \notin M$.

4. Local endomorphism rings. With the help of the foregoing theorems, it is now possible to ask and answer some questions concerning whether or not $\mathrm{End}_R(R/I)$ is local when R/I is hollow. The obvious conjectures and the main difficulties may be described as follows. We know that $\mathrm{End}_R(R/I)$ is isomorphic to $\mathscr{I}(I)/I$. We would thus expect that if $\mathrm{End}_R(R/I)$ is a local ring, its unique maximal ideal would be $M \cap \mathscr{I}(I)/I$ where M is the unique maximal left ideal containing I. This is indeed so in the cases we will consider. In the general case, however, the following difficulty arises. Suppsse $a \in \mathscr{I}(I)$ and $a \notin M$. We would like to show that a induces an isomorphism on R/I. The obvious proof to try is as follows. Since $a \notin M$, we know $Ra + I = R$. Thus there is $r \in R$ and $ra + i = 1$. We would then like to say that r induces an inverse map for a on R/I. Unfortunately, we also need to know that $Ir \subseteqq I$ and possibly this is not so. Thus there is a possibility that $\mathrm{End}_R(R/I)$ is not local and we must specialize.

The following observations are also useful. Suppose $a \in \mathscr{I}(I)$, $a \notin M$. Then there is $r \in R$ and $ra + i_1 = 1$ for some i_1. By Proposition 3.1, $r \notin M$ since, otherwise $1 \in M$. Thus $Rr + 1 = R$ so there is $x \in R$ with $xr + i_2 = 1$. We then have $xra + xi_1 = x$ and $xra + i_2a = a$. Thus $x = a - i_2a + xi_1 = a + xi_3$. Thus $x \in \mathscr{I}(I)$ and induces the same endomorphism on R/I that a does.

THEOREM 4.1. *If R/I is hollow and I is an ideal, then* $\mathrm{End}_R(R/I)$ *is local.*

Proof. It is easy to see that $\mathrm{End}_R(R/I) \cong R/I$ is a ring and that R/I has M/I as its unique maximal ideal where M is the unique maximal ideal containing I.

We recall that, if $a \in R$, $(I: a) = \{x \in R \mid xa \in I\}$ is a left ideal. If $a \in \mathscr{I}(I)$, then $(I: a)$ contains I.

THEOREM 4.2. *Let I be a left ideal of R which is contained uniquely in the maximal left ideal M of R. Suppose R satisfies the following chain condition. For every $a \in \mathscr{I}(I)$, the sequence of left ideals*

$$I \subseteqq (I: a) \subseteqq (I: a^2) \subseteqq \cdots$$

eventually stabilizes. Then $\mathrm{End}_R(R/I)$ *is local.*

Proof. If M is a left R-module and $f: M \to M$ is an epimorphism such that the sequence of submodules

$$\ker f \subseteqq \ker f^2 \subseteqq \cdots$$

eventually stabilizes, then a standard argument shows that f is an isomorphism. Moreover, it is easy to see that, if M is a hollow module with the property that every epimorphism from M to M is an isomorphism, then $\mathrm{End}_R(M)$ is local. We need only note that, in that case, nonunits would form an ideal. Thus it is enough to show that, for any endomorphism $f: R/I \to R/I$, the above sequence becomes stable. If $f \in \mathrm{End}_R(R/I)$, then there exists $a \in \mathscr{I}(I)$ such that $f(r + I) = ra + I$ for all $r + I \in R/I$. Then for any $n \geqq 1$, $\mathrm{Ker}\, f^n = (I: a^n)/I$ and since the sequence

$$(I: a) \subseteqq (I: a^2) \subseteqq \cdots$$

stabilizes, the proof is complete.

We note that if M is any indecomposable R-module such that $\mathrm{End}_R(M)$ is semi-perfect (in particular, an artinian ring), then $\mathrm{End}_R(M)$ is local (see [4], p. 76, Corollary 1). Since any hollow module is indecomposable, the above also holds true for hollow modules.

THEOREM 4.4. *Let R/I be hollow. If R/I is either projective or injective, then* $\mathrm{End}_R(R/I)$ *is local.*

Proof. The first part follows from Theorem 4.2 of Ware [7]. The second part follows since it is well-known that an indecomposable injective has a local ring of endomorphisms.

THEOREM 4.5. *If I is divisible as a right $\mathscr{I}(I)$ module, and R/I is hollow, then* $\mathrm{End}_R(R/I)$ *is local.*

Proof. Let a represent a map from R/I to R/I and $a \notin M$. We would like to show that the map induced by a has an inverse. Since $a \notin M$, there is $r \in R$, $i_1 \in I$ and $ra + i_1 = 1$. By the remarks preceding Theorem 4.1, we may as well assume there is $i_2 \in I$ and $ra + i_2 = 1$. If $Ir \subseteq I$, we are done since then r would induce the required inverse for a. Now if $ir \in Ir$, then $i = i_3 a$ since I is divisible in its idealizer. So $ir = i_3 ar = i_3 - i_3 i_2$ which implies $Ir \subseteq I$.

COROLLARY 4.6. *If I is injective in its idealizer and R/I is hollow, then* $\mathrm{End}_R(R/I)$ *is local.*

THEOREM 4.7. *If $I \cap Ra = Ia$ for all $a \in \mathscr{I}(I)$ and R/I is hollow, then* $\mathrm{End}_R(R/I)$ *is local.*

Proof. Let $a \notin M$ induce a map from R/I to R/I. We know the map induced by a is epic. If $xa \in I$, then $xa = i_1 a$ by hypothesis. Thus $(x - i_1)a = 0$. From the remarks preceding Theorem 4.1, we can assume there is r with $ar = 1 - i_2$, i_2 I. Thus $(x - i_1)ar = 0 = (x - i_1)(1 - i_2)$. So $x = i_1(1 - i_2) + xi_2 \in I$.

5. Final remarks. Proposition 3.1 and Corollary 3.2 are interesting because they give relationships between the idealization of I and the idealization of the unique maximal ideal containing I. There are more theorems like this which relate I, M, and the left annihilator of I.

PROPOSITION 5.1. *Let I be a left ideal of R which is uniquely contained in the maximal left ideal M. If either I contains no idempotents or $I^2 \neq I$, then* $(0\colon I) \subseteq M$.

Proof. Suppose $(0\colon I) \not\subseteq M$. Then there is $\alpha \in (0\colon I)$ and $i \in I$ and $\alpha + i = 1$ by Theorem 2.3. Thus $i = (\alpha + i)i = \alpha i + i^2 = i^2$. Thus i contains an idempotent. Furthermore, if $i_1 \in I$, then $i_1 = (\alpha + i)i_1 = ii_1$ so $i_1 \in I^2$. Thus $I = I^2$.

PROPOSITION 5.2. *Let I, M, and R be as above. Then, if M is not an ideal, $IR = R$ and* $(0\colon I) = 0$.

Proof. The ideal IR contains I so either it is contained in M or it is equal to R. By Proposition 3.1, if $IR \subseteq M$, then $\mathscr{I}(IR) \subseteq \mathscr{I}(M)$. But $\mathscr{I}(IR) = R$ and $\mathscr{I}(M) \neq R$ by hypothesis. Thus $IR = R$.

Now suppose $\alpha \in (0: I)$. Since $IR = R$, there are elements $i_1, \cdots, i_n \in I, r_1, \cdots, r_n \in R$ and $i_1r_1 + \cdots + i_nr_n = 1$. So $\alpha = \alpha i_1r_1 + \cdots + \alpha i_nr_n = 0$. Thus $(0: I) = 0$.

Finally, we will close by showing that some hollow modules are not cyclic. For example, if $p \in Z$ is a prime, consider $Z_{(p)}$ the abelian group of p-adic integers. We can show this is hollow in the following way. Suppose $Z_{(p)} = A_1 + A_2$ where A_1 and A_2 are both proper subgroups of $Z_{(p)}$. Let π_n be the canonical projection of $Z_{(p)}$ on Z/p^nZ. We know each Z/p^nZ is hollow. Thus, since $Z/pZ = \pi_nA_1 + \pi_nA_2$ we must have one, say π_nA_1, equal to Z/p^nZ. But this would imply $\pi_iA_1 = Z/p^iZ$ for all $i < n$. Thus $\pi_nA_1 = Z/p^nZ$ for all n. But then $A_1 = Z_{(p)}$ and we are done.

As another, and easier, example of an infinitely generated hollow module, we can consider the abelian group $Z(p^\infty)$ for any prime p. This is an infinite group but it must be hollow since every proper subgroup of $Z(p^\infty)$ is finite while the group itself is infinite.

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